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Kinematic shakedown by the Norton–Hoff–Friaa regularising method and augmented Lagrangian

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Abstract The shakedown analysis of elastic perfectly plastic structures is formulated as a discrete nonlinear mathematical programming problem by means of the finite element technique. The kinematical problem is regularized through the introduction of the Norton–Hoff viscoplastic material to overcome the non-differentiability of the objective function, and can be solved numerically by the augmented Lagrangian technique. ***To cite this article: M.A. Hamadouche, C. R. Mecanique 330 (2002) 305–311.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

computational solid mechanics / shakedown / viscoplasticity / optimization

Adaptation cinématique par la methode de régularisation de Norton–Hoff–Friaa et du Lagrangien augmenté

Résumé L'analyse d'adaptation des structures élastique parfaitement plastique est formulée comme un problème de programmation mathématique non-linear discret, en utilisant la méthode des éléments finis. Le problème cinématique est régularisé par l'introduction d'un matériaux viscoplastique de Northon–Hoff pour rendre la fonction objectif différentiable, et peut etre résolu numériquement par la technique du Lagrangien augmenté. ***Pour citer cet article : M.A. Hamadouche, C. R. Mecanique 330 (2002) 305–311.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

mécanique des solides numérique / adaptaion / viscoplasticité / optimisation

Version française abrégée

La théorie de l'adaptation plastique (mieux connue sous le terme anglais de « shakedown ») étudie les conditions de ruine d'une structure élastoplastique soumise à des charges variables entre des bornes qui sont fixes, d'une manière indépendante les unes des autres. Si la structure s'adapte le travail dissipé plastiquement dans toute la structure est fini, les déformations plastiques tendent vers une limite, la réponse en contraintes tend vers une réponse purement élastique. Au contraire si elle ne s'adapte pas deux modes de ruine sont possibles : la ruine par déformation plastique progressive (rochet) qui correspond à une accumulation démesurée de la déformation plastique mettant l'ouvrage en péril ; ou la ruine par déformation plastique alternée (fatigue cyclique) qui entame à la longue l'endurance locale du matériau.

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La théorie de l'adaptation plastique a connu un développement considérable ces dernières années. Sans fournir un aperçu général et complet de la littérature sur le sujet qui sortirait du cadre de ce travail, je signale que le théorème statique est donné par Melan [5] et le théorème cinématique par Koiter [6]. Ils donnent respectivement la borne inférieure et supérieure de la charge limite d'adaptation, en utilisant la méthode des éléments finis en couplage avec la programmation mathématique linéaire ou non linéaire.

Le théorème statique a fait l'objet de plusieurs études numériques et applications. Cependant l'application du théorème cinématique reste difficile et ses applications numériques se font rares.

L'approche qui consiste à linéariser le critère de résistance [1] produit une source d'erreurs et introduit un très grand nombre de variables, si bien que la programmation mathématique linéaire est considérée numériquement inefficace.

L'utilisation du critère de von Mises conduit à une fonction objective, qui est ici la dissipation plastique, non linéaire et non différentiable à l'origine. Un algorithme direct peut être utilisé pour la solution du problème de minimisation [2]. Une autre alternative consiste à utiliser une méthode de régularisation pour rendre la fonction objective différentiable. Dans cette étude la méthode de Norton–Hoff–Friaa qui est déjà appliquée avec succés pour le cas de l'analyse limite est adoptée [3]. Elle consiste à remplacer dans le calcul de la puissance dissipée, le matériau parfaitement plastique par un matériau viscoplastique de Norton–Hoff à faible viscosité.

