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Second-order work and dissipation on indirect paths

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Abstract The concept of the second-order dissipation on arbitrary complex paths in the space of internal state variables is developed. A general framework for time-independent dissipative solids is adopted which encompasses plasticity, micro-cracking or martensitic phase transformation. Circumstances are established in which the dissipation evaluated to the second order is minimized on a radial path. As an application, this minimum property is used to simplify a sufficiency condition for stability of equilibrium. ***To cite this article: H. Petryk, C. R. Mecanique 330 (2002) 121–126.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

solids and structures / dissipative materials / energy / path-dependence / stability

Travail du deuxième ordre et dissipation sur des chemins indirects

Résumé Nous développons le concept de dissipation du deuxième ordre sur des chemins complexes arbitraires dans l'espace des variables d'état internes. On adopte un cadre général pour les solides dissipatifs indépendants du temps qui contient la plasticité, la micro-fissuration et les changements de phase mortensitiques. On établit les circonstances dans lesquelles la dissipation évaluée au deuxième ordre est minimisée sur un chemin radical. En tant qu'application on utilise cette propriété de minimum pour simplifier une condition suffisante pour le stabilité de l'équilibre. ***Pour citer cet article : H. Petryk, C. R. Mecanique 330 (2002) 121–126.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

The second-order work of deformation in time-independent solids is a classical concept which has played an important role in classifying elastic-plastic materials and investigating stability [1,2]. In a geometrically exact treatment, a correct formula for the second-order work per unit *reference* volume is obtained by using a *work-conjugate* pair $(\mathbf{e}, \mathbf{t})$ of arbitrarily chosen Lagrangean measures of strain $\mathbf{e}$ and stress $\mathbf{t}$. Along a *direct* smooth path of negligible curvature, the second-order work per unit reference volume is $\Delta_2 w = \frac{1}{2}\Delta\mathbf{t}\cdot\Delta\mathbf{e}$, where a prefix Δ denotes a final small increment and a central dot denotes full contraction. The value of the second-order work is sensitive to the choice of the pair $(\mathbf{e}, \mathbf{t})$ [3].

The concept of the second-order work has been extended in [4] to arbitrarily circuitous strain paths whose complexity is preserved as their length tends to zero. Such paths are called *indirect* paths; they are *not* fixed

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as the Euclidean norm of a strain increment $|\Delta\mathbf{e}| \to 0$ but, for instance, can be scaled down proportionally to a final $\Delta\mathbf{e}$. This generalizes the earlier concepts of the second-order work done on *small* closed cycles in stress [1] or strain [3]. Under the assumption (to be relaxed in this paper) of continuing equilibrium *within* the material element, the second-order work $\Delta_2 w$ per unit reference volume is defined by the work decomposition

$$\Delta w = \Delta_1 w + \Delta_2 w + \mathrm{o}\big((\Delta l)^2\big), \quad l = \int (\mathrm{d}\mathbf{e}\cdot\mathrm{d}\mathbf{e})^{1/2} \tag{1}$$

$\mathrm{o}(\cdot)$ being the usual order symbol such that $\mathrm{o}(r^2)/r^2 \to 0$ as $r \to 0$. If the stress varies continuously along any path then the first-order work reads $\Delta_1 w = \mathbf{t}^0 \cdot \Delta\mathbf{e}$, where $\mathbf{t}^0$ is the stress at the beginning of the increment. The second-order work density $\Delta_2 w$ is generally *path-dependent*, i.e., depends on the *family* of paths for which the quotient $(\Delta w - \mathbf{t}^0 \cdot \Delta\mathbf{e})/(\Delta l)^2$ has a well-defined limit value as $\Delta l \to 0$.

For a class of multi-mode elastoplastic solids obeying the normality and symmetry postulates it has been shown [4] that $\Delta_2 w = \frac{1}{2}\Delta\mathbf{t}\cdot\Delta\mathbf{e} - \frac{1}{2}\sum_K \Delta f_K \Delta\gamma_K$ where f_K is a yield function and γ_K is a time integral of the plastic multiplier for a mode index K. Circumstances have also been established in which $\Delta_2 w$ corresponding to a given $\Delta\mathbf{e}$ is minimized on a direct strain path.

