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Boundary layer sensitivity and receptivity

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Abstract The relation between the receptivity and the sensitivity of the incompressible flow in the boundary layer over a flat plate to harmonic perturbations is determined. Receptivity describes the *birth* of a disturbance, whereas sensitivity is a concept of larger breath, describing the modification incurred by the state of a system as a response to parametric variations. The governing equations ruling the system's state are the non-local stability equations. Receptivity and sensitivity functions can be obtained from the solution of the adjoint system of equations. An application to the case of Tollmien–Schlichting waves spatially developing in a flat plate boundary layer is studied. ***To cite this article: C. Airiau et al., C. R. Mecanique 330 (2002) 259–265.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

fluid mechanics / stability / adjoint / receptivity / sensitivity

Sensibilité et réceptivité d'une couche limite

Résumé La relation entre les fonctions de réceptivité et de sensibilité d'une couche limite incompressible à des perturbations du type sources harmoniques est démontrée. La réceptivité décrit la *naissance* d'une onde d'instabilité alors que la sensibilité représente la *modification* de l'état d'un système à une variation d'un de ses paramètres. L'évolution spatiale des ondes d'instabilité (l'état du système) est donnée par la solution des équations de stabilité non locales. Les fonctions de réceptivité et de sensibilité sont déduites de la solution des équations adjointes. La théorie est appliquée aux ondes de Tollmien–Schlichting qui se développent spatiallement dans la couche limite de Blasius. ***Pour citer cet article : C. Airiau et al., C. R. Mecanique 330 (2002) 259–265.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

La *réceptivité* d'une couche limite laminaire est le processus qui donne *naissance* à une onde d'instabilité. Classiquement on distingue la *réceptivité naturelle* de la *réceptivité forcée*. La première forme de réceptivité est expliquée [1,5] comme l'interaction entre une perturbation environnementale, de très grande longueur d'onde telle qu'une onde acoustique ou de vorticité, et le bord d'attaque d'une aile ou une inhomogénéité de la paroi (élément de rugosité, aspiration stationnaire). La réceptivité forcée par

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une source instationnaire harmonique locale ou distribuée a été étudiée théoriquement dans [2–4,6]. Ce second processus de génération d'une onde ne requiert pas d'interaction, et peut être déclenché par une inhomogénéité instationnaire harmonique à la paroi (soufflage/aspiration, vibration) ou à l'intérieur même de la couche limite (ruban vibrant, force électro-magnétique). Ces deux processus sont habituellement caractérisés par des fonctions de réceptivité exprimant la phase et l'amplitude initiales de l'onde à la position de la source. La *sensibilité* est l'étude des variations de l'état d'un système aux variations de ses paramètres. Elle apparaît en particulier dans la théorie du contrôle des écoulements [7] et elle correspond au gradient de la fonctionnelle coût qu'on cherche à minimiser. Dans le cadre de la théorie linéaire de l'instabilité, on considère que l'état est l'onde d'instabilité (ici, les ondes de Tollmien–Schlichting, TS). Cet état est caractérisé par le vecteur des inconnues : perturbations de vitesse et de pression ; les paramètres sont, soit des conditions aux limites pariétales inhomogènes, soit des termes sources de masse ou de quantité de mouvement. En général ces deux études sont dissociées. Nous allons montrer ici, que dans le cas de forçages harmoniques d'une couche limite, une relation mathématique simple existe entre les fonctions caractérisant la réceptivité et la sensibilité.