Le taux d'énergie totale de dissipation $D(\dot{\varepsilon}^P)$ égale $+\infty$ à moins que le matériau soit incompressible. C'est pour cette raison qu'on a besoin d'utiliser des éléments finis incompressibles pour le calcul du facteur de sécurité à l'adaptation. Deux alternatives permettent le traitement de cette incompressibilité, l'utilisation des éléments finis mixtes [4] ou bien la méthode de pénalité [2]. Cette dernière semble attrayante, elle est plus facile à appliquer numériquement. L'inconvénient de cette méthode concerne le choix du paramètre de pénalisation, des valeurs relativement petites entraînent une faible approximation de l'incompressibilité, des valeurs grandes impliquent des difficultés numériques. Cette carence est comblée par la méthode du Lagrangien augmenté qui combine l'utilisation des multiplicateurs de Lagrange avec les fonctions de pénalités [3].

En conclusion le problème cinématique initial, régularisé par l'introduction d'un matériau de Norton–Hoff, est formulé comme un problème de programmation mathématique non linéaire discret, en utilisant la méthode des éléments finis, puis solutionner par la technique du lagrangien augmenté en jonction avec l'algorithme d'Uzawa. La validation est faite sur un problème déjà étudié du point de vue numérique : une plaque carrée mince, en matériau de von Mises, trouée en son centre et soumise à une traction uniforme P_1 et P_2 sur ses bords respectivement dans les directions x_1 et x_2 (Fig. 1). L'accord entre les résultats obtenus par cette méthode et les résultats des auteurs [1,2] sont raisonnables.

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1. Introduction

The shakedown theorems provide static and kinematic approaches to the question of whether or not shakedown will occur for a given structure under a given cyclic loading. Many developments in computational mechanics deal with the static approach, but there is a lack in numerical solutions based on kinematic theorem. In previous kinematic shakedown analyses, linear programming techniques were developed by using linearized criterion [1]. This procedure produces a source of inaccuracies and introduces an extremely large number of variables, so that linear programming formulations were considered of limited computational interest. The use of von Mises yielding criterion leads to non-differentiable and non-linear mathematical programming. This property becomes the main difficulty in the upper bound shakedown analysis. A direct algorithm can be used to solve the problem of minimization of the objective function [2]. Another alternative consists in using a regularization procedure which involves the use of a fictitious viscous plastic material instead of ideal plastic material. Then, the corresponding viscous-plastic dissipation function becomes differentiable everywhere [3].

The total energy dissipation rate $D(\dot{\varepsilon}^P)$ equals $+\infty$ unless $\operatorname{tr}\dot{\varepsilon}^P = 0$. The incompressibility condition can be treated by using the mixed finite element [4] or by the penalty method [2]. Of these alternative the penalty method seems particularly attractive in that it has the advantage of being the single field formulation, permitting the use of displacement finite element code with only slight modification. The inconvenience of the penalty method deals with the choice of penalization parameter; relatively small values involve poor approximation of incompressibility; large values may imply numerical difficulties in general. To get an unconstrained problem where the function to minimize does not suffer from the ill-conditioning it is required to use the augmented Lagrangian method which combines the use of Lagrange multipliers with the ones of penalty functions. In this study the Norton–Hoff–Friaa regularizing method proposed by Gennouni et al. [3] is adopted. It leads to an efficient numerical computation of the shakedown upper bound via the augmented Lagrangian technique.

2. Problem formulation by a kinematic approach

Consider a body B of volume V with surface S composed of the disjoint parts S_u and S_T, where respectively kinematic and static conditions are prescribed. The body is subjected to body forces $F_i(x,t)$ in V, tractions $T_i(x,t)$ on S_T and surface displacements $U_i(x,t)$ imposed on S_u. It is assumed that: the kinematic relations are linear; the loads vary slowly in time; the material is elastic-perfectly plastic and stable in Druker's sense.

