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On loading criteria in plasticity

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Abstract The loading criteria of the Lagrangian strain-space formulation of rate-independent plasticity are compared with those of Nguyen and Bui and those of Kuhn–Tucker type. When the latter two sets of conditions are expressed in a fully strain-space form, their relationship to the loading criteria of the strain-space formulation becomes transparent. ***To cite this article: J. Casey, C. R. Mecanique 330 (2002) 285–290.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

continuum mechanics / plasticity / loading criteria

Sur des critères de chargement en plasticité

Résumé Les critères de chargement dans la formulation dans l'espace des déformations de Lagrange pour la plasticité indépendante du taux de déformation sont comparés à deux autres types de conditions, celle de Nguyen et Bui et celle du type Kuhn–Tucker. Quand ces dernières sont exprimées entièrement dans l'espace des déformations, leur relation au critère de chargement en formulation d'espace de déformations devient transparent. ***Pour citer cet article : J. Casey, C. R. Mecanique 330 (2002) 285–290.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

For general strain-hardening behavior of elastic-plastic materials (i.e., including softening and perfectly plastic behavior), the inadequacy of the classical stress-space loading criteria is well known [1,2]. In 1975, Naghdi and Trapp [2] proposed an alternative formulation of plasticity theory in which loading criteria are formulated with reference to loading surfaces in strain space. Casey and Naghdi [3] showed that when the strain-space loading criteria are adopted as primary, the associated conditions in stress space lead to a geometrically appealing characterization of hardening, softening, and perfectly plastic behavior. Related material is contained in [4–11].

In 1974, Nguyen and Bui [1], wishing to accommodate softening behavior in infinitesimal plasticity, suggested new loading criteria in terms of the yield function in stress space. These authors did not employ a yield function in strain space, but they did derive a loading criterion involving the strain rate tensor. In the present Note, a reinterpretation of the Nguyen–Bui conditions is given in strain space – where they attain their simplest form – and a comparison is made with the loading criteria of the strain-space formulation.

Another widely used set of loading criteria, the "Kuhn–Tucker conditions" [12–15], are also discussed. The presentation by Simo and Hughes [15], although it is confined to infinitesimal plasticity, is especially

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relevant. These authors describe their formulation as "strain-driven" – strain rather than stress is the basic independent variable. They employ yield surfaces only in stress space, but the strain rate tensor eventually appears in their loading criteria. Simo [14] expresses his yield function in strain-space form, and then employs Kuhn–Tucker loading conditions.

The equations in Section 2 hold equally well in the various formulations discussed in Sections 3, 4, and 5. In Sections 3–5, one needs to distinguish carefully between different characterizations of "plastic loading".

2. Basic equations

Let the Lagrangian strain and plastic strain tensors be $\boldsymbol{E}$ and $\boldsymbol{E}_p$, respectively, and let $\boldsymbol{S}$ be the symmetric Piola–Kirchhoff stress tensor. We regard $\boldsymbol{E}$ and $\boldsymbol{E}_p$ as points in a six-dimensional Euclidean space (strain space), and $\boldsymbol{S}$ as a point in another six-dimensional Euclidean space (stress space). Further, let the scalar κ and the second-order tensor $\boldsymbol{\alpha}$ be hardening parameters. For brevity, represent the list of variables $(\boldsymbol{E}_p, \kappa, \boldsymbol{\alpha})$ by the symbol Z. Assume a smooth stress response function of the form $\boldsymbol{S} = \widehat{\boldsymbol{S}}(\boldsymbol{E}, Z)$. Equivalently, $\boldsymbol{S}$ may be expressed as a function of the strain difference $\boldsymbol{E} - \boldsymbol{E}_p$ and Z. (Alternatively, a multiplicative decomposition of the deformation gradient may be employed, but, in general, the intermediate configuration is not stress-free [16].) Denote the fourth-order elasticity tensor $\partial\widehat{\boldsymbol{S}}/\partial\boldsymbol{E}$ by $\boldsymbol{C}$. If $f(\boldsymbol{S}, Z)$ is the yield function in stress space, the yield function in strain space is defined by

$$f(\boldsymbol{S}, Z) = f\big(\widehat{\boldsymbol{S}}(\boldsymbol{E}, Z), Z\big) = g(\boldsymbol{E}, Z) \tag{1}$$

