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Numerical implementation of composite Koiter shells including membrane/bending coupling coefficients

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Abstract We propose a way for determining the generalized coefficients of rigidity – some of which are membrane/bending coupling coefficients – which appear in the deformation energy of the Koiter model of thin shells. This is concerned with a heterogeneous material in the thickness direction. A new program to compute these coefficients is implemented in the finite element code Modulef, in order to simulate problems of thin multilayered shells with linearly elastic anisotropic layers. We propose an example of an inhibited multilayered thin shell, with hyperbolic middle surface, involving a composite material with unidirectional fibres. ***To cite this article: H. Ranarivelo, C. R. Mecanique 330 (2002) 273–278.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

computational solid mechanics / Koiter's model / heterogeneity / anisotropy / asymptotic methods / Modulef

Implémentation numérique des coques composites de Koiter prenant en considération les coefficients de couplage membrane/flexions

Résumé Nous proposons une démarche de calcul des coefficients généralisés de rigidité – dont ceux de couplage membrane/flexion – qui apparaîssent dans l'énergie de déformation d'un modèle de coque mince de Koiter. C'est le cas lorsque le matériau est hétérogène dans le sens de l'épaisseur. Un nouveau module de calcul de ces coefficients est implémenté dans le code de calcul élément fini Modulef, permettant la simulation de problèmes de coques minces multicouches à plis élastiques linéaires anisotropes. Nous proposons l'exemple d'une coque mince multicouche inhibée, à surface moyenne hyperbolique, en composite à fibres unidirectionnelles. ***Pour citer cet article : H. Ranarivelo, C. R. Mecanique 330 (2002) 273–278.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

mécanique des solides numérique / modèle de Koiter / hétérogénéité / anisotropie / méthodes asymptotiques / Modulef

1. Introduction

A thin shell is a three dimensional body C, the thickness of which is small compared with the other dimensions. The shape of C is then close to its middle surface S. The small thickness is denoted 2ε ($\varepsilon \ll 1$).

S is defined by a map application $\vec{\varphi}$:

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$$\vec{\varphi}: \quad \Omega \subset \mathcal{R}^2 \longrightarrow \mathcal{R}^3$$
$$\left(y^1, y^2\right) \longmapsto \vec{\varphi}\left(y^1, y^2\right)$$

Greek indices take values in $\{1,2\}$ and latin indices in $\{1,2,3\}$. Let $P(y^1, y^2) \in S$.

The covariant local basis at P is $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ such as $\vec{a}_\alpha = \partial\vec{\varphi}/\partial y^\alpha$ and $\vec{a}_3 = \vec{a}_1 \wedge \vec{a}_2 / |\vec{a}_1 \wedge \vec{a}_2|$.

The contravariant local basis at P is $(\vec{a}^1, \vec{a}^2, \vec{a}^3)$ such as $\vec{a}^\alpha . \vec{a}_\beta = \delta^\alpha_\beta$ where δ^α_β are Kronecker's symbols. Let $(O, \vec{i}, \vec{j}, \vec{k})$ an orthonormal axis system of R^3. The thin shell C can be then defined as:

$$C = \left\{ Q\left(y^1, y^2, y^3\right) \mid \vec{OQ} = \vec{\varphi}\left(y^1, y^2\right) + y^3 \vec{a}_3,\ y^3 \in [-\varepsilon; +\varepsilon] \right\} \tag{1}$$

Different mechanical models for such a thin body were proposed [1]. In the linear case, the justification of the so-called Koiter's model is now well established [2–4], by the use of asymptotic methods associated with multiple scales (instead of the kinematical hypothesis). The main idea of asymptotic methods is to study the behaviour of C when ε gets smaller and smaller. Two different limit schemes then appear when S is or not geometrically rigid: the inhibited case or the non-inhibited case.

