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Recovery of small electromagnetic inhomogeneities from partial boundary measurements

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Abstract We consider for the inverse problem of identifying locations and certain properties of the shapes of small dielectric inhomogeneities in a homogeneous background medium from boundary measurements on part of the boundary or dynamic boundary measurements for a finite time interval. Using as weights particular background solutions we develop asymptotic methods based on appropriate averaging of the data. ***To cite this article: H. Ammari, A.G. Ramm, C. R. Mecanique 330 (2002) 199–205.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

continuum mechanics / Helmholtz equation / reconstruction problem / small dielectric inhomogeneities / partial measurements

Identification de petites inhomogénéités diélectriques à partir de mesures partielles

Résumé Nous considérons deux problèmes d'identification de petites inhomogénéités diélectriques à partir de mesures incomplètes. Pour chaque problème, nous construisons une fonction dont la transformée de Fourier inverse permet de localiser les petites inhomogénéités. ***Pour citer cet article : H. Ammari, A.G. Ramm, C. R. Mecanique 330 (2002) 199–205.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soit Ω un ouvert borné de $\mathbf{R}^d$, $d \geqslant 2$, de classe C^2. Supposons que Ω contient m inhomogénéités, $\{z_j + \alpha B_j\}_{j=1}^m$, où α est un petit paramètre, $B_j \subset \mathbf{R}^d$ est un ouvert borné et les points $\{z_j\}_{j=1}^m$ vérifient les hypothèses (1). Soient la perméabilité magnétique μ_α et la permittivité électrique ε_α de forme (2).

La première partie de cette Note concerne le problème de reconstruction des points $\{z_j\}_{j=1}^m$ et des tenseurs de polarization $\{M_j\}_{j=1}^m$ des domaines $\{B_j\}_{j=1}^m$, définis par (6), à partir de la mesure de la dérivée normale du champ électrique E_α, solution de l'équation de Helmholtz (3), sur une partie $\Gamma_1 \Subset \partial\Omega$. Grâce à un résultat de densité, établi dans la Proposition 1.1, et à la formule asymptotique (5), nous réduisons ce

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problème inverse au calcul de la transformée de Fourier inverse de la fonction écart à la réciprocité Λ_α, définie dans (8). Cette partie généralise la méthode introduite dans [5] aux situations où on ne disposerait pas de la mesure de la dérivée normale du champ électrique E_α sur tout le bord $\partial\Omega$. À notre connaissance, ce résultat constitue, parmi [7] et [8], l'un des tous premiers résultats de reconstruction de coefficients dans une EDP elliptique à partir de mesures incomplètes sur le bord.

Dans la seconde partie, nous considérons pour l'équation des ondes, le problème d'identification des petites inhomogénéités diélectriques à partir de la mesure de la dérivée normale du champ électrique E_α, solution de l'équation des ondes (10), sur $\partial\Omega \times (0, T)$ où $T < +\infty$. Compte-tenu du fait que les mesures sont faites sur un intervalle de temps fini, nous ne pouvons pas réduire ce probème inverse à celui pour l'équation de Helmholtz, étudié dans [5] et [1], en prenant la transformée de Fourier en temps. Notre idée est d'intégrer l'équation des ondes (10) contre des solutions explicites de l'équation homogène (9) et d'intégrer par parties afin de former des quantités simples, fonctions de la mesure $\partial E_\alpha/\partial\nu|_{\partial\Omega\times(0,T)}$ et de la donnée $E_\alpha|_{\partial\Omega\times(0,T)}$, et dont la transformée de Fourier inverse permet de localiser les petites inhomogeneités diélectriques en tant que support d'une certaine distribution, avec une erreur qui tend vers zéro lorsque la taille caractéristique de ces inhomogénéités diélectriques converge vers zéro. Nous commençons par réécrire notre problème sous la forme (12). Ensuite, grâce aux formules asymptotiques (13), nous réduisons ce problème inverse dynamique au calcul de la transformée de Fourier inverse de la fonction écart à la reciprocité Λ_α^T, définie dans (16).