2. Théorie

L'évolution des ondes d'instabilités présentes dans la couche limite est décrite par les équations de stabilité non-locales PSE [9]. La perturbation bidimensionnelle $\tilde{\mathbf{q}}(x, y, t)$ est modélisée par l'équation (1) ; x et y sont respectivement, les directions longitudinale et normale à la paroi, t est le temps et le comportement temporel de $\tilde{\mathbf{q}}$ est en $\mathrm{e}^{-\mathrm{i}\omega t}$ avec ω une pulsation. On définit une amplitude locale et une énergie de perturbation par $\hat{A}'(x)$ et $E(x)$. On suppose que la couche limite est perturbée par des sources instationnaires périodiques, localisées ou distribuées, à la paroi ou dans l'écoulement (équation (2)). Le problème consiste à trouver les nouvelles amplitude et phase d'une onde existante et soumise, en aval, à une source donnée (étude de sensibilité), puis de résoudre la réceptivité forcée associée. L'approche originale proposée consiste à calculer le gradient d'une fonctionnelle (équation (3)) par rapport aux termes de forçage prédéfinis. La fonctionnelle décrit une mesure de l'état du système (ici l'énergie de l'onde à la position finale x_f du domaine Ω), à laquelle sont ajoutées les équations d'état, multipliées scalairement par des multiplicateurs de Lagrange, solutions de l'état adjoint [7]. La variation de la fonctionnelle par rapport à chacune de ses variables, considérées indépendantes, produit les équations régissant l'état adjoint ainsi que les gradients de la fonctionnelle par rapport aux sources, i.e., les fonctions de sensibilité, qui s'expriment à travers les variables adjointes (équation (6)). Une équation de l'état adjoint lie, par intégration (équation (7)), l'amplitude locale d'une onde aux conditions en x_f et aux différents forçages. En considérant l'exemple d'un forçage ponctuel à la paroi $\hat{v}_w$, d'amplitude ε, l'équation (8) donne l'amplitude complexe modifiée de l'onde à partir de la fonction de sensibilité. La fonction de réceptivité $\tilde{\Lambda}_{\hat{v}_w}$ est alors déduite par l'équation (9), ce qui constitue l'objectif recherché.

3. Application et conclusion

Des fonctions de sensibilité pour différents forçages sont calculées pour une couche limite de Blasius. La Fig. 1(a) met en évidence l'avantage du gain en temps de calcul et en précision, de l'utilisation d'une approche variationelle avec l'état adjoint, par rapport à des résolutions multiples des équations directes. Le maximum de sensibilité est trouvé, quel que soit le type de forçage, à proximité de la première branche de la courbe de stabilité neutre, et au voisinage de la couche critique (Figs. 1(a) et 2(b)). Des fonctions de réceptivité locale pour des forçages à la paroi sont représentées sur la Fig. 1(b).

En conclusion, l'approche variationnelle a permis de mettre en relation et d'étudier la sensibilité et la réceptivité d'une couche limite soumis à différents types de forçages harmoniques. L'extension à des écoulements de couche limite tridimensionnelle et compressible est possible, sinon aisée.

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1. Introduction

Receptivity studies aim to model and provide the initial conditions of a linear stability problem for convective-type instabilities, such as those arising in boundary layers. It is customary to distinguish the *natural* receptivity process [1], where an external forcing of very long wavelength, such as one acoustic or one vorticity wave, interacts with the leading edge of a plate, a roughness element, a wavy wall or a steady blowing/suction distribution [2–6] in such a way as to produce an instability wave, from the *forced* receptivity process where one harmonic forcing [2–4,6], at the wall (vibration, unsteady blowing/suction) or inside the boundary layer (vibrating ribbon, electro-magnetic force) can directly excite the instability wave. Usually the receptivity coefficients are complex functions expressing the local phase and amplitude of the disturbance at the location of the *source* of the instability. When the source is distributed along the wall the receptivity functions are given at a certain abscissa x downstream of the wall inhomogeneity.

Recently, a variational approach using the Parabolized Stability Equations (PSE) was developed to investigate the active control of Tollmien–Schlichting (TS) waves in boundary layers [7], inspired by a similar work which employed the Navier–Stokes equations [8]. In [7], the *state* of the system (i.e., the vector of unknowns: disturbance velocity and pressure, in the present case) was optimally modified by blowing and suction at the wall. The optimality procedure was based on the minimization of a cost functional, which comprised a measure of the state (the disturbance kinetic energy) and a measure of the cost of the control (the energy of the blowing/suction velocity at the wall). To reach the objective of minimizing the cost, a direct/adjoint iteration technique was employed, based on the gradients of the cost function with respect to the control variables. These gradients are commonly referred to as sensitivity functions in optimal control theory.

To set ideas the following definitions are provided here:

(i) *Receptivity* is the process through which one environmental disturbance (or several interacting disturbances) produces an instability wave in the boundary layer.

(ii) *Sensitivity* considers the effects on the system's state of a variation in one of its parameters. In the context of linear stability theory, the state in object can be either the base flow or the disturbance field. The *parameters* that influence the state are, for example, the Reynolds number, possible modifications in fluid properties, the smoothness of the wall, perturbations in the outer stream, etc.