Further, assume that $\partial f/\partial\boldsymbol{S} \neq 0$. It is clear that

$$\begin{aligned} &\frac{\partial g}{\partial \boldsymbol{E}} = \boldsymbol{C}^{\mathrm{T}}\left[\frac{\partial f}{\partial \boldsymbol{S}}\right] \neq 0, \qquad &&\frac{\partial g}{\partial \boldsymbol{E}_p} - \frac{\partial f}{\partial \boldsymbol{E}_p} = \left(\frac{\partial \widehat{\boldsymbol{S}}}{\partial \boldsymbol{E}_p}\right)^{\mathrm{T}}\left[\frac{\partial f}{\partial \boldsymbol{S}}\right] \\ &\frac{\partial g}{\partial \kappa} - \frac{\partial f}{\partial \kappa} = \frac{\partial \widehat{\boldsymbol{S}}}{\partial \kappa}\cdot\frac{\partial f}{\partial \boldsymbol{S}}, \qquad &&\frac{\partial g}{\partial \boldsymbol{\alpha}} - \frac{\partial f}{\partial \boldsymbol{\alpha}} = \left(\frac{\partial \widehat{\boldsymbol{S}}}{\partial \boldsymbol{\alpha}}\right)^{\mathrm{T}}\left[\frac{\partial f}{\partial \boldsymbol{S}}\right] \end{aligned} \tag{2}$$

where the superscript T denotes transposition and $\boldsymbol{C}^{\mathrm{T}}[\partial f/\partial\boldsymbol{S}]$ has a component representation $C_{MNKL}\partial f/\partial S_{MN}$, summation being performed on repeated indices.

The elastic region in stress space corresponds to $f < 0$, and the elastic region in strain space to $g < 0$. The yield surface in stress space corresponds to $f = 0$, and the yield surface in strain space to $g = 0$. The point $\boldsymbol{E}_p$ belongs to the region $g < 0$ in strain space, according to the prescription given in [16]. It follows from (1) that during any motion of the continuum, the material derivatives of f and g coincide:

$$\dot{f} = \dot{g} \tag{3}$$

Hence,

$$\widehat{f} + \frac{\partial f}{\partial \boldsymbol{E}_p}\cdot\dot{\boldsymbol{E}}_p + \frac{\partial f}{\partial \kappa}\dot{\kappa} + \frac{\partial f}{\partial \boldsymbol{\alpha}}\cdot\dot{\boldsymbol{\alpha}} = \widehat{g} + \frac{\partial g}{\partial \boldsymbol{E}_p}\cdot\dot{\boldsymbol{E}}_p + \frac{\partial g}{\partial \kappa}\dot{\kappa} + \frac{\partial g}{\partial \boldsymbol{\alpha}}\cdot\dot{\boldsymbol{\alpha}} \tag{4}$$

where

$$\widehat{f} = \frac{\partial f}{\partial \boldsymbol{S}}\cdot\dot{\boldsymbol{S}}, \qquad \widehat{g} = \frac{\partial g}{\partial \boldsymbol{E}}\cdot\dot{\boldsymbol{E}} \tag{5}$$

In view of $(2)_1$ and $(5)_2$, $\widehat{g}$ can also be written as

$$\widehat{g} = \boldsymbol{C}^{\mathrm{T}}\left[\frac{\partial f}{\partial \boldsymbol{S}}\right]\cdot\dot{\boldsymbol{E}} = \frac{\partial f}{\partial \boldsymbol{S}}\cdot\boldsymbol{C}\big[\dot{\boldsymbol{E}}\big] \tag{6}$$

But, it is still the strain rate rather than the stress rate, that appears in (6).

In all of the formulations discussed below, despite differences in their characterization of "plastic loading", there is agreement on the following points (a)–(c):

(a) An "elastic state" is characterized by $f < 0$, and, equivalently by $g < 0$; an "elastic–plastic state" is characterized by $f = 0$, and equivalently by $g = 0$. No other states are attainable. In an elastic–plastic state, the quantity $\widehat{f}$ represents the inner product between the tangent to a stress path and the outward normal to the yield surface in stress space; likewise, $\widehat{g}$ represents the inner product between the tangent to a strain path and the outward normal to the yield surface in strain space.

(b) If $f < 0$ (or, equivalently $g < 0$), then

$$\dot{\boldsymbol{E}}_p = \boldsymbol{0}, \quad \dot{\kappa} = 0, \quad \dot{\boldsymbol{\alpha}} = \boldsymbol{0} \tag{7}$$

Hence, in this case, both the yield surface in stress space and the yield surface in strain space are stationary. Moreover, it follows from (3), (4), and $(7)_{1,2,3}$ that in this case

$$\widehat{f} = \widehat{g} = \dot{f} = \dot{g} \tag{8}$$

which can be positive, zero, or negative.