1. The Helmholtz equation

1.1. Problem formulation

Let Ω be a bounded C^2-domain in $\mathbf{R}^d$, $d \geqslant 2$, and ν be the outward unit normal to $\partial\Omega$. Assume that Ω contains a finite number of inhomogeneities, each of the form $z_j + \alpha B_j$, where $B_j \subset \mathbf{R}^d$ is a bounded, smooth domain containing the origin. The total collection of inhomogeneities is $\mathcal{B}_\alpha = \bigcup_{j=1}^m (z_j + \alpha B_j)$. The points $z_j \in \Omega$, $j = 1, \ldots, m$, which determine the location of the inhomogeneities, are assumed to satisfy the following inequalities:

$$|z_j - z_l| \geqslant c_0 > 0, \quad \forall j \neq l \quad \text{and} \quad \operatorname{dist}(z_j, \partial\Omega) \geqslant c_0 > 0, \quad \forall j \tag{1}$$

Assume that $\alpha > 0$, the common order of magnitude of the diameters of the inhomogeneities, is sufficiently small, that these inhomogeneities are disjoint, and that their distance to $\mathbf{R}^d \setminus \overline{\Omega}$ is larger than $c_0/2$. Let μ_0 and ε_0 denote the permeability and the permittivity of the background medium, and assume that $\mu_0 > 0$ *and* $\varepsilon_0 > 0$ *are positive constants*. Let $\mu_j > 0$ and $\varepsilon_j > 0$ denote the permeability and the permittivity of the j-th inhomogeneity, $z_j + \alpha B_j$, these are also assumed to be positive constants. Introduce the piecewise-constant magnetic permeability

$$\mu_\alpha(x) = \begin{cases} \mu_0, & x \in \Omega \setminus \overline{\mathcal{B}}_\alpha, \\ \mu_j, & x \in z_j + \alpha B_j, \ j = 1, \ldots, m \end{cases} \tag{2}$$

If we allow the degenerate case $\alpha = 0$, then the function $\mu_0(x)$ equals the constant μ_0. The piecewise constant electric permittivity, $\varepsilon_\alpha(x)$ is defined analogously.

Consider solutions to the time-harmonic Maxwell's equations with $\exp(-\mathrm{i}\omega t)$ time dependence. Let E_α be the electric field in the presence of the inhomogeneities. It solves the Helmholtz equation

$$\nabla \cdot \left(\frac{1}{\mu_\alpha} \nabla E_\alpha\right) + \omega^2 \varepsilon_\alpha E_\alpha = 0 \quad \text{in } \Omega \tag{3}$$

with the boundary condition $E_\alpha = f$ on $\partial\Omega$, where $\omega > 0$ is a given frequency. The electric field, E_0, in the absence of any inhomogeneities, satisfies the following equation:

$$\Delta E_0 + k^2 E_0 = 0 \quad \text{in } \Omega \tag{4}$$

where $k^2 = \omega^2 \mu_0 \varepsilon_0$, with $E_0 = f$ on $\partial\Omega$. In order to insure well-posedness (also for the α-dependent case for α sufficiently small) we shall assume that k^2 is not an eigenvalue for the operator $-\Delta$ in $\mathrm{L}^2(\Omega)$ with the Dirichlet boundary conditions. It has been shown in [1] that the following asymptotic formula holds uniformly on $\partial\Omega$

$$\begin{aligned}
&\frac{\partial E_\alpha}{\partial \nu}(x) - \frac{\partial E_0}{\partial \nu}(x) - 2\int_{\partial\Omega} \left(\frac{\partial E_\alpha}{\partial \nu} - \frac{\partial E_0}{\partial \nu}\right)(y) \frac{\partial G(x,y)}{\partial \nu}\,\mathrm{d}s(y) \\
&\quad = 2\alpha^d \sum_{j=1}^m \left(\frac{\mu_j}{\mu_0} - 1\right) \nabla_y \frac{\partial G(x,z_j)}{\partial \nu(x)} \cdot M_j\left(\frac{\mu_j}{\mu_0}\right) \nabla E_0(z_j) \\
&\qquad - 2\alpha^d k^2 \sum_{j=1}^m \left(1 - \frac{\varepsilon_j}{\varepsilon_0}\right) \frac{\partial G(x,z_j)}{\partial \nu(x)} |B_j| E_0(z_j) + \mathrm{o}\big(\alpha^d\big)
\end{aligned} \tag{5}$$

