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Diffusion and wave behaviour in linear Voigt model

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Abstract A boundary value problem $\mathcal{P}_\varepsilon$ related to a third order parabolic equation with a small parameter ε is analized. This equation models the one-dimensional evolution of many dissipative media as viscoelastic fluids or solids, viscous gases, superconducting materials, incompressible and electrically conducting fluids. Moreover, the third order parabolic operator regularizes various nonlinear second order wave equations. In this paper, the hyperbolic and parabolic behaviour of the solution of $\mathcal{P}_\varepsilon$ is estimated by means of *slow time* $\tau = \varepsilon t$ and *fast time* $\theta = t/\varepsilon$. As consequence, a rigorous asymptotic approximation for the solution of $\mathcal{P}_\varepsilon$ is established. To cite this article: M. De Angelis, P. Renno, C. R. Mecanique 330 (2002) 21–26. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

continuum media / partial different equations / viscoelasticity / superconductivity / boundary layer

Diffusion et comportement ondulatoire dans le modèle linéaire de Voigt

Résumé On analyse un problème aux limites $\mathcal{P}_\varepsilon$ pour une équation parabolique du troisième ordre. Cette équation décrit l'évolution monodimensionnelle de beaucoup de materiaux dissipatifs comme les fluides ou les solides viscoélastiques, les gaz visqueux, les materiels superconducturs, les fluides incompressibles conducteurs de l'électricité. De plus l'opérateur parabolique du troisième ordre regularise divers equations non lineaires des ondes du deuxième ordre. On examine dans ce travail le comportment hyperbolique ou parabolique de la solution du problème $\mathcal{P}_\varepsilon$ à l'aide des temps lent et rapide. En conséquence, on donne une approximation asymptotique rigooreose de la solution du problème $\mathcal{P}_\varepsilon$. Pour citer cet article : M. De Angelis, P. Renno, C. R. Mecanique 330 (2002) 21–26. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

milieux continus / équations aux dérivées partielles / viscoéllasticité / supraconductivité

1. Introduction

The parabolic equation

$$\mathcal{L}_\varepsilon u \equiv \varepsilon \partial_{xxt} u + c^2 \partial_{xx} u - \partial_{tt} u = -f \tag{1.1}$$

describes a great deal of models of applied sciences and represents a typical example of hyperbolic equations perturbed by viscous terms.

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According to the meaning of f, examples of dissipative phenomena related to (1.1) are: motions of viscoelastic fluids or solids [1–3], heat conduction at low temperature [4,5], sound propagation in viscous gases [6], propagation of plane waves in perfect incompressible and electrically conducting fluids [7]. Moreover, when $f = au_t + \sin u - \gamma$, the equation (1.1) is the *perturbed sine-Gordon equation* which models the Josephson tunnel effect in superconductivity [8]. Further applications of (1.1) arise in the study of viscoelastic plates with memory, when the relaxation function is given by an exponential function [9]. At last, one remarks that even the Navier–Stokes equations for a compressible gas with small viscosity, in Lagrangian coordinates, can be reduced to (1.1) with $f = f(u_t, u_x, u_{xt}, u_{xx})$ [10].

Also the meaningful analytical results concerning the qualitative analysis of (1.1) are very numerous and one can refer to an extensive bibliography (e.g., [10–18]). In particular, the behaviour of solutions of (1.1) when $\varepsilon \to 0$ has been analized in various applications of artificial viscosity method to non linear second order wave equation [19,20]. But, from a physical point of view, it would be interesting to estimate the time – intervals where the hyperbolic or parabolic behaviour prevails, evaluating so the influence of dissipative causes on the wave propagation.

These aspects are analyzed in this paper referring to the strip problem $\mathcal{P}_\varepsilon$ for equation (1.1) with a linear f. The Green function G related to $\mathcal{P}_\varepsilon$ has already been determined in [21] by means of a rapidly decreasing Fourier series, and its asymptotic behaviour for $t \to \infty$ has been obtained, too.