Von Mises yield criterion is adopted $F(\sigma) = \sqrt{J2} - \sigma_Y/\sqrt{3}$ where σ_Y denotes the yield stress in uniaxial tests and J_2 the second stress invariant. In view of associative flow rule ($\dot{\varepsilon}^P = \dot{\lambda}\partial F/\partial\sigma$) and von Mises yield function, the plastic dissipation is expressed as $D(\dot{\varepsilon}^P) = \sqrt{2/3}\,\sigma_Y(\dot{\varepsilon}^P\dot{\varepsilon}^P)^{1/2}$ which is a non-negative scalar convex function and homogenous of degree one. The admissible set of stresses in the static formulation is unbounded: the addition of a scalar function in the diagonal of σ, corresponding to adding hydrostatic pressure, does not affect the yield condition. The dual property in the kinematic formulation is that the total energy dissipation rate $D(\dot{\varepsilon}^P)$ equals $+\infty$ unless $\operatorname{tr}\dot{\varepsilon}^P = 0$. That is why we need incompressible finite elements to perform the calculation of shakedown limit load factor α_{SD}, which is the solution of the following optimization problem [2]:

$$\alpha_{\mathrm{SD}} = \min_{\varepsilon^p,\Delta u} \sqrt{\frac{2}{3}}\sigma_Y \int_0^T\int_{(V)} \left(\dot{\varepsilon}^{PT}\mathbf{X}\dot{\varepsilon}^P\right)^{1/2} \mathrm{d}V\,\mathrm{d}t \tag{1a}$$

$$\text{subject to} \quad \int_0^T\int_{(V)} \sigma^{eT}\dot{\varepsilon}^P\,\mathrm{d}V\,\mathrm{d}t = 1 \tag{1b}$$

$$\mathbf{Y}^T\dot{\varepsilon}^P = 0 \quad \text{in } V,\ \forall t \tag{1c}$$

$$\Delta\varepsilon^P = \int_0^T \dot{\varepsilon}^P\,\mathrm{d}t = \Re(\Delta\mathbf{u}) \quad \text{in } V \tag{1d}$$

$$\Delta\mathbf{u} = 0 \quad \text{on } S_u \tag{1e}$$

where $\dot{\varepsilon}^P$ and $\Delta\varepsilon^P$ denote respectively the plastic strain rates and plastic strain increments; $\Delta\mathbf{u}$ is the increments of residual displacements; σ^e is the fictitious elastic stresses caused by external body forces $\mathbf{f}$ and surface tractions $\mathbf{P}$. The period of cyclic loading programs is denoted by T, $\Re$ is the (linear) compatibility differential operator, $\mathbf{Y}^T = \{1\,1\,1\,0\,0\,0\}$ is a constant vector and $\mathbf{X} = \operatorname{diag}[\mathbf{I}, \mathbf{I}/2]$, $\mathbf{I}$ being the identity matrix of order 3. Eq. (1c) expresses the plastic incompressibility implied by Mises model.

3. Regularization by the Norton–Hoff–Friaa method

The objective function being nondifferentiable at $\dot{\varepsilon}^P = 0$. A regularized method is used to overcome this difficulty. It consists in replacing the dissipated power $D(\dot{\varepsilon}^P)$ of the perfectly plastic material by the limits, when the viscosity tends to zero, of the dissipated power Norton–Hoff viscoplastic material $[D(\dot{\varepsilon}^P)]^{\mathrm{NH}}$ [3]. For the case of von Mises yield criterion, the Norton–Hoff viscoplastic dissipation is given by:

$$\left[D\left(\dot{\varepsilon}^P\right)\right]^{\mathrm{NH}} = \frac{1}{P}\left[D\left(\dot{\varepsilon}^P\right)\right]^P = \frac{1}{P}\left[\sqrt{\frac{2}{3}}\sigma_Y\right]^P \left(\dot{\varepsilon}^P \dot{\varepsilon}^P\right)^{P/2} \tag{2}$$

where $P > 1$ is the viscosity parameter. The viscoplastic dissipation tends to the plastic dissipation when the viscosity parameter tends to one.

4. Discretization

The presence of time integrals in the direct formalization of Koiter theorem [6] leads to difficulties in applications. According to the following two theorems [7], we can consider a cyclic loading instead of an arbitrarily loading history and consider the stress and the strain rate field only at every vertex instead of the integration over the time cycle.

THEOREM 1. – *Shakedown will happen for a loading domain* **D** *if and only if it occurs on its convex envelope* **Đ**.