We consider the case of material heterogeneities, only in the thickness direction: then, C can be studied as a thin multilayered shell. The mechanical behaviour of each (homogeneous) layer $C_k,\ k = 1, N$, is assumed to be elastic linearly anisotropic. C_k is contained between the two surfaces $y^3 = y_k^{3-}$ and $y^3 = y_k^{3+}$.

At each $Q(y^1, y^2, y^3) \in C_k$, we choose a local orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ such that $\vec{e}_3(Q) = \vec{a}_3(y^1, y^2)$ and $\vec{e}_1(Q)$ defines a particular direction for the material, such as a natural direction of anisotropy. Let $\Theta_k = (\widehat{\vec{e}_1, \vec{a}_1})$ be the orientation of the local orthonormal basis.

Using components in the orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, let us define:

$$\bar{\sigma}_{ij} = \bar{a}^k_{ijhl}\left(y^1, y^2\right)\bar{\gamma}_{hl} \quad \text{at each } Q\left(y^1, y^2, y^3 \in \left[y_k^{3-}; y_k^{3+}\right]\right) \tag{2}$$

where $\bar{\sigma}_{ij}$ and $\bar{\gamma}_{hl}$ are the components of the three-dimensional stress tensor and the three-dimensional linearized strain tensor, respectively, in the orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. Moreover, $\bar{a}^k$ denotes the symmetric tensor of rigidity for the C_k's material. It can be considered as completely known by the identification of the 21 intrinsic mechanical characteristics of the elastic linear anisotropic material of C_k [5]. For instance, some of these constants are Young modulus $E_i,\ i = 1, 3$, in the three orthotropic directions $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

Using now the components in the local curvilinear basis $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ or $(\vec{a}^1, \vec{a}^2, \vec{a}_3)$, we have:

$$\sigma^{ij} = a^{ijhl^k}\left(y^1, y^2\right)\gamma_{hl} \quad \text{at each } Q\left(y^1, y^2, y^3 \in \left[y_k^{3-}; y_k^{3+}\right]\right) \tag{3}$$

Let $\eta \equiv (\eta^1, \eta^2, 0) = (y^1, y^2, 0)$ the global curvilinear space variables for the asymptotic analysis and $z \equiv (0, 0, z^3) = (0, 0, y^3/\varepsilon)$ the local ones. Let $y \equiv (y^1, y^2, y^3)$. The displacement is denoted $\vec{u}^{(\varepsilon)}(y)$.

Instead of the three-dimensional local stress tensor σ and three-dimensional local linearized strain tensor γ, the asymptotic methods lead to Koiter's model, which only involves *two dimensional* entities, defined on S: the *generalized two dimensional stress* $\mathcal{T}^{\alpha\beta}$ *and moments* $\mathcal{M}^{\alpha\beta}$ (in contravariant form):

$$\mathcal{T}^{\alpha\beta}(\eta) = \int_{-1}^{+1} \varepsilon\sigma^{\alpha\beta(1)}(\eta, z)\sqrt{a}\,\varepsilon\,\mathrm{d}z \quad \text{and} \quad \mathcal{M}^{\alpha\beta}(\eta) = \int_{-1}^{+1} \varepsilon z\,\varepsilon\sigma^{\alpha\beta(1)}(\eta, z)\sqrt{a}\,\varepsilon\,\mathrm{d}z \tag{4}$$

where $\varepsilon\sigma^{\alpha\beta(1)}(\eta, z)$ represents the first nonvanishing term in the asymptotic expansion of $\sigma^{\alpha\beta(\varepsilon)}$ and $\sqrt{a} = \sqrt{|\vec{a}_1 \wedge \vec{a}_2|}$.