Now, in the hypotesis of ε vanishing, appropriate estimates of G by the *slow time* εt and the *fast time* t/ε will be established. As consequence, the main result is a rigorous approximation for the solution of the problem $\mathcal{P}_\varepsilon$ which holds for all $t < \varepsilon^{-\eta}$ ($\eta > 0$).

2. Statement of the problem

If T is a positive constant and

$$D = \left\{(x,t) : 0 \leqslant x \leqslant l,\ 0 < t \leqslant T\right\}$$

let $u(x,t)$ the regular solution of the boundary initial value problem:

$$\begin{cases} \partial_{xx}(\varepsilon u_t + c^2 u) - \partial_{tt} u = f(x,t), & (x,t) \in D \\ u(x,0) = f_0(x), \quad u_t(x,0) = f_1(x), & x \in [0,l] \\ u(0,t) = 0, \quad u(l,t) = 0, & 0 < t \leqslant T \end{cases} \tag{2.1}$$

where $f(x,t)$ is an arbitrary specified function.

Now, denote with $w(x,t)$ the solution of the reduced problem obtained by (2.1) with $\varepsilon = 0$. To establish a rigorous asymptotic approximation for $u(x,t)$ when $\varepsilon \to 0$, we put:

$$u(x,t,\varepsilon) = \mathrm{e}^{-\varepsilon t} w(x,t) + r(x,t,\varepsilon) \tag{2.2}$$

where the *error* $r(x,t,\varepsilon)$ must be estimated.

By means of standard computations one verifies that $r(x,t,\varepsilon)$ is the solution of the problem:

$$\begin{cases} \partial_{xx}(\varepsilon r_t + c^2 r) - \partial_{tt} r = F(x,t,\varepsilon), & (x,t) \in D \\ r(x,0) = 0, \quad r_t(x,0) = 0, & x \in [0,l] \\ r(0,t) = 0, \quad r(l,t) = 0, & 0 < t \leqslant T \end{cases} \tag{2.3}$$

where the source term $F(x,t,\varepsilon)$ is:

$$F(x,t,\varepsilon) = f(x,t)\left(1 - \mathrm{e}^{-\varepsilon t}\right) + \mathrm{e}^{-\varepsilon t}\left[-\varepsilon \lambda_t + \varepsilon^2 (w + w_{xx})\right] \tag{2.4}$$

with $\lambda = 2w + w_{xx}$.

The problem (2.3) has already been solved in [21] and the solution is given by:

$$r(x,t,\varepsilon) = -\int_0^l \mathrm{d}\xi \int_0^t F(\xi,\tau,\varepsilon) G(x,\xi,t-\tau)\,\mathrm{d}\tau \tag{2.5}$$

where $G(x,\xi,t)$ is:

$$G(x,\xi,t) = \frac{2}{l}\sum_{n=1}^{\infty} H_n(t)\sin\gamma_n x\sin\gamma_n\xi \tag{2.6}$$

with

$$H_n(t) = \frac{\mathrm{e}^{-bn^2 t}}{bn^2\sqrt{1-(k/n)^2}}\sinh\left\{bn^2 t\sqrt{1-(k/n)^2}\right\} \tag{2.7}$$

and

$$b = \frac{\pi^2}{2l^2}\varepsilon = q\varepsilon, \quad k = \frac{2cl}{\pi\varepsilon}, \quad \gamma_n = \frac{\pi}{l}n \tag{2.8}$$