The aim of this paper is to extend the concept of the path-dependent second-order work to the second-order dissipation on indirect paths in the space of internal state variables $\boldsymbol{\alpha}$. The investigations are not restricted to plasticity; other time-independent sources of dissipation like micro-cracking or martensitic phase transformation also fall within the framework used below.

2. Work and dissipation

Suppose that the state of a material sample $\mathcal{M}$ being in mechanical and thermal equilibrium is characterized, to a suitable approximation, by a triple $(\mathbf{e}, T, \boldsymbol{\alpha})$, where $\mathbf{e}$ is a macroscopic strain relative to a fixed reference configuration, T is a uniform temperature, and $\boldsymbol{\alpha}$ stands for a collection of internal state variables, considered in the *reference* configuration, that characterize internal structural rearrangements within $\mathcal{M}$. For simplicity, the volume occupied by $\mathcal{M}$ in the reference configuration is taken as unity. In the following the temperature T is assumed to be fixed and is removed from the list of variables. The analysis and results can be straightforwardly extended to more general systems with $(\mathbf{e}, \mathbf{t})$ as a work-conjugate pair of finite dimensional vectors of generalized displacements and forces.

An increment Δw of work (per unit reference volume) to be supplied to $\mathcal{M}$ in isothermal transition between two states $(\mathbf{e}^0, \boldsymbol{\alpha}^0)$ and $(\mathbf{e}, \boldsymbol{\alpha})$ of internal mechanical equilibrium is classically split into two parts

$$\Delta w = \Delta\phi + \Delta\mathcal{D} \tag{2}$$

where ϕ is the Helmholtz free energy, assumed to be a twice Fréchet differentiable function of $(\mathbf{e}, \boldsymbol{\alpha})$, and $\mathcal{D}$ is the intrinsic dissipation, being a path-dependent functional. If mechanical equilibrium within $\mathcal{M}$ is perturbed (at a micro-level) then two extra terms enter the energy balance: kinetic energy $K \geqslant 0$ and an excess $\phi^{\mathrm{act}} - \phi$ of the actual free energy integral ϕ^{act} within $\mathcal{M}$, of the integrand defined with the help of the axiom of local state, over its equilibrium value $\phi(\mathbf{e}, \boldsymbol{\alpha})$. We return to these extra terms in Section 4 when discussing material stability aspects.

The stress $\mathbf{t}$ conjugate to $\mathbf{e}$, and thermodynamic forces conjugate to $\boldsymbol{\alpha}$, denoted collectively by $\boldsymbol{A}$, are determined from

$$\mathbf{t} = \frac{\partial\phi}{\partial\mathbf{e}}(\mathbf{e}, \boldsymbol{\alpha}), \qquad \boldsymbol{A} = -\frac{\partial\phi}{\partial\boldsymbol{\alpha}}(\mathbf{e}, \boldsymbol{\alpha}) \tag{3}$$

Accordingly,

$$\Delta\phi = \Delta_1\phi + \Delta_2\phi + \mathrm{o}\big(\rho^2\big), \quad \rho = a|\Delta\mathbf{e}| + \|\Delta\boldsymbol{\alpha}\|, \quad 0 < a = \mathrm{const} \tag{4}$$

where

$$\Delta_1\phi = \mathbf{t}^0 \cdot \Delta\mathbf{e} - \boldsymbol{A}^0 \cdot \Delta\boldsymbol{\alpha}, \qquad \Delta_2\phi = \frac{1}{2}\Delta\mathbf{t} \cdot \Delta\mathbf{e} - \frac{1}{2}\Delta\boldsymbol{A} \cdot \Delta\boldsymbol{\alpha} \tag{5}$$

and $\|\Delta\boldsymbol{\alpha}\| = (\Delta\boldsymbol{\alpha} \cdot \Delta\boldsymbol{\alpha})^{1/2}$ is a norm in the Hilbert space assumed to contain all $\boldsymbol{\alpha}$.