where the remainder $\mathrm{o}(\alpha^d)$ is independent of the set of points $\{z_j\}_{j=1}^m$ provided that (1) holds, $G(x,y)$ is a free space Green's function for $\Delta + k^2$, and each M_j is a $d \times d$, symmetric, positive definite matrix associated with the j-th inhomogeneity, called the polarizability tensor, which is given by

$$(M_j)_{ll'} = |B_j|\delta_{ll'} + \left(\frac{\mu_j}{\mu_0} - 1\right) \int_{\partial B_j} y_l \frac{\partial \phi_{l'}^+}{\partial \nu_j}\,\mathrm{d}s(y) \tag{6}$$

where, for $1 \leqslant l' \leqslant d$, $\phi_{l'}(y)$ is the unique function which satisfies

$$\begin{cases}
\Delta \phi_{l'} = 0 \quad \text{in } B_j \text{ and } \mathbf{R}^d \setminus \overline{B_j} \\
\dfrac{\mu_j}{\mu_0} \dfrac{\partial \phi_{l'}^-}{\partial \nu_j} - \dfrac{\partial \phi_{l'}^+}{\partial \nu_j} = -\nu_j \cdot e_{l'} \quad \text{on } \partial B_j
\end{cases}$$

with $\phi_{l'}$ continuous across ∂B_j and $\lim_{|y|\to\infty} \phi_{l'}(y) = 0$. Here $\{e_{l'}\}_{l'=1}^d$ is an orthonormal basis of $\mathbf{R}^d$, ν_j denotes the outward unit normal to ∂B_j, superscripts $-$ and $+$ indicate the limiting values as the point approaches ∂B_j from outside B_j, and from inside B_j, respectively.

1.2. Identification procedure

Before describing our identification procedure, let us introduce the sets $N(\Omega) = \{v : v \in \mathrm{H}^1(\overline{\Omega}) \cap \mathrm{H}^2(\Omega), \Delta v + k^2 v = 0 \text{ in } \Omega\}$ and $\widetilde{N}(\Omega) = \{v : v \in \mathrm{H}^1(\overline{\Omega}) \cap \mathrm{H}^2(\Omega), \Delta v + k^2 v = 0 \text{ in } \Omega, v = 0 \text{ on } \Gamma_2\}$, where $\Gamma_2 = \partial\Omega \setminus \overline{\Gamma_1}$, where Γ_1 is an open in $\partial\Omega$ subset.

The general approach we use to recover the locations and the polarizability tensors of the small inhomogeneities is to integrate the solution E_α against special test functions in the set $\widetilde{N}(\Omega)$. This approach is similar to the ideas used by Calderón [2] in his proof of uniqueness of the linearized conductivity problem and later, by Sylvester and Uhlmann in their important work [3] on uniqueness of the three-dimensional inverse conductivity problem. See also Isaacson and Isaacson [4] for exact calculation of the Calderón's approximation for the case of homogeneous concentric disks in $\mathbf{R}^2$.

Let v be any function in $\widetilde{N}(\Omega)$. As in [5], the following estimate can be derived from (5):

$$\int_{\Gamma_1} \frac{\partial E_\alpha}{\partial \nu} v \, \mathrm{d}s - \int_{\partial\Omega} \frac{\partial v}{\partial \nu} E_\alpha \, \mathrm{d}s = \alpha^d \sum_{j=1}^m \left(1 - \frac{\mu_j}{\mu_0}\right) \nabla E_0(z_j) \cdot M_j\left(\frac{\mu_j}{\mu_0}\right) \nabla v(z_j) + \alpha^d k^2 \sum_{j=1}^m \left(1 - \frac{\varepsilon_j}{\varepsilon_0}\right) |B_j| E_0(z_j) v(z_j) + \mathrm{o}\big(\alpha^d\big) \tag{7}$$

where $|B_j|$ stands for the volume of the set B_j. We want to make suitable choices for the test functions v in $\widetilde{N}(\Omega)$ and the boundary condition $E_\alpha|_{\partial\Omega}$ in order to get simple equations for the unknown parameters, namely, for the points $\{z_j\}_{j=1}^m$ and matrices $\{M_j\}_{j=1}^m$. Similar idea was used and the associated numerical experiments have been successfully conducted in the case of the conductivity problem [5] with boundary measurements on all of $\partial\Omega$.