Moreover, we have the following *reduced local constitutive law at* $Q \in C_k$:

$$\varepsilon\sigma^{\alpha\beta(1)} = c^{\alpha\beta\lambda\delta^k}(\eta)\left[\gamma_{\alpha\beta}(\vec{s}) + \varepsilon z \rho^\star_{\alpha\beta}(\vec{s})\right] \quad \text{where } z \in \left[z_k^-; z_k^+\right]. \tag{5}$$

We note that (5) involves reduced local coefficient of rigidity $c^{\alpha\beta\lambda\delta^k}(\eta)$ at each $Q \in C_k$. It also involves *generalized strains* (in contravariant form) which are: the membrane strain $\gamma_{\alpha\beta}(\vec{s})$ and curvature changes $\rho^\star_{\alpha\beta}(\vec{s})$, where $\vec{s}(y^1, y^2) = \vec{u}^\varepsilon(y^1, y^2, 0)$ represents the displacement on S:

$$\gamma_{\alpha\beta}(\vec{s}) = \frac{D_\alpha s_\beta + D_\beta s_\alpha}{2} \quad \text{and} \quad \rho^\star_{\alpha\beta}(\vec{s}) = -\rho_{\alpha\beta}(\vec{s}) = \delta b_{\alpha\beta}\left(y^1, y^2\right) \tag{6}$$

where D_α is the covariant derivation and $\delta b_{\alpha\beta}$ the variation of the coefficients of the second fundamental form of S.

The strain energy form of the Koiter's model is then:

$$a(\vec{s}, \vec{s}) = \int_\Omega \left(\mathcal{T}^{\alpha\beta}\gamma_{\alpha\beta}(\vec{s}) + \mathcal{M}^{\alpha\beta}\rho^\star_{\alpha\beta}\right) \mathrm{d}\Omega \tag{7}$$

Using results established in [6], for an heterogeneous material, when observed in the thickness direction, we have:

$$\mathcal{T}^{\alpha\beta} = \varepsilon M^{\alpha\beta\lambda\delta}\gamma_{\lambda\delta} + \varepsilon^2 C^{\alpha\beta\lambda\delta}\rho^\star_{\lambda\delta} \tag{8}$$

$$\mathcal{M}^{\alpha\beta} = \varepsilon^2 C^{\alpha\beta\lambda\delta}\gamma_{\lambda\delta} + \varepsilon^3 F^{\alpha\beta\lambda\delta}\rho^\star_{\lambda\delta} \tag{9}$$

$M^{\alpha\beta\lambda\delta}$ and $F^{\alpha\beta\lambda\delta}$ are the generalized coefficient of membrane rigidity and of bending rigidity, respectively. $C^{\alpha\beta\lambda\delta}$ are the *coupling membrane/bending coefficients.*

Our purpose is to compute these generalized coefficients, in particular the coupling ones.

2. Computation of the generalized coefficients of rigidity

The *local* reduced coefficients in curvilinear components at $Q(\eta, z) \in C$ can be written as:

$$c^{\alpha\beta\lambda\delta}(\eta, z) = \left\{\begin{array}{ll} c^{\alpha\beta\rho\delta^1}(\eta) & \text{si } z_1^- \leqslant z \leqslant z_1^+ \\ \ldots & \\ c^{\alpha\beta\rho\delta^k}(\eta) & \text{si } z_k^- \leqslant z \leqslant z_k^+ \\ \ldots & \\ c^{\alpha\beta\rho\delta^N}(\eta) & \text{si } z_N^- \leqslant z \leqslant z_N^+ \end{array}\right\} \tag{10}$$

Indeed, we have first to compute $c^{\alpha\beta\rho\delta^k}(\eta)$ relative to each layer C_k. For that, we assume that the analogous reduced components $\bar{c}^{\alpha\beta\rho\delta^1}(\eta)$ in the orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ are completely known.