We shall assume that the knowledge of the *forward rate* $\dot{\boldsymbol{\alpha}}$ (right-hand time derivative) of $\boldsymbol{\alpha}$ in a given state determines the intrinsic dissipation rate $\dot{\mathcal{D}}$ in the material sample $\mathcal{M}$. Moreover, the intrinsic dissipation rate is assumed to be time-independent and insensitive to elastic strain or stress. It means that *the dissipation function* D of $(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha})$ [5] is given such that

$$\dot{\mathcal{D}} = D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) \geqslant 0, \quad D(r\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) = rD(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) \quad \text{for all } r > 0 \tag{6}$$

Since $\dot{\boldsymbol{\alpha}}$ represents *virtual* internal structural rearrangements in $\mathcal{M}$ which can take place under suitable perturbing forces within the material sample, the function D defines a *virtual* dissipation rate. An important consequence is that Δw defined by (2) can differ from that in (1) by the amount of work supplied to $\mathcal{M}$ directly by a disturbing agency acting *between* the initial and final states. Only in the absence of such disturbances we must have $\dot{w} = \mathbf{t} \cdot \dot{\mathbf{e}}$ and $D = \boldsymbol{A} \cdot \dot{\boldsymbol{\alpha}}$.

3. Decomposition of intrinsic dissipation

We shall consider a class of piecewise-smooth paths $\boldsymbol{\alpha}(\zeta)$ of a current length ζ and a small final length $\Delta\zeta = \zeta^\Delta - \zeta^0$ in the $\boldsymbol{\alpha}$-space, leading from a given $\boldsymbol{\alpha}^0 = \boldsymbol{\alpha}(\zeta^0)$ to any $\boldsymbol{\alpha}^0 + \Delta\boldsymbol{\alpha} = \boldsymbol{\alpha}(\zeta^\Delta)$. The forward rate $\dot{\boldsymbol{\alpha}}$ is taken, without loss of generality, with respect to ζ as a time-like parameter. The dissipation $\Delta\mathcal{D}$ along such a path, defined by time integration of the dissipation function (6), is decomposed as follows:

$$\Delta\mathcal{D} = \Delta_1\mathcal{D} + \Delta_2\mathcal{D} + \mathrm{o}\big((\Delta\zeta)^2\big), \quad \zeta = \int (\mathrm{d}\boldsymbol{\alpha} \cdot \mathrm{d}\boldsymbol{\alpha})^{1/2} \tag{7}$$

where *both* terms $\Delta_1\mathcal{D}$ and $\Delta_2\mathcal{D}$ are generally *path-dependent*. The first-order dissipation is defined by fixing the argument $\boldsymbol{\alpha} = \boldsymbol{\alpha}^0$ of D, viz.

$$\Delta_1\mathcal{D} = \int_{\zeta^0}^{\zeta^\Delta} D\big(\dot{\boldsymbol{\alpha}}(\theta), \boldsymbol{\alpha}^0\big)\,\mathrm{d}\theta \tag{8}$$

The second-order dissipation $\Delta_2\mathcal{D}$ is defined by linearizing the dependence of D on $\boldsymbol{\alpha}$ (but with *no* linearization with respect to $\dot{\boldsymbol{\alpha}}$), that is,

$$\Delta_2\mathcal{D} = \int_{\zeta^0}^{\zeta^\Delta} D_1\big(\dot{\boldsymbol{\alpha}}(\theta), \hat{\boldsymbol{\alpha}}(\theta)\big)\,\mathrm{d}\theta, \quad \hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha} - \boldsymbol{\alpha}^0, \quad D_1\big(\dot{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}}\big) \equiv \hat{\boldsymbol{\alpha}} \cdot D_{,\boldsymbol{\alpha}}\big(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}^0\big) \tag{9}$$