THEOREM 2. – *Shakedown will happen for an arbitrarily varying load path within domain* **D** *if it happens for a cycle of loading passing through all vertices* P_k, $k = 1, \ldots, m$, *of the frontier of* **Đ**.

The plastic strain rates $\dot{\varepsilon}^P$ can differ from zero at a point of the body only if the stress state σ^k corresponding to the loading corner k attains the yield surface. Let us denote, respectively, the elastic stress and the plastic strain subincrements during loading at corner k of domain **Đ** by σ_k^e and

$$\varepsilon_k^P(x) = \int_{\tau_k} \dot{\varepsilon}^P(x, t)\,\mathrm{d}t \tag{3a}$$

$$\Delta\varepsilon^P = \sum_{k=1}^{m} \varepsilon_k^P \tag{3b}$$

in which τ_k is the time interval staying at the stress state σ^k. It is important to note that the summation of the sub increments (3b) is a compatible strain field. As a consequence of the above statement, the minimization (1) concerning the elastic plastic continuum can be cast into the form:

$$\alpha_{\mathrm{SD}} = \min_{\varepsilon_k^P, \Delta u} \frac{1}{P}\left[\sqrt{\frac{2}{3}}\sigma_Y\right]^P \sum_{k=1}^{m} \int_{(V)} \left(\varepsilon_k^{PT}\mathbf{X}\varepsilon_k^P\right)^{P/2} \mathrm{d}V \tag{4a}$$

$$\text{subject to} \quad \sum_{k=1}^{m} \int_{(V)} \sigma_k^{eT} \varepsilon_k^P \,\mathrm{d}V = 1 \tag{4b}$$

$$\mathbf{Y}^T \varepsilon_k^P = 0 \quad \text{in } V,\ k = 1, \ldots, m \tag{4c}$$

$$\Delta\varepsilon^P = \sum_{k=1}^{m} \varepsilon_k^P = \mathfrak{R}(\Delta\mathbf{u}) \quad \text{in } V \tag{4d}$$

$$\Delta\mathbf{u} = 0 \quad \text{on } S_u \tag{4e}$$

In order to discretize in space the problem (4), the volume of the body V is divided into **n** elements. In each element, the following relations holds

$$\Delta \mathbf{u}^e = \mathbf{N}^e(x)\Delta \mathbf{U}^e \tag{5a}$$

$$\Delta \varepsilon^{eP}(x) = \mathbf{B}^e(x)\Delta \mathbf{U}^e \tag{5b}$$

where $\Delta \mathbf{U}^e$ is the vector of nodal displacements, $\mathbf{N}^e$ is the shape function matrix and $\mathbf{B}^e$ the consequent compatibility matrix. By summation over all elements and using the Gaussian integration method, we obtain:

$$\alpha_{\mathrm{SD}} = \min_{\varepsilon_{kr}^P,\Delta U} \frac{1}{P}\left[\sqrt{\frac{2}{3}}\sigma_Y\right]^P \sum_{k=1}^{m}\sum_{r=1}^{n} W_r|\mathbf{J}|_r\left(\varepsilon_{kr}^{PT}\mathbf{X}\varepsilon_{kr}^P\right)^{P/2} \tag{6a}$$

$$\text{subject to} \quad \sum_{k=1}^{m}\sum_{r=1}^{n} W_r|J|_r\sigma_k^{eT}\varepsilon_{kr}^P = 1 \tag{6b}$$

$$\mathbf{Y}^T\varepsilon_{kr}^P = 0, \quad r = 1,\ldots,n,\ k = 1,\ldots,m \tag{6c}$$

$$\Delta\varepsilon_r^P = \sum_{k=1}^{m}\varepsilon_{kr}^P = \mathbf{B}_r\Delta\mathbf{U}, \quad r = 1,\ldots,n \tag{6d}$$

where: index r runs over the set G of the **n** Gauss integration points $(r = 1,\ldots,n)$; W and $|\mathbf{J}|$ represent the Gauss integration weight and the determinant of the Jacobian matrix of the map respectively; vector $\Delta\mathbf{U}$ contains all unconstrained nodal displacements of the finite element model, $\mathbf{B}_r$ is the assembled compatibility matrix for strains at Gauss point r.