3. Estimates of the Green function by fast and slow times

When $\varepsilon \to 0$, two characteristic times affect the behaviour of G, i.e., $\tau = \varepsilon t$ (*slow time*), and $\theta = t/\varepsilon$ (*fast time*). To point out the different contributions an appropriate form of G we will considered. For this, if $N = [2cl/(\pi\varepsilon)]$, the G-function can be given the form:

$$G = \frac{2}{l}\left\{\sum_{n=1}^{N} + \sum_{N+1}^{\infty}\right\} H_n(t)\sin(\gamma_n x)\sin(\gamma_n\xi) = G_1 + G_2 \tag{3.1}$$

where, for $n < N$, the functions H_n are:

$$H_n(t) = \frac{\mathrm{e}^{-bn^2 t}}{bn^2\sqrt{(k/n)^2-1}}\sin\left\{bn^2 t\sqrt{(k/n)^2-1}\right\} \tag{3.2}$$

If α is an arbitrary constant such that:

$$1/2 < \alpha < 1, \qquad N_\alpha = \left[\frac{2cl}{\pi\varepsilon^\alpha}\right] \tag{3.3}$$

the term G_1 can be written:

$$G_1(x,\xi,t) = \frac{2}{l}\left\{\sum_{n=1}^{N_\alpha} H_n(t) + \sum_{N\alpha+1}^{N} H_n(t)\right\}\sin(\gamma_n x)\sin(\gamma_n\xi) \tag{3.4}$$

It is easy to prove that when $1 \leqslant n \leqslant N_\alpha$ it results:

$$\sqrt{\left(\frac{k}{n}\right)^2 - 1} \geqslant \frac{\sqrt{1-\varepsilon^{2(1-\alpha)}}}{\varepsilon^{1-\alpha}}, \qquad \mathrm{e}^{-bn^2 t} \leqslant \mathrm{e}^{-qt\varepsilon} \tag{3.5}$$

Otherwise, if $N_\alpha + 1 \leqslant n \leqslant N$, one has $N = k - \beta$ with $0 < \beta < 1$, and it is:

$$\sqrt{\left(\frac{k}{n}\right)^2 - 1} \geqslant \frac{\sqrt{\pi\varepsilon\beta}\,\sqrt{4cl-\beta\pi\varepsilon}}{2cl-\pi\varepsilon\beta}, \qquad \mathrm{e}^{-bn^2 t} \leqslant \mathrm{e}^{-2c^2 t/\varepsilon^{2\alpha-1}} \tag{3.6}$$

In particular, when k is an integer one has $\beta = 1$ and the term $t\,\mathrm{e}^{-2c^2 t/\varepsilon}$ must be considered too.

For all $\varepsilon \in]0, \varepsilon_0]$ ($\varepsilon_0 < 1$), the formulae (3.5), (3.6) allow to obtain the following estimate for G_1:

$$\left|G_1(x,\xi,t)\right| \leqslant A_0\varepsilon^{-\alpha}\,\mathrm{e}^{-qt\varepsilon} + A_1\varepsilon^{-3/2}\,\mathrm{e}^{-c^2t/\varepsilon^{2\alpha-1}} \tag{3.7}$$

where the constants A_0, A_1 do not depend on ε. As $\varepsilon < \varepsilon^{1-2\alpha}$, the prevailing term in (3.7) is related to the slow time $\tau = \varepsilon t$. So, the circular component G_1 is controlled by the slow time. On the contrary, the hyperbolic component G_2 is characterized only by the fast time θ. In fact, let:

$$C = \frac{\pi(1-\beta)4cl}{2cl+\pi(1-\beta)}, \qquad C_1 = \frac{2\zeta(2)}{qlC} \tag{3.8}$$

with $\beta \equiv 0$ if k is an integer. Observing that $\forall n \geqslant N+1$, one has (see, f.i. [21]):

$$bn^2t\left(1 \pm \sqrt{1-\left(\frac{k}{n}\right)^2}\right) \geqslant \frac{c^2t}{\varepsilon}, \qquad \sqrt{1-\left(\frac{k}{n}\right)^2} \geqslant \varepsilon C \tag{3.9}$$

and consequently it results:

$$\left|G_2(x,\xi,t)\right| \leqslant C_1\varepsilon^{-2}\,\mathrm{e}^{-c^2t/\varepsilon} \tag{3.10}$$