Let us introduce the following notations:

$$\{\vec{e}\}^t = \langle \vec{e}_1\ \vec{e}_2\ \vec{e}_3 \rangle, \quad \{\vec{a}_{\text{cov}}\}^t = \langle \vec{a}_1\ \vec{a}_2\ \vec{a}_3 \rangle, \quad \{\vec{a}_{\text{cont}}\}^t = \langle \vec{a}^1\ \vec{a}^2\ \vec{a}^3 \rangle$$

$$\left\{\bar{\sigma}^{(1)}\right\} = \langle \bar{\sigma}^{(1)}_{11}\ \bar{\sigma}^{(1)}_{22}\ \bar{\sigma}^{(1)}_{12} \rangle, \quad \left\{\bar{\gamma}^{(1)}\right\} = \langle \bar{\gamma}^{(1)}_{11}\ \bar{\gamma}^{(1)}_{22}\ 2\bar{\gamma}^{(1)}_{12} \rangle, \quad \text{so that} \quad \left\{\bar{\sigma}^{(1)}\right\} = [\bar{c}]^k\left\{\bar{\gamma}^{(1)}\right\}$$

$$\left\{\sigma^{(1)}\right\} = \langle \sigma^{11^{(1)}}\ \sigma^{22^{(1)}}\ \sigma^{12^{(1)}} \rangle, \quad \left\{\gamma^{(1)}\right\} = \langle \gamma^{(1)}_{11}\ \gamma^{(1)}_{22}\ 2\gamma^{(1)}_{12} \rangle$$

We can write $\{\vec{e}\} = [T]^k\{\vec{a}_{\text{cov}}\}$ where $[T]^k$ depends on the orientation Θ_k. Then, we have the relation between cartesian and curvilinear components of the local stress:

$$\left\{\sigma^{(1)}\right\} = [U]^k\left\{\bar{\sigma}^{(1)}\right\} \quad \text{where } [U]^k \text{ depends on } [T]^k$$

We can also write $\{\vec{e}\} = [R]^k\{\vec{a}_{\text{cont}}\}$. Then, we have the relation between cartesian and curvilinear components of the local strain: $\{\gamma^{(1)}\} = [V]^k\{\bar{\gamma}^{(1)}\}$ where $[V]^k$ depends on $[R]^k$.

Finally, we obtain:

$$\left\{\sigma^{(1)}\right\} = [c]^k \left\{\gamma^{(1)}\right\} \quad \text{where } [c]^k = [U]^k.[\bar{c}]^k.[V]^{k^{-1}} \tag{11}$$

Once $c^{\alpha\beta\rho\delta^k}(\eta)$ is computed by (11), we can express the generalized two-dimensional and tangential stress $T^{\alpha\beta}$ for the Koiter's model using its definition in (4) and Eq. (5):

$$\begin{aligned} T^{\alpha\beta}(\eta) &= \varepsilon \int_{-1}^{1} c^{\alpha\beta\lambda\delta}(\eta, z)\left[\gamma_{\lambda\delta}(\vec{s}) + \varepsilon z \rho^{\star}_{\lambda\delta}(\vec{s})\right]\sqrt{a}\,\mathrm{d}z \\ &= \varepsilon M^{\alpha\beta\lambda\delta}(\eta)\gamma_{\lambda\delta}(\vec{s}) + \varepsilon^2 C^{\alpha\beta\lambda\delta}(\eta)\rho^{\star}_{\lambda\delta}(\vec{s}) \end{aligned} \tag{12}$$

where we have defined *the generalized coefficient of membrane rigidity*:

$$M^{\alpha\beta\lambda\delta}(\eta) = \sqrt{a} \int_{-1}^{1} c^{\alpha\beta\lambda\delta}(\eta, z)\,\mathrm{d}z = \sqrt{a} \sum_{k=1}^{N} c^{\alpha\beta\lambda\delta^k}.\left(z_k^+ - z_k^-\right) \tag{13}$$

and *the generalized coupling coefficient of membrane/bending rigidity*:

$$C^{\alpha\beta\lambda\delta}(\eta) = \sqrt{a} \int_{-1}^{1} z c^{\alpha\beta\lambda\delta}(\eta, z)\,\mathrm{d}z = \sqrt{a} \sum_{k=1}^{N} c^{\alpha\beta\lambda\delta^k}.\frac{(z_k^+)^2 - (z_k^-)^2}{2} \tag{14}$$