(c) Consistency condition: "Plastic loading" from an elastic–plastic state always leads to an elastic–plastic state. Consequently, during "plastic loading", it is necessary that

$$\dot{f} = 0 \tag{9}$$

By virtue of (3), the latter condition is equivalent to

$$\dot{g} = 0 \tag{10}$$

3. Loading criteria of the strain-space formulation

In the strain-space formulation of plasticity, due to Naghdi and Trapp [2], the following definitions occur (in addition to (a) above):

(NT 1) "unloading from an elastic–plastic state" is defined by the conditions

$$g = 0, \qquad \widehat{g} < 0 \tag{11}$$

(NT 2) "neutral loading from an elastic–plastic state" is defined by

$$g = 0, \qquad \widehat{g} = 0 \tag{12}$$

(NT 3) "loading from an elastic–plastic state" is defined by

$$g = 0, \qquad \widehat{g} > 0 \tag{13}$$

Geometrically, (11), (12), and (13) correspond respectively to the situations in which the tangent to a strain path makes an angle smaller than, equal to, or greater than 90° with the outward normal to the yield surface in strain space.

It is assumed that in cases (11) and (12), the rates of the variables Z vanish, as in $(7)_{1,2,3}$. It then follows from (3) and (4) that during "unloading from an elastic–plastic state",

$$\dot{f} = \widehat{f} = \dot{g} = \widehat{g} < 0 \tag{14}$$

whereas during "neutral loading from an elastic-plastic state",

$$\dot{f} = \widehat{f} = \dot{g} = \widehat{g} = 0 \tag{15}$$

It is further assumed that in case (13), the rates of the variables Z are linear in the rate of $\boldsymbol{E}$. The classical continuity argument of Prager then leads to relations of the form

$$\dot{\boldsymbol{E}}_p = \pi\widehat{g}\boldsymbol{\rho}, \quad \dot{\kappa} = \pi\widehat{g}\lambda, \quad \dot{\boldsymbol{\alpha}} = \pi\widehat{g}\boldsymbol{\beta} \tag{16}$$

where the constitutive functions $\boldsymbol{\rho}$, λ, $\boldsymbol{\beta}$ depend on the variables $(\boldsymbol{E}, Z)$, and π is a Lagrange multiplier that may depend on the variables $(\boldsymbol{E}, Z)$. The extra multiplier π is introduced only for the convenience it affords in satisfying a restriction that will be derived momentarily; a development without the extra multiplier is contained in [5, pp. 235–236].

The consequence (10) of the consistency condition (c), together with $(5)_2$, and $(16)_{1,2,3}$, leads to

$$\widehat{g}\left\{1 + \pi\left(\frac{\partial g}{\partial \boldsymbol{E}_p} \cdot \boldsymbol{\rho} + \frac{\partial g}{\partial \kappa}\lambda + \frac{\partial g}{\partial \boldsymbol{\alpha}} \cdot \boldsymbol{\beta}\right)\right\} = 0 \tag{17}$$

And, since by the definition NT 3 of "loading", $\widehat{g}$ is positive, it follows immediately from (17) that

$$1 + \pi\left(\frac{\partial g}{\partial \boldsymbol{E}_p} \cdot \boldsymbol{\rho} + \frac{\partial g}{\partial \kappa}\lambda + \frac{\partial g}{\partial \boldsymbol{\alpha}} \cdot \boldsymbol{\beta}\right) = 0 \tag{18}$$

It is therefore impossible for π to be zero, and without any loss in generality, we may take it to be positive. This choice is equivalent to the convention that $\boldsymbol{\rho}$, λ, $\boldsymbol{\beta}$ in $(16)_{1,2,3}$ have the same sense as the corresponding rates. Thus, in the strain-space formulation of plasticity one obtains

$$\frac{1}{\pi} = -\left(\frac{\partial g}{\partial \boldsymbol{E}_p} \cdot \boldsymbol{\rho} + \frac{\partial g}{\partial \kappa}\lambda + \frac{\partial g}{\partial \boldsymbol{\alpha}} \cdot \boldsymbol{\beta}\right) > 0 \tag{19}$$

With the help of $(2)_{2,3,4}$, (19) may be expressed in the equivalent form

$$\frac{1}{\pi} = -\left(\frac{\partial f}{\partial \boldsymbol{E}_p} \cdot \boldsymbol{\rho} + \frac{\partial f}{\partial \kappa}\lambda + \frac{\partial f}{\partial \boldsymbol{\alpha}} \cdot \boldsymbol{\beta}\right) - \frac{\partial f}{\partial \boldsymbol{S}} \cdot \left\{\frac{\partial \widehat{\boldsymbol{S}}}{\partial \boldsymbol{E}_p}[\boldsymbol{\rho}] + \frac{\partial \widehat{\boldsymbol{S}