So, if $M_0 = \max\{A_1, C_1\}$ and $1/2 < \alpha < 1$, the following theorem holds:

THEOREM 3.1. – *For all $\varepsilon \in (0, \varepsilon_0]$ ($\varepsilon_0 < 1$) and $(x,t) \in D$, the Green function $G(x,\xi,t)$ verifies the following estimate*:

$$\left|G(x,\xi,t)\right| \leqslant A_0\varepsilon^{-\alpha}\,\mathrm{e}^{-qt\varepsilon} + M_0\varepsilon^{-3/2}\,\mathrm{e}^{-c^2t/\varepsilon^{2\alpha-1}} \tag{3.11}$$

where the constants A_0, M_0 do not depend on ε.

4. On the behaviour of the solution

Now, the remainder term $r(x,t,\varepsilon)$ of (2.2) can be estimated. Referring to the function f defined in (2.4), let

$$\left\|F\right\| = \max\left\{\sup_D \left|f(x,t)\right|, \sup_D \left[|\lambda_t| + \varepsilon|\lambda - u|\right]\right\} \tag{4.1}$$

Then, one has the following theorem:

THEOREM 4.1. – *Let $F(x,t,\varepsilon) \in C^1(D)$ and let F, F_x, F_t bounded for all t. Then, the error term $r(x,t,\varepsilon)$ verifies the estimate*:

$$\left|r(x,t,\varepsilon)\right| < k\|F\|\left(\varepsilon^{\eta}t\right)^2 \tag{4.2}$$

where the constants k and η do not depend on ε and $\eta \in (0, 1/2)$.

Proof. – First, by (2.4), (2.5) one deduces:

$$\left|r(x,t,\varepsilon)\right| \leqslant l\varepsilon \int_0^t \mathrm{e}^{-\varepsilon\tau}\left\{\left|\lambda_t(x,\tau)\right| + \varepsilon|\lambda - u|\right\}\left|G(x,\xi,t-\tau)\right| \mathrm{d}\tau$$
$$+ l\int_0^t \left|f(x,\tau)\right|\left|1-\mathrm{e}^{-\varepsilon\tau}\right|\left|G(x,\xi,t-\tau)\right| \mathrm{d}\tau \tag{4.3}$$

Further, by means of the well-known inequality [22]:

$$\mathrm{e}^{-x} \leqslant \left[\frac{\gamma}{\mathrm{e}x}\right]^{\gamma} \quad \forall\gamma > 0,\ \forall x > 0 \tag{4.4}$$

by (3.10) and (4.3) one can deduce:

$$|r| \leqslant \|F\| l t^2 \left\{\frac{3}{2}A_0\varepsilon^{1-\alpha} + \frac{2M_0}{(\mathrm{e}c^2)^{\gamma}(1-\gamma)}\frac{\varepsilon^{(2\alpha-1)\gamma}}{\sqrt{\varepsilon}}\right\} \tag{4.5}$$

So, if α and γ are such that $3/4 < \alpha < 1$ and $[2(2\alpha-1)]^{-1} < \gamma < 1$, it suffices to put

$$2\eta = \min\left\{(2\alpha-1)\gamma - \frac{1}{2},\ 1-\alpha\right\}, \quad k = \max\left\{\frac{3}{2}A_0,\ \frac{2M_0}{(\mathrm{e}c^2)^{\gamma}(1-\gamma)}\right\} \tag{4.6}$$

to deduce (4.2).

As consequence of Theorem 4.1, finally we can observe that:

When $\varepsilon \to 0$, *the solution of the problem* (2.1) *can be approximatated by means of the following formula*:

$$u(x,t,\varepsilon) = \mathrm{e}^{-\varepsilon t} w(x,t) + r(x,t,\varepsilon) \tag{4.7}$$

where the error $r(x,t,\varepsilon)$ *is bounded for all* $t < (1/\varepsilon)^{\eta}$.

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