Hence, a sensitivity study is not, in general, concerned with the *birth* of an instability wave, unlike a receptivity study. However, on the basis of the definitions above, it is clear that a relationship exists between appropriately defined receptivity and sensitivity functions, whenever the system's state is the disturbance vector, and the parameter that – through its variations – affects the state, is an outer forcing with a physical origin.

2. Sensitivity functions

The theory is based on the nonlocal stability equations [9]. The linear development of a TS wave is modeled by

$$\tilde{\mathbf{q}}(x,y,t) = \mathbf{q}(x,y)\chi(x)\,\mathrm{e}^{-\mathrm{i}\omega t} + c.c. \qquad \chi(x) = \mathrm{e}^{\mathrm{i}\int_{x_0}^{x}\alpha(\xi)\,\mathrm{d}\xi} \tag{1}$$

where $\mathbf{q}$ is the dimensionless amplitude function, $\mathbf{q} = (u, v, p)$, α may be associated to a complex wave number, and ω is the frequency. $c.c.$ denotes the complex conjugate; x and y are, respectively, the streamwise and vertical coordinates, nondimensionalized by $\delta_0 = (\nu x_0^\star/U_\infty)^{1/2}$, a length proportional to the boundary layer thickness, with $x_0^\star$ a dimensional distance away from the plate's leading edge. The velocity is made dimensionless by the upstream velocity U_∞, ν is the kinematic viscosity, and the fluid is assumed incompressible. The reference Reynolds number is $R_0 = U_\infty\delta_0/\nu$ and the local Reynolds number is $R_\delta = U_\infty\delta/\nu = (U_\infty x^\star/\nu)^{1/2}$, with $x^\star$ the streamwise position. The domain investigated is

$\Omega = \{x \in [x_0, x_f];\ y \in [0, \infty[\}$. We define the spatial part of the disturbance by $\hat{\mathbf{q}} = \chi\mathbf{q}$, the complex disturbance amplitude $\hat{A}'(x) = \max_y |\hat{u}|$ and a disturbance energy as $E(x) = \int_0^\infty |\hat{u}|^2\,\mathrm{d}y$.

We consider the case of a boundary layer excited by an unsteady forcing at the wall (subscript 'w') or elsewhere within the boundary layer (momentum or mass flow sources):

$$\tilde{\mathbf{u}}_w(x,t) = \hat{\mathbf{u}}_w(x)\,\mathrm{e}^{-\mathrm{i}\omega t} + c.c., \qquad \tilde{\mathbf{s}}(x,y,t) = \hat{\mathbf{s}}(x,y)\,\mathrm{e}^{-\mathrm{i}\omega t} + c.c. \tag{2}$$

The problem consists in finding how some functional (linked to the disturbance) is affected by the forcing terms above (sensitivity problem), and the initial phase and amplitude of the disturbance for the related receptivity problem. The functional chosen is a measure of the magnitude of the disturbance at the output location x_f, i.e., $E(x_f) = E_f$. The original approach proposed here is based on the determination of the variation $\delta\mathcal{L}$ of an augmented (Lagrangian) functional $\mathcal{L}$ with respect to the given forcing terms. For more details on the definition of $\mathcal{L}$ the reader is referred to [8,7]; it suffices here to say that the arguments of $\mathcal{L}$ are, by definition, independent of one another, and variations of $\mathcal{L}$ are equal to variations of E_f, with $\mathcal{L}$ given by:

$$\begin{aligned}\mathcal{L}\big(\mathbf{q},\alpha,\hat{\mathbf{s}},\hat{\mathbf{u}}_w,\hat{\mathbf{q}}^*,\gamma^*,\hat{\mathbf{r}}^*\big) = E_f\big(\mathbf{q},\alpha,\hat{\mathbf{s}},\hat{\mathbf{u}}_w\big) - \Big\{\big\langle\hat{\mathbf{q}}^*, \chi\mathcal{L}_{\mathrm{PSE}}\,\mathbf{q} - \hat{\mathbf{s}}\big\rangle_\Omega \\ - \left\langle\gamma^*, \int_0^\infty \bar{u}\frac{\partial u}{\partial x}\,\mathrm{d}y\right\rangle_\Gamma + \big\langle\hat{\mathbf{r}}^*, \chi\,\mathbf{u}(x,0) - \hat{\mathbf{u}}_w(x)\big\rangle_\Gamma + c.c.\Big\}\end{aligned} \tag{3}$$