Let us describe our inversion method. Take η to be a vector in $\mathbf{R}^d$, $\eta^\perp$ a unit vector in $\mathbf{R}^d$ which is orthogonal to η, and γ a complex number. Then $\mathrm{e}^{\mathrm{i}(\eta+\gamma\eta^\perp)\cdot x}$ is a solution to the Helmholtz equation in $\mathbf{R}^d$ if and only if $\gamma^2 = k^2 - |\eta|^2$ and in this case $\mathrm{e}^{\mathrm{i}(\eta-\gamma\eta^\perp)\cdot x}$ is also a solution to the Helmholtz equation in $\mathbf{R}^d$. For simplicity, let us consider the case where all the B_j are balls. In this case all the matrices M_j are multiples of the identity matrix which makes our analysis simpler.

If $\partial E_\alpha/\partial\nu$ is known on the whole boundary $\partial\Omega$ then taking $E_\alpha = \mathrm{e}^{\mathrm{i}(\eta+\gamma\eta^\perp)\cdot x}$ on $\partial\Omega$ and $v = \mathrm{e}^{\mathrm{i}(\eta-\gamma\eta^\perp)\cdot x}$ in Ω we know from [5] that

$$\int_{\partial\Omega} \frac{\partial E_\alpha}{\partial \nu}(y) \, \mathrm{e}^{\mathrm{i}(\eta-\gamma\eta^\perp)\cdot y} \, \mathrm{d}s(y) - \int_{\partial\Omega} \frac{\partial}{\partial \nu}\big(\mathrm{e}^{\mathrm{i}(\eta-\gamma\eta^\perp)\cdot y}\big) E_\alpha(y) \, \mathrm{d}s(y) = \alpha^d \sum_{j=1}^m \mathrm{e}^{2\mathrm{i}\eta\cdot z_j}\left[\left(1 - \frac{\mu_j}{\mu_0}\right) M_j\left(\frac{\mu_j}{\mu_0}\right)\big(2|\eta|^2 - k^2\big) + k^2\left(1 - \frac{\varepsilon_j}{\varepsilon_0}\right)|B_j|\right] + \mathrm{o}\big(\alpha^d\big)$$

The main difficulty in generalizing this approach to the case when $\partial E_\alpha/\partial\nu$ is known only on a part $\Gamma_1 \Subset \partial\Omega$ is to construct a function $w_\alpha(x)$ in $\widetilde{N}(\Omega)$, that is asymptotically $\mathrm{e}^{\mathrm{i}(\eta-\gamma\eta^\perp)\cdot x}$ as α approaches 0. The following lemma holds.

LEMMA 1.1. – *Let $\Omega' \Subset \Omega$ be a C^2-domain. Let $\eta \in \mathbf{R}^d$ and $\eta^\perp$ be a unit vector in $\mathbf{R}^d$ that is orthogonal to η. There exists $w_\alpha \in \widetilde{N}(\Omega)$ such that*

$$w_\alpha(x) = \mathrm{e}^{\mathrm{i}(\eta-\gamma\eta^\perp)\cdot x} + \mathrm{o}\big(\alpha^d\big) \quad \textit{and} \quad \nabla w_\alpha(x) = \mathrm{i}\big(\eta - \gamma\eta^\perp\big)\,\mathrm{e}^{\mathrm{i}(\eta-\gamma\eta^\perp)\cdot x} + \mathrm{o}\big(\alpha^d\big)$$

uniformly in Ω'.

This lemma is an immediate corollary of the following general density result.