under the assumption that the function $D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha})$ possesses the Fréchet derivative $D_{,\boldsymbol{\alpha}} = \partial D/\partial\boldsymbol{\alpha}$ with respect to $\boldsymbol{\alpha}$, so that

$$D\big(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}^0 + \hat{\boldsymbol{\alpha}}\big) - D\big(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}^0\big) - \hat{\boldsymbol{\alpha}} \cdot D_{,\boldsymbol{\alpha}}\big(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}^0\big) = \|\dot{\boldsymbol{\alpha}}\|\,\mathrm{o}\big(\|\hat{\boldsymbol{\alpha}}\|\big) \quad \text{for all } \hat{\boldsymbol{\alpha}},\ \dot{\boldsymbol{\alpha}} \tag{10}$$

Clearly, $D_1(r\dot{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}}) = rD_1(\dot{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})$ for every $r > 0$ and $D_1(\dot{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})$ is linear with respect to $\hat{\boldsymbol{\alpha}}$.

The *proof* of the decomposition in (7) follows by the rearrangement of a time integral of D with the help of the estimates (10) and $\|\hat{\boldsymbol{\alpha}}\| \leqslant \Delta\zeta$.

The integrand in (9) can be transformed as follows

$$2D_1\big(\dot{\boldsymbol{\alpha}}(\theta), \hat{\boldsymbol{\alpha}}(\theta)\big) = D_1\big(\dot{\boldsymbol{\alpha}}(\theta), \hat{\boldsymbol{\alpha}}(\theta)\big) + \int_{\zeta^0}^{\theta} D_1\big(\dot{\boldsymbol{\alpha}}(\theta), \dot{\boldsymbol{\alpha}}(\zeta)\big)\,\mathrm{d}\zeta$$

$$= D_1\big(\dot{\boldsymbol{\alpha}}(\theta), \hat{\boldsymbol{\alpha}}(\theta)\big) + \int_{\zeta^0}^{\theta} D_1\big(\dot{\boldsymbol{\alpha}}(\zeta), \dot{\boldsymbol{\alpha}}(\theta)\big)\,\mathrm{d}\zeta + D_1^{\mathrm{a}}(\theta)$$

$$= \frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\zeta^0}^{\theta} D_1\big(\dot{\boldsymbol{\alpha}}(\zeta), \hat{\boldsymbol{\alpha}}(\theta)\big)\,\mathrm{d}\zeta + D_1^{\mathrm{a}}(\theta) \tag{11}$$

where

$$\mathcal{D}_1^{\mathrm{a}}(\theta) = \int_{\zeta^0}^{\theta}\big(D_1\big(\dot{\boldsymbol{\alpha}}(\theta), \dot{\boldsymbol{\alpha}}(\zeta)\big) - D_1\big(\dot{\boldsymbol{\alpha}}(\zeta), \dot{\boldsymbol{\alpha}}(\theta)\big)\big)\,\mathrm{d}\zeta \tag{12}$$

On substituting (11) into (9) we obtain

$$\Delta_2\mathcal{D} = \frac{1}{2}\int_{\zeta^0}^{\zeta^\Delta} D_1\big(\dot{\boldsymbol{\alpha}}(\theta), \Delta\boldsymbol{\alpha}\big)\,\mathrm{d}\theta + \frac{1}{2}\int_{\zeta^0}^{\zeta^\Delta} \mathcal{D}_1^{\mathrm{a}}(\theta)\,\mathrm{d}\theta \tag{13}$$

The last term vanishes by (12) if the dissipation function possesses the symmetry property

$$D_1\big(\dot{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}}\big) = D_1\big(\hat{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}}\big) \quad \text{for all } \dot{\boldsymbol{\alpha}},\ \hat{\boldsymbol{\alpha}} \tag{14}$$

The dissipation (7) evaluated to second order along a straight segment initiated at $\boldsymbol{\alpha}^0$ in $\boldsymbol{\alpha}$-space, called *a radial path*, after straightforward integration with the use of (6) and (10) is specified as the sum of

$$\Delta_1\mathcal{D}^{\mathrm{d}} = D\big(\Delta\boldsymbol{\alpha}, \boldsymbol{\alpha}^0\big) \quad \text{and} \quad \Delta_2\mathcal{D}^{\mathrm{d}} = \frac{1}{2}D_1(\Delta\boldsymbol{\alpha}, \Delta\boldsymbol{\alpha}) \tag{15}$$