5. Augmented Lagrangian form of the kinematic shakedown

The augmented Lagrangian form of the problem defined by the equations (6a)–(6d) is as follows:

$$\begin{aligned}
L_a\left(\varepsilon_{kr}^P,\Delta U,\lambda,\mu,L_r,\rho\right) &= \frac{1}{P}\left[\sqrt{\frac{2}{3}}\sigma_Y\right]^P \sum_{k=1}^{m}\sum_{r=1}^{n} W_r|\mathbf{J}|_r\left(\varepsilon_{kr}^{PT}\mathbf{X}\varepsilon_{kr}^P\right)^{P/2} \\
&+ \lambda\left(1-\sum_{k=1}^{m}\sum_{r=1}^{n} W_r|J|_r\sigma_{kr}^{eT}\varepsilon_{kr}^P\right) + \mu\left(\sum_{k=1}^{m}\sum_{r=1}^{n} W_r|\mathbf{J}|_r\mathbf{Y}^T\varepsilon_{kr}^P\right) + \sum_{r=1}^{n}\mathbf{L}_r\left(\sum_{k=1}^{m}\varepsilon_{kr}^P - \mathbf{B}_r\Delta\mathbf{U}\right) \\
&+ \frac{\rho}{2}\left(1-\sum_{k=1}^{m}\sum_{r=1}^{n} W_r|J|_r\sigma_{kr}^{eT}\varepsilon_{kr}^P\right)^2 + \frac{\rho}{2}\left(\sum_{k=1}^{m}\sum_{r=1}^{n} W_r|\mathbf{J}|_r\varepsilon_{kr}^{PT}\mathbf{Y}\mathbf{Y}^T\varepsilon_{kr}^P\right) \\
&+ \frac{\rho}{2}\left(\sum_{k=1}^{m}\varepsilon_{kr}^P - \mathbf{B}_r\Delta\mathbf{U}\right)\left(\sum_{r=1}^{m}\varepsilon_{kr}^P - \mathbf{B}_r\Delta\mathbf{U}\right)
\end{aligned} \tag{7}$$

where λ, μ, L_r, $r = 1,\ldots,n$, are the Lagrange multipliers and ρ is the penalty parameter. This optimization problem can be solved by the Uzawa algorithm [3]:

Set $N = 0$. Estimate $\mathring{\varepsilon}^P$, $\mathring{\lambda}$, $\mathring{\mu}$, $\mathring{L}_r$, $r = 1,\ldots,n$, and scalar ρ.

Minimise L_a $(\varepsilon^{PN},\Delta U,\lambda^N,\mu^N,L_r^N,\ r = 1,\ldots,n)$, let ΔU^N the solution. (8)

Minimise L_a $(\varepsilon^P,\Delta U^N,\lambda^N,\mu^N,L_r^N,\ r = 1,\ldots,n)$, let ε^{PN+1} the solution. (9)

Update

$$\lambda^{N+1} \text{ by } \lambda^{N+1} = \lambda^N + C^N \left(1 - \sum_{k=1}^{m} \sum_{r=1}^{n} W_r |J|_r \sigma_{kr}^{eT} \varepsilon_{kr}^{PN} \right)$$
$$\mu^{N+1} \text{ by } \mu^{N+1} = \mu^N + C^N \left(\sum_{k=1}^{m} \sum_{r=1}^{n} W_r |\mathbf{J}|_r \mathbf{Y}^T \varepsilon_{kr}^{PN} \right) \tag{10}$$
$$L_r^{N+1} \text{ by } L_r^{N+1} = L_r^N + C^N \left(\sum_{k=1}^{m} \varepsilon_{kr}^{PN} - \mathbf{B}_r \Delta U^N \right), \quad r = 1, \ldots, n$$

C^N is a positive parameter. For $N = N + 1$, repeat the procedure (8)–(10). The final solution obtained by this procedure is $(\overset{*}{\varepsilon}{}^P, \Delta \overset{*}{U}, \overset{*}{\lambda}, \overset{*}{\mu}, \overset{*}{L})$ where $(\overset{*}{\varepsilon}{}^P, \Delta \overset{*}{U})$ is the solution of the problem (6).

6. Application

The problem that was solved by [1] and [2] is considered here. A thin square plate with a central circular hole as shown in Fig. 1. The loading consisted of two uniform surface tractions, P_1 and P_2, in the x_1 and x_2 directions respectively. Both P_1 and P_2 were allowed to vary between zero and the maximum tensile loads. The loading program was square in the load space as shown in Fig. 2. The main assumptions are: plane-stress states; Mises elastic–plastic material model; four-node isoparametric finite elements. The plane stress hypothesis makes the plastic incompressibility constraint implicit, however the constraint (6c) vanish in Eq. (7). In the curves of the “shakedown domain” each point corresponds to the maximum values that P_1, P_2 can attain in prescribed loading program giving rise to shakedown. Results are gathered in Fig. 3, the difference may be due to piecewise linearization of the yield criterion and linear programming used by [1], and the weak convergence of the iterative procedure used by [2]. However in general the agreement is reasonable as shown in Table 1.

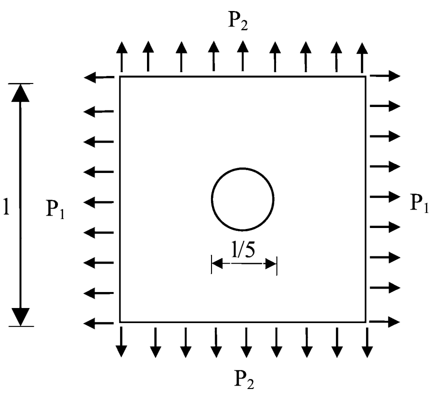

Figure 1. Thin plate with central hole.

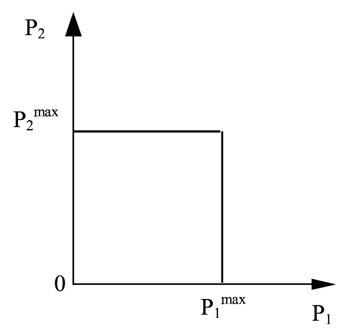

Figure 2. Loading domain.

Table 1. Shakedown analysis of square plate with circular hole.

Tableau 1. *Analyse d’adaptation d’une plaque carée trovée au centre.*

Author	$P_1^{\max} = P_2^{\max}$	$P_1^{\max}$ ($P_2^{\max} = 0$)
Corradi et al. [1]	0.518	0.690
Carvelli et al. [2]	0.504	0.654
Present method	0.490	0.623

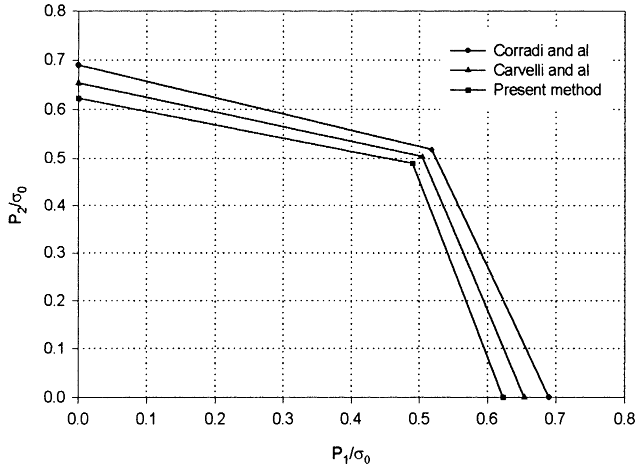

Figure 3. Shakedown domain.

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