PROPOSITION 1.1. – *The set $\widetilde{N}(\Omega)$ is dense, in the $\mathrm{L}^2(\Omega')$ norm, in the set $N(\Omega)$.*

Proof. – Assume the contrary and let $v \in N(\Omega)$ be an element which cannot be approximated in $\mathrm{L}^2(\Omega')$ by the functions from $\widetilde{N}(\Omega)$ with a prescribed accuracy. Then there is an element in $N(\Omega)$, which we denote again v, such that $\int_{\Omega'} vw \, \mathrm{d}x = 0$, $\forall w \in \widetilde{N}(\Omega)$. Let G_0 be the Dirichlet Green's function in Ω:

$$\begin{cases} \Delta G_0 + k^2 G_0 = \delta_y(x) & \text{in } \Omega \\ G_0 = 0 & \text{on } \partial\Omega \end{cases}$$

Define $\widetilde{\mathrm{H}}^{3/2}(\Gamma_1) = \{p \in \mathrm{H}^{3/2}(\Gamma_1)$ such that there exists $\widetilde{p} \in \mathrm{H}^{3/2}(\partial\Omega)$, $\widetilde{p}|_{\Gamma_2} = 0$, $\widetilde{p}|_{\Gamma_1} = p\}$. Since any $w \in \widetilde{N}(\Omega)$ can be represented as follows

$$w(x) = \int_{\Gamma_1} \frac{\partial G_0(x,y)}{\partial \nu(y)} p(y) \, \mathrm{d}s(y), \quad x \in \Omega$$

where $p \in \widetilde{\mathrm{H}}^{3/2}(\Gamma_1)$ is arbitrary, we have

$$\int_{\Omega'} v(y) \frac{\partial G_0(x,y)}{\partial \nu(x)} \, \mathrm{d}y = 0, \quad \forall x \in \Gamma_1$$

Introducing $u(x) := \int_{\Omega'} v(y) G_0(x,y)\, \mathrm{d}y$ and using the unique continuation principle we can prove that $v = 0$ in Ω. □

Now, if we choose $E_\alpha = \mathrm{e}^{\mathrm{i}(\eta + \gamma\eta^\perp)\cdot x}$ on $\partial\Omega$ and $v = w_\alpha$ in Ω then, since the points $\{z_j\}_{j=1}^m$ are away from the boundary $\partial\Omega$, it follows from (7) and Lemma 1.1 that the following asymptotic expansion holds:

$$\begin{aligned}\Lambda_\alpha(\eta) &= \int_{\Gamma_1} \frac{\partial E_\alpha}{\partial \nu} w_\alpha \, \mathrm{d}s - \int_{\partial\Omega} \frac{\partial w_\alpha}{\partial \nu} E_\alpha \, \mathrm{d}s \\ &= \alpha^d \sum_{j=1}^m \mathrm{e}^{2\mathrm{i}\eta\cdot z_j} \left[\left(1 - \frac{\mu_j}{\mu_0}\right) M_j \left(\frac{\mu_j}{\mu_0}\right) \left(2|\eta|^2 - k^2\right) + k^2 \left(1 - \frac{\varepsilon_j}{\varepsilon_0}\right) |B_j| \right] + \mathrm{o}\left(\alpha^d\right) \end{aligned} \tag{8}$$

2. The wave equation

Let $\tau = \sqrt{\varepsilon_0 \mu_0}$. For an arbitrary (nonnull) $\eta \in \mathbf{R}^d$, the function $E_0(x,t) = \mathrm{e}^{\mathrm{i}(\eta\cdot x + \frac{|\eta|}{\tau} t)}$ satisfies the following (unperturbed) wave equation:

$$\tau^2 \partial_t^2 E_0 - \Delta E_0 = 0 \quad \text{in } \Omega \times \mathbf{R} \tag{9}$$

Consider now the initial boundary value problem for the wave equation in the presence of the small inhomogeneities

$$\begin{cases} \varepsilon_\alpha \partial_t^2 E_\alpha - \operatorname{div}\left(\dfrac{1}{\mu_\alpha} \operatorname{grad} E_\alpha\right) = 0 \quad \text{in } \Omega \times (0,T) \\ E_\alpha|_{t=0} = E_0|_{t=0}, \quad \partial_t E_\alpha|_{t=0} = \partial_t E_0|_{t=0} \quad \text{in } \Omega \\ E_\alpha|_{\partial\Omega\times(0,T)} = E_0|_{\partial\Omega\times(0,T)} \end{cases} \tag{10}$$

Here $T > 0$ is a final observation time. It can be shown that there exists a unique solution $E_\alpha \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega)) \cap \mathrm{W}^{1,\infty}(0,T;\mathrm{L}^2(\Omega))$ to the initial boundary value problem (10), and this solution satisfies the following.