The inner products are defined as:

$$\langle a,b\rangle_\Gamma = \int_{x_0}^{x_f} \bar{a}b\,\mathrm{d}x, \qquad \langle c,d\rangle_\Omega = \int_{x_0}^{x_f}\int_0^\infty \bar{c}d\,\mathrm{d}y\,\mathrm{d}x$$

overbar denoting complex conjugation. The first inner product on the right-hand side of equation (3) contains the state equation for $\chi\mathbf{q}$, scalarly multiplied by the Lagrange multiplier $\hat{\mathbf{q}}^*$. In the following, the role of $\hat{\mathbf{q}}^*$ as adjoint function will become clear. The second inner product on the right-hand side accounts for the normalization condition imposed on the PSE. The third inner product allows to satisfy the constraint on the wall forcing $\hat{\mathbf{u}}_w$. The Lagrange multipliers, $\hat{\mathbf{q}}^*$ and γ^* are solutions of an adjoint PSE system. The adjoint equations can be recovered by setting to zero the variations of the Lagrangian functional with respect to the direct variables. Some integrations by parts over the domain Ω (or Γ) are required. The adjoint PSE (APSE), including boundary and terminal conditions are given in [7]. Incidentally, note that [7] employed the same numerical tools as here, albeit in a flow control context, but it did not address receptivity/sensitivity issues.

The variation of the output energy with respect to any forcing term c is given by the following Fréchet differential:

$$\delta E_f = \lim_{\epsilon\to 0}\frac{\mathcal{L}(c+\epsilon\delta c) - \mathcal{L}(c)}{\epsilon} \tag{4}$$

c representing the real and imaginary part of the components of $\hat{\mathbf{u}}_w$ and $\hat{\mathbf{s}}$, i.e., $\hat{u}_w, \hat{v}_w, \hat{s}_x, \hat{s}_y, \hat{s}_m$. The variation of the functional may be written as:

$$\delta E_f = \big\langle S_{\hat{u}_w}, \delta\hat{u}_w\big\rangle_\Gamma + \big\langle S_{\hat{v}_w}, \delta\hat{v}_w\big\rangle_\Gamma + \big\langle S_{\hat{s}_x}, \delta\hat{s}_x\big\rangle_\Omega + \big\langle S_{\hat{s}_y}, \delta\hat{s}_y\big\rangle_\Omega + \big\langle S_{\hat{s}_m}, \delta\hat{s}_m\big\rangle_\Omega \tag{5}$$

The sensitivity functions S_c are obtained from the solution of the adjoint equations and are given by:

$$\begin{aligned} &S_{\hat{u}_w}(x) = \frac{1}{R_0}\frac{\partial\hat{u}^*}{\partial y}(x,0), \qquad S_{\hat{v}_w}(x) = \hat{p}^*(x,0), \\ &S_{\hat{s}_x}(x,y) = \hat{u}^*(x,y), \qquad S_{\hat{s}_y}(x,y) = \hat{v}^*(x,y), \qquad S_{\hat{s}_m}(x,y) = \hat{p}^*(x,y)\end{aligned} \tag{6}$$

They relate the variation in final disturbance energy to possible variations in the wall boundary conditions or in momentum and mass source terms.

3. Receptivity functions

The relation between the sensitivity functions, the disturbance $\hat{\mathbf{q}}$ at the generic location x and the forcing terms is obtained by setting to zero the variation of $\mathcal{L}$ with respect to the complex wave number α:

$$\tilde{J}(x) = \tilde{J}(x_f) + \int_{x_f}^{x} \left(\bar{S}_{\hat{u}_w}\hat{u}_w + \bar{S}_{\hat{v}_w}\hat{v}_w\right)\mathrm{d}x + \int_{x_f}^{x}\int_0^{\infty}\left(\bar{S}_{\hat{s}_x}\hat{s}_x + \bar{S}_{\hat{s}_y}\hat{s}_y + \bar{S}_{\hat{s}_m}\hat{s}_m\right)\mathrm{d}y\,\mathrm{d}x \tag{7}$$