Validity of (15) can be extended to a class of *direct* paths whose tangent tends uniformly to that of a radial path as $\|\Delta\boldsymbol{\alpha}\| \to 0$, implying that $\mathcal{D}_1^{\mathrm{a}} = \mathrm{o}(\|\Delta\boldsymbol{\alpha}\|)$; hence the superscript 'd'.

It is well known [5] that if

$$D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) \text{ is convex in } \dot{\boldsymbol{\alpha}} \tag{16}$$

for every $\boldsymbol{\alpha}$ then

$$\int_{\zeta^0}^{\zeta^\Delta} D\big(\dot{\boldsymbol{\alpha}}(\theta), \boldsymbol{\alpha}^0\big)\,\mathrm{d}\theta \geqslant D\big(\Delta\boldsymbol{\alpha}, \boldsymbol{\alpha}^0\big) \tag{17}$$

A simple proof is obtained by noting that $D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) + D(\dot{\boldsymbol{\alpha}}^*, \boldsymbol{\alpha}) \geqslant D(\dot{\boldsymbol{\alpha}} + \dot{\boldsymbol{\alpha}}^*, \boldsymbol{\alpha})$ for all $\dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}}^*, \boldsymbol{\alpha}$ and extending the inequality to an infinite sum. It follows that

$$\Delta\mathcal{D} \geqslant \Delta_1\mathcal{D}^{\mathrm{d}} + \mathrm{o}\big((\Delta\zeta)^2\big) \quad \text{if } D_1 \equiv 0 \tag{18}$$

The minimum property (18) can be extended to a class of dissipation functions D dependent on $\boldsymbol{\alpha}$. Namely, the following proposition holds true.

PROPOSITION 1. – *If the dissipation function satisfies* (6), (10), (16) *and the symmetry condition* (14) *then the dissipation* (7) *evaluated to the second order is minimized on a radial path, i.e.,*

$$\Delta\mathcal{D} \geqslant \Delta_1\mathcal{D}^{\mathrm{d}} + \Delta_2\mathcal{D}^{\mathrm{d}} + \mathrm{o}\big((\Delta\zeta)^2\big) \tag{19}$$

In *proof*, the last integral in (13) vanishes on substituting (14). By combining the expressions (13) and (8), from (7) we arrive at

$$\Delta\mathcal{D} = \int_{\zeta^0}^{\zeta^\Delta} D\left(\dot{\boldsymbol{\alpha}}(\theta), \boldsymbol{\alpha}^0 + \frac{1}{2}\Delta\boldsymbol{\alpha}\right)\mathrm{d}\theta + \mathrm{o}\big((\Delta\zeta)^2\big) \tag{20}$$

On using (17) with $\boldsymbol{\alpha}^0 + \frac{1}{2}\Delta\boldsymbol{\alpha}$ in place of $\boldsymbol{\alpha}^0$ as a consequence of (16), from (20) it follows that

$$\Delta\mathcal{D} \geqslant D\left(\Delta\boldsymbol{\alpha}, \boldsymbol{\alpha}^0 + \frac{1}{2}\Delta\boldsymbol{\alpha}\right) + \mathrm{o}\big((\Delta\zeta)^2\big) = D\big(\Delta\boldsymbol{\alpha}, \boldsymbol{\alpha}^0\big) + \frac{1}{2}D_1(\Delta\boldsymbol{\alpha}, \Delta\boldsymbol{\alpha}) + \mathrm{o}\big((\Delta\zeta)^2\big)$$

which shows validity of (19).