Finally, the generalized bidimensional and tangentiel moments $\mathcal{M}^{\alpha\beta}$ for the Koiter's model using its definition in (4) are:

$$\begin{aligned} \mathcal{M}^{\alpha\beta}(\eta) &= \varepsilon \int_{-1}^{1} \varepsilon z c^{\alpha\beta\lambda\delta}(\eta, z)\left[\gamma_{\lambda\delta}(\vec{s}) + \varepsilon z \rho^{\star}_{\lambda\delta}(\vec{s})\right]\sqrt{a}\,\mathrm{d}z \\ &= \varepsilon^2 C^{\alpha\beta\lambda\delta}(\eta)\gamma_{\lambda\delta}(\vec{s}) + \varepsilon^3 F^{\alpha\beta\lambda\delta}(\eta)\rho^{\star}_{\lambda\delta}(\vec{s}) \end{aligned} \tag{15}$$

where we have defined *the generalized coefficient of bending rigidity*:

$$F^{\alpha\beta\lambda\delta}(\eta) = \sqrt{a} \int_{-1}^{1} z^2 c^{\alpha\beta\lambda\delta}(\eta, z)\,\mathrm{d}z = \sqrt{a} \sum_{k=1}^{N} c^{\alpha\beta\lambda\delta^k}.\frac{(z_k^+)^3 - (z_k^-)^3}{3} \tag{16}$$

3. Numerical simulation with Modulef

We propose an example of a simulation with the finite element code Modulef. We use the so-called Ganev–Argyris element [7] so that the meshing is done on the domain of the curvilinear parameters Ω, instead of on the surface S. The simulation *has previously needed the implementation of a new program to compute the generalized coefficients of rigidity* as explained before. These coefficients are calculated at each Gauss point of the numerical integration scheme.

3.1. Data for the problem

We consider a thin shell with a thickness 2ε where $\varepsilon = 10^{-3}$. It is constituted of three layers made of a composite material with unidirectional fibres: epoxy/carbon with 60% of fibres [8]. Let $\vec{e}_1$ indicating the fibers direction at each $Q \in C_k,\ k = 1, 3$:

$$\Theta_1 = 45^\circ, \quad z_-^1 = -1; \quad z_+^1 = -0.5/\Theta_2 = 0^\circ, \quad z_-^2 = -0.5$$
$$z_+^2 = 0.5/\Theta_3 = 30^\circ, \quad z_-^3 = 0.5; \quad z_+^3 = +1$$

In this case, the expression of $\bar{c}^k_{\alpha\beta\lambda\delta}$ only needs four mechanical constants: $E_1 = 84$ GPa, $E_2 = 5.6$ GPa the Young's modulus; $G_{12} = 2.1$ GPa the shear modulus; $\nu_{12} = 0.34$ the Poisson's modulus. The domain of parameters Ω is: $[0; 1] \times [0; 0.5]$. The map application $\vec{\varphi}$ is defined by:

$$\vec{\varphi}(y^1, y^2) = y^1\vec{i} + y^2\vec{j} + y^1y^2\vec{k}$$

The boundaries $y^1 = 0$ and $y^2 = 0$ are clamped. The shell is an inhibited one. External force $\vec{f} = -\varepsilon\vec{a}^3$ is applied.