where $\tilde{J}(x) = \int_0^{+\infty}\{(U + 2\mathrm{i}\alpha/R_0)(\bar{\hat{u}}^*\hat{u} + \bar{\hat{v}}^*\hat{v}) + \bar{\hat{u}}^*\hat{p} + \bar{\hat{p}}^*\hat{u}\}\,\mathrm{d}y$, and U is the boundary layer streamwise mean flow velocity. By assuming that source terms change the amplitude and the phase of the disturbance, but do not influence its shape, it is possible to extract the complex amplitude from $\tilde{J}(x)$, by writing $\tilde{J}(x) = \hat{A}'(x)J(x)$, i.e., by normalizing $\tilde{J}(x)$ with the local complex wave amplitude $\hat{A}'(x)$. The complex function $J(x)$ is calculated from the reference, unforced case. Hence, Eq. (7) provides a relation between the amplitude at the position x, and the function J in the homogeneous case, i.e.,

$$\hat{A}'(x)J(x) = \hat{A}'(x_f)J(x_f) = \hat{A}'(x_0)J(x_0)$$

In the inhomogeneous case, when considering as only forcing a harmonic, point-source wall perturbation at x_s, of magnitude ε, i.e., $\tilde{v}_w = \varepsilon\delta_D(x - x_s)\exp(\mathrm{i}\omega t) + c.c.$, with $\delta_D(\cdot)$ the Dirac distribution, the response to the forcing can be obtained by the use of the sensitivity function as

$$\hat{A}'(x_f) = \hat{A}'(x_0)\frac{J(x_0)}{J(x_f)} + \varepsilon\frac{\bar{S}_{\hat{v}_w}(x_s)}{J(x_f)} \tag{8}$$

The receptivity function $\tilde{\Lambda}_{\hat{v}_w}(x_s)$ is usually defined [2–4,6] as the ratio between the complex amplitude of the TS wave which would be induced at the location of the source x_s, in the absence of incoming disturbances ($\hat{A}'(x_0) = 0$), and the amplitude of the forcing ε. We can, therefore, write the relation between receptivity and sensitivity, generalized to any impulsive-type source, as

$$\tilde{\Lambda}_c(x_s) = \frac{\bar{S}_c(x_s)}{J(x_s)} \tag{9}$$

When the sources are distributed along the wall or in the boundary layer, relations (8) and (9) must be carefully written with the integral over Γ or over Ω.

4. Application to the Blasius boundary layer

The previous section provides a way of studying the non-parallel receptivity of a boundary layer to time-periodic sources. Here, we consider the case of the two-dimensional flate plate boundary layer, modeled by the Blasius self-similar profile. Fig. 1(a) shows the modulus of the sensitivity function $S_{\hat{v}_w}$, normalized by $|J(x_f)|$ versus the local Reynolds number R_δ, with $R_\delta \in [350, 760]$. This distribution may also be obtained by solving N times the PSE with a pointwise perturbed wall velocity (N is the number of wall mesh points) and by computing the sensitivity function point-by-point with some finite difference expression. This way to proceed is time consuming and less accurate, by comparison with solving *once* the PSE and the APSE.

The receptivity functions to a wall forcing are plotted in Fig. 1(b) using relation (9). The receptivity to a vertical velocity at the wall, $\hat{v}_w$, is about one order of magnitude larger than the receptivity to $\hat{u}_w$. Because the receptivity functions represent local amplitudes, the curves do not exhibit a maximum close to branch I.

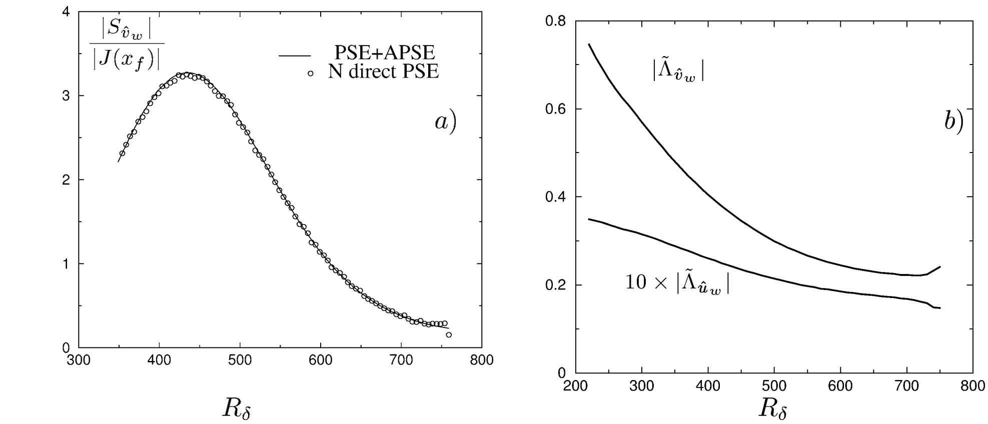

Figure 1. a) Sensitivity to $\hat{v}_w$, comparison with direct calculations, $F = \omega/R_0 = 86 \times 10^{-6}$; b) Receptivity to wall velocity perturbations, $\hat{u}_w$ and $\hat{v}_w$, $F = 100 \times 10^{-6}$.

Figure 1. a) *Sensibilité à* $\hat{v}_w$*, comparaison avec le calcul direct,* $F = \omega/R_0 = 86 \times 10^{-6}$; b) *Receptivité à une perturbation de vitesse pariétale,* $\hat{u}_w$ *et* $\hat{v}_w$*,* $F = 100 \times 10^{-6}$.

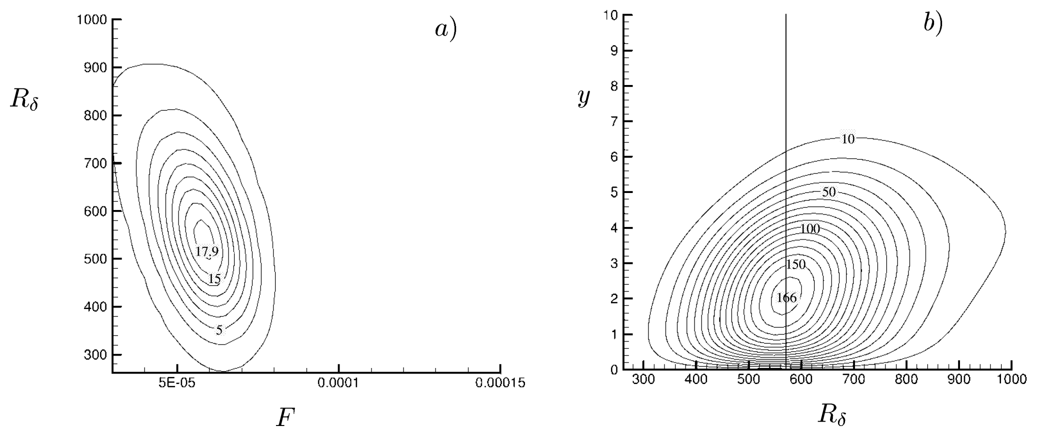

Figure 2. Isocontours of: a) $|S_{\hat{v}_w}/J(x_f)|$ as function of F, b) $|S_{\hat{S}_x}/J(x_f)|$ at $F_{\max}$. The vertical line denotes the location of the first point of the neutral curve.

Figure 2. *Lignes isomodule de* : a) $|S_{\hat{v}_w}/J(x_f)|$ *en fonction de* F, b) $|S_{\hat{S}_x}/J(x_f)|$ *à* $F_{\max}$. *La ligne verticale donne la position du premier point de la courbe neutre.*

For a given domain $R_\delta \in [260, 1000]$, the largest sensitivity occurs at the frequency $F_{\max} = 6 \times 10^{-5}$ for which there is a maximum amplification of the TS waves at x_f, as shown for example on Fig. 2(a), where sensitivity to $\hat{v}_w$ is plotted. The maximum of the modulus of the sensitivity to a streamwise source of momentum (Fig. 2(b)) is found close to branch I of the neutral stability curve and in close proximity of the critical layer as already indicated previously [6]. More importantly, the general result provided by figures such as Figs. 1 and 2, is that, by employing Eq. (7), the receptivity function to any type of forcing, in particular distributed source terms, can be computed by taking one simple inner product.

5. Conclusion

In this work we have tried to elucidate differences and analogies between the concepts of receptivity and sensitivity. We have considered the response of a two-dimensional boundary layer to monochromatic forcing terms, imposed at the wall or within the flow, on the spatial growth of TS waves. A vertical velocity at the wall (mimicking harmonic blowing/suction or a vertical vibration of a smooth wall) has an effect on the instability wave which is one order of magnitude larger than an horizontal velocity disturbance (this latter models, e.g., unsteady roughness or longitudinal vibrations of the wall). The most effective volume term is found to correspond to a momentum source term in the x direction. These considerations extend beyond receptivity and can be directly employed in the definition of effective means of flow control.

The extension of the present work to three-dimensional compressible boundary layers and to other types of exponentially growing flow instabilities is perhaps tedious but straightforward.

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