In applications, it is more convenient to have an estimate of the higher-order terms of $\Delta\mathcal{D}$ expressed through the direct final distance $\|\Delta\boldsymbol{\alpha}\|$ rather than through the path length $\Delta\zeta$. That replacement is not immediate since the ratio $\Delta\zeta/\|\Delta\boldsymbol{\alpha}\|$ may be unbounded in the limit as $\|\Delta\boldsymbol{\alpha}\| \to 0$. However, it suffices to add a natural assumption that the dissipation function is bounded from below and above through

$$b_1\|\dot{\boldsymbol{\alpha}}\| \leqslant D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) \leqslant b_2\|\dot{\boldsymbol{\alpha}}\| \tag{21}$$

where b_1 and b_2 are positive constants. Then

$$\Delta\mathcal{D} \geqslant b_1\Delta\zeta, \qquad \Delta_1\mathcal{D}^{\mathrm{d}} + \Delta_2\mathcal{D}^{\mathrm{d}} \leqslant b_2\|\Delta\boldsymbol{\alpha}\|\big(1 + b_3\|\Delta\boldsymbol{\alpha}\|\big) \tag{22}$$

for some positive constant b_3, which leads to the following result.

PROPOSITION 2. – *Under the assumptions of Proposition* 1*, if* (21) *holds then*

$$\Delta\mathcal{D} \geqslant \Delta_1\mathcal{D}^{\mathrm{d}} + \Delta_2\mathcal{D}^{\mathrm{d}} + \mathrm{o}\big(\|\Delta\boldsymbol{\alpha}\|^2\big) \tag{23}$$

In fact, in the case when $\Delta\zeta/\|\Delta\boldsymbol{\alpha}\| \to \infty$ as $\|\Delta\boldsymbol{\alpha}\| \to 0$ the estimate (23) is true by (22), and otherwise $\Delta\zeta$ in (19) can be replaced by $\|\Delta\boldsymbol{\alpha}\|$ being then of the same order.

It is emphasized that the expressions on both sides in (19) or (23) are evaluated for *the same* final increment $\Delta\boldsymbol{\alpha}$ in the limit as $\|\Delta\boldsymbol{\alpha}\| \to 0$. In the special case when $D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha})$ is *linear* in $\boldsymbol{\alpha}$, the higher-order terms indicated by the symbol $\mathrm{o}(\cdot)$ disappear and all the above expressions and inequalities for dissipation become exact for a finite $\Delta\boldsymbol{\alpha}$.

It is worth mentioning that for validity of (23) it is not necessary that the symmetry property (14) hold in the whole space of $\dot{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\alpha}}$ as assumed in the propositions above. The possibility of relaxing the assumption (14) is not examined here.

4. Application to material stability

The concept of the second-order work and dissipation on arbitrary indirect paths can be used to establish conditions sufficient for stability of equilibrium. This is illustrated below by the example of material stability against internal structural rearrangements at a fixed macroscopic strain $\mathbf{e}$. The amount of energy (per unit reference volume) to be supplied to the material sample $\mathcal{M}$ in isothermal transition from an equilibrium state $(\mathbf{e}, \boldsymbol{\alpha}^0)$ to a perturbed state reads

$$\Delta V = \Delta w + \big(\phi^{\mathrm{act}} - \phi\big) + K \tag{24}$$

where the quantities involved are specified in the paragraph that follows (2). Clearly, $\Delta V = \mathrm{const}$ in a motion free of current (persistent) disturbances at fixed $\mathbf{e}$. A condition for *elastic stability* within $\mathcal{M}$ is that $\phi^{\mathrm{act}} > \phi$ outside mechanical equilibrium, which we assume as granted. Then, if

$$\Delta w \geqslant C\|\Delta\boldsymbol{\alpha}\|^2 \quad \text{as long as} \quad \|\Delta\boldsymbol{\alpha}\| < R \tag{25}$$

for some positive constants C, R then ΔV defined by (24) possesses all properties of a (path-dependent) Lyapunov functional in a neighborhood of $\boldsymbol{\alpha}^0$. Consequently, by the classical Lyapunov–Movchan theory, (25) is sufficient for stability of equilibirum within $\mathcal{M}$ (at fixed $\mathbf{e}$ and for infinitesimal disturbances) with respect to two metrics, ΔV for the disturbance strength and $\|\Delta\boldsymbol{\alpha}\|$ for the distance from the examined equilibrium state. Note that (25) is sufficient for stability against both initial and persistent disturbances.