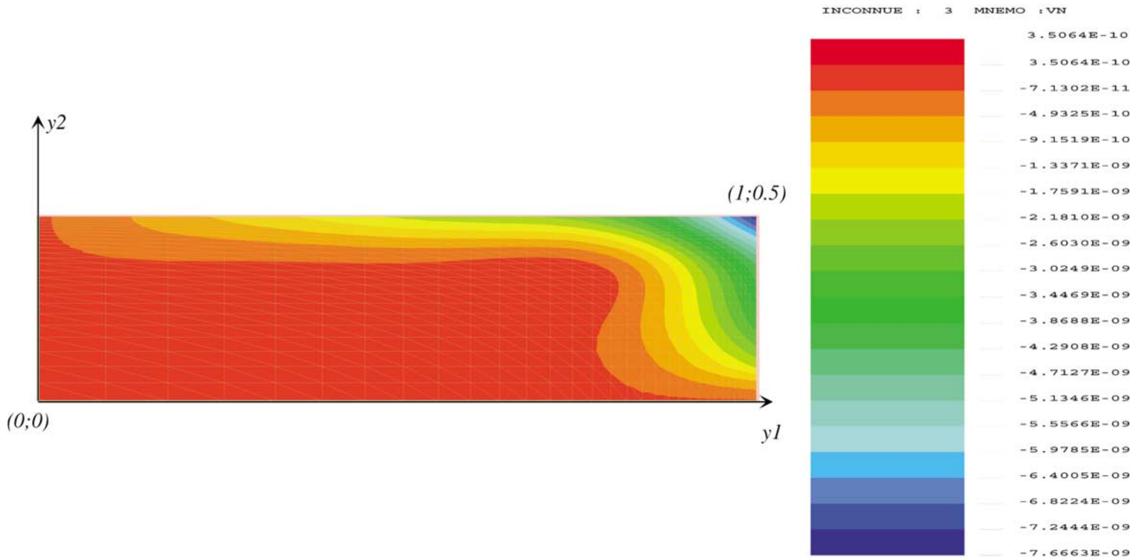

Figure 1. S_3 on Θ.

Figure 1. S_3 *sur* Θ.

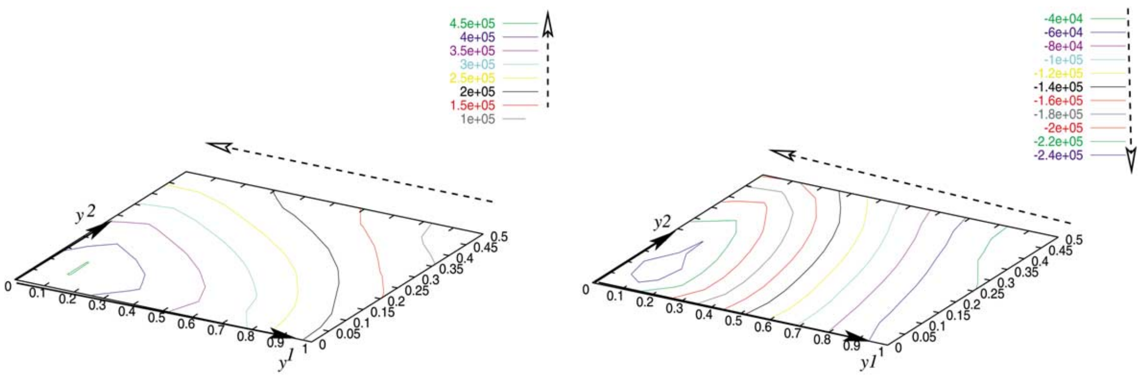

Figure 2. Isolines of (left) C^{1111} and (right) C^{2222} at Gauss points on Ω.

Figure 2. *Isovaleurs de* (*gauche*) C^{1111} *et* (*droite*) C^{2222} *aux points de Gauss sur* Ω.

3.2. Results

Fig. 1 illustrates the normal displacement s_3 on Ω after a finite element analysis with adapted Modulef. This final result was obtained after several meshing tests in order to take account of the existence of boundary layers along free edges $y^1 = 1$ and $y^2 = 0.5$: s_3 takes singular large values inside these narrow regions, whereas the rest of Ω still remains quite rigid. Details of this boundary phenomenon can be seen in [9,10].

Fig. 2 illustrates some of the generalized coupling coefficients (C^{1111} and C^{2222}), calculated at each Gauss point of the numerical integration scheme.

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