LEMMA 2.1. – *The following estimate as $\alpha \to 0$ holds*:

$$\|E_\alpha - E_0\|_{\mathrm{L}^\infty(0,T;\mathrm{H}_0^1(\Omega))} + \left\|\partial_t(E_\alpha - E_0)\right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} + \left\|\varepsilon_\alpha \partial_t^2(E_\alpha - E_0)\right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{-1}(\Omega))} \leqslant C\alpha \tag{11}$$

where the constant C is independent of α and the set of points $\{z_j\}_{j=1}^m$ provided that assumption (1) *holds. Also, $\partial E_\alpha/\partial\nu|_{\partial\Omega\times(0,T)}$ belongs to* $\mathrm{L}^2(0,T;\mathrm{L}^2(\partial\Omega))$.

Let us now rewrite the first equation in (10) as follows

$$\tau^2 \partial_t^2 E_\alpha - \Delta E_\alpha = \sum_{j=1}^m \left(\tau^2 - \tau_j^2\right) \partial_t^2 E_\alpha \chi(z_j + \alpha B_j) - \left[\frac{\partial E_\alpha}{\partial \nu}\right]_{\partial(z_j+\alpha B_j)} \delta_{\partial(z_j+\alpha B_j)} \quad \text{in } \Omega \times (0,T) \tag{12}$$

where $\tau_j = \sqrt{\varepsilon_j \mu_j}$, $\chi(z_j + \alpha B_j)$ is the characteristic function of the domain $z_j + \alpha B_j$, $[\partial E_\alpha/\partial\nu]_{\partial(z_j+\alpha B_j)}$ denotes the jump of $\partial E_\alpha/\partial\nu$ across $\partial(z_j + \alpha B_j)$, and $\delta_{\partial(z_j+\alpha B_j)}$ is the surface Dirac measure at $\partial(z_j + \alpha B_j)$. We now reduce our inverse problem to an inverse source one by expanding the right-hand side of (12) in terms of the small parameter α. The leading order term in this asymptotic expansion contains information on the locations $\{z_j\}_{j=1}^m$ and the domains $\{B_j\}_{j=1}^m$ and thus, our inverse problem becomes to obtain this information from knowledge of $\partial E_\alpha/\partial\nu|_{\partial\Omega\times(0,T)}$ (where $E_\alpha|_{\partial\Omega\times(0,T)}$ is given).

As in [6], the following asymptotic expansions can be derived.

LEMMA 2.2. – *We have the following asymptotic expansions*

$$\begin{aligned}&\partial_t^2 E_\alpha \chi(z_j+\alpha B_j) = -\frac{|\eta|^2}{\tau^2}\,\mathrm{e}^{\mathrm{i}(\eta\cdot x+\frac{|\eta|}{\tau}t)}\chi(z_j+\alpha B_j)+\mathrm{o}(1)\\ &\left[\frac{\partial E_\alpha}{\partial\nu}\right]_{\partial(z_j+\alpha B_j)} = i\left(1-\frac{\mu_0}{\mu_j}\right)\left[\nu_j\cdot\eta+\left(\frac{\mu_j}{\mu_0}-1\right)\frac{\partial\Phi_j}{\partial\nu_j}\bigg|_+\left(\frac{x-z_j}{\alpha}\right)\cdot\eta\right]\mathrm{e}^{\mathrm{i}(\eta\cdot z_j+\frac{|\eta|}{\tau}t)}+\mathrm{o}(1)\end{aligned}\tag{13}$$

for all $x\in\Omega$, $t\in(0,T)$, *where the remainder* $\mathrm{o}(1)$ *is independent of the set of points* $\{z_j\}_{j=1}^m$.