The left-hand term Δw in (25) is generally a path-dependent functional. However, if the basic estimate (23) holds, as implied by the assumptions of Proposition 2, then on using (2) and (4) the stability condition (25) reduces to

$$\Delta_1 w^{\mathrm{d}} + \Delta_2 w^{\mathrm{d}} \geqslant C \|\Delta \boldsymbol{\alpha}\|^2 \tag{26}$$

where the left-hand side expressions

$$\Delta_1 w^{\mathrm{d}} = \Delta_1 \phi + \Delta_1 \mathcal{D}^{\mathrm{d}}, \qquad \Delta_2 w^{\mathrm{d}} = \Delta_2 \phi + \Delta_2 \mathcal{D}^{\mathrm{d}} \tag{27}$$

defined relative to a given equilibrium state, are only functions of a final increment $\Delta\boldsymbol{\alpha}$ in internal state variables (for $\Delta\mathbf{e} = \mathbf{0}$). This step is essential in arriving at a verifiable stability condition. It was earlier justified for state-independent dissipation functions [6,5] (and obviously for path-independent dissipation); the present justification is extended to a wider class of dissipation functions dependent non-linearly on $\boldsymbol{\alpha}$. It should be added that (26) is to be verified only in the limit as $\|\Delta\boldsymbol{\alpha}\| \to 0$. The same essential step can be done in a more general stability analysis of a continuum for various boundary conditions.

A prerequisite for (26) is that

$$\Delta_1 w^{\mathrm{d}} = D\big(\Delta\boldsymbol{\alpha}, \boldsymbol{\alpha}^0\big) - \boldsymbol{A}^0 \cdot \Delta\boldsymbol{\alpha} \geqslant 0 \quad \text{for all } \Delta\boldsymbol{\alpha} \tag{28}$$

Under the convexity assumption (16), inequality (28) is satisfied for $\boldsymbol{A}^0$ from the admissible domain defined as the subdifferential of $D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}^0)$ with respect to $\dot{\boldsymbol{\alpha}}$ at $\dot{\boldsymbol{\alpha}} = \mathbf{0}$ [7]. In this case, the condition (25), sufficient for stability of equilibrium against internal structural rearrangements within $\mathcal{M}$ at a fixed macroscopic strain $\mathbf{e}$, can be reduced further to

$$2\Delta_2 w^{\mathrm{d}} = D_1(\Delta\boldsymbol{\alpha}, \Delta\boldsymbol{\alpha}) - \Delta\boldsymbol{A} \cdot \Delta\boldsymbol{\alpha} \geqslant 2C\|\Delta\boldsymbol{\alpha}\|^2 \quad \text{for all } \Delta\boldsymbol{\alpha} \text{ such that } \Delta_1 w^{\mathrm{d}}(\Delta\boldsymbol{\alpha}) = 0 \tag{29}$$

with $\Delta\boldsymbol{A} = -\phi_{,\boldsymbol{\alpha}\boldsymbol{\alpha}} \cdot \Delta\boldsymbol{\alpha}$ from (3). In fact, by continuity argument, (29) can be extended with a smaller positive constant C to a larger domain (cone) where $\Delta_1 w^{\mathrm{d}}(\Delta\boldsymbol{\alpha}) \leqslant k\|\Delta\boldsymbol{\alpha}\|$ for a sufficiently small positive constant k. Then, (29) is still sufficient for (26) to hold within this larger cone as follows from (5), (15), (27) and (28), while outside this cone (26) is implied by positive definiteness of the first-order term $\Delta_1 w^{\mathrm{d}}$.

Stability conditions analogous to (29) or more general were formulated earlier [6,8,5] but without the present sufficiency proof for a class of non-linearly state-dependent dissipation functions.

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