Suppose that $T>\tau\,\mathrm{diam}(\Omega)$. We introduce $\varphi\in C_0^\infty(\mathbf{R})$ satisfying

$$0\leqslant\varphi\leqslant1,\qquad \int_{\mathbf{R}}\varphi\,\mathrm{d}t=1,\qquad \varphi=0\quad\text{for } |t|\geqslant T-\tau\,\mathrm{diam}(\Omega)$$

to obtain the approximation of the plane wave $\delta(\frac{t}{\tau}-x\cdot\omega)$ of direction ω:

$$\frac{1}{\alpha}\varphi\left(\frac{t/\tau-x\cdot\omega}{\alpha}\right)\to\delta\left(\frac{t}{\tau}-x\cdot\omega\right)\quad\text{as }\alpha\to0\tag{14}$$

for any unit vector $\omega\in\mathbf{R}^d$. Note that $\delta(\frac{T}{\tau}-x\cdot\omega)=0$ for all $x\in\Omega$.

Multiplying (12) by $\frac{1}{\alpha}\varphi(\frac{t/\tau-x\cdot\omega}{\alpha})$, integrating by parts over $\Omega\times(0,T)$ and using (13) and (14), one gets:

$$\begin{aligned}&-\int_0^T\int_{\partial\Omega}\frac{\partial}{\partial\nu}(E_\alpha-E_0)(y)\left(\frac{1}{\alpha}\varphi\left(\frac{t/\tau-y\cdot\omega}{\alpha}\right)\right)\mathrm{d}s(y)\,\mathrm{d}t=\alpha^d\sum_{j=1}^m\left[\left(\frac{\tau_j^2}{\tau^2}-1\right)|\eta|^2+\left(1-\frac{\mu_0}{\mu_j}\right)\frac{|\eta|}{\tau}\eta\right.\\ &\quad\left.\times\int_{\partial B_j}\left(\left(\nu_j+\left(\frac{\mu_j}{\mu_0}-1\right)\frac{\partial\Phi_j}{\partial\nu_j}\bigg|_+(y)\right)y\,\mathrm{d}s(y)\right)\cdot\omega\right]\mathrm{e}^{\mathrm{i}(\eta+\frac{|\eta|}{\tau}\omega)\cdot z_j}+\mathrm{o}\big(\alpha^d\big)\end{aligned}\tag{15}$$

Recall that $\partial E_\alpha/\partial\nu|_{\partial\Omega\times(0,T)}$ belongs to $L^2(0,T;L^2(\partial\Omega))$. Thus, if we choose $\omega=\eta/|\eta|$, it follows from (15) and (14) that the following theorem holds:

THEOREM 2.1. – *Suppose that* $T>\tau\,\mathrm{diam}(\Omega)$. *The following asymptotic expansion holds*:

$$\begin{aligned}\Lambda_\alpha(\eta)&=\int_0^T\int_{y\in\partial\Omega,\ y\cdot\eta/|\eta|=t/\tau}\frac{\partial}{\partial\nu}(E_\alpha-E_0)(y)\,\mathrm{d}s(y)\,\mathrm{d}t\\ &=\alpha^d\sum_{j=1}^m\left[\left(\frac{\tau_j^2}{\tau^2}-1\right)|\eta|^2-\frac{1}{\tau}\left(1-\frac{\mu_0}{\mu_j}\right)\left(M_j\left(\frac{\mu_j}{\mu_0}\right)\eta\right)\cdot\eta\right]\mathrm{e}^{\mathrm{i}(1+\frac{1}{\tau})\eta\cdot z_j}+\mathrm{o}\big(\alpha^d\big)\end{aligned}\tag{16}$$

where the remainder $\mathrm{o}(\alpha^d)$ *is independent of the set of points* $\{z_j\}_{j=1}^m$ *provided that* (1) *holds.*

Given $E_\alpha|_{\partial\Omega\times(0,T)}$ Theorem 2.1 permits to reconstruct the locations of the small inhomgeneities and their polarization tensors from measurements of $\partial E_\alpha/\partial\nu|_{\partial\Omega\times(0,T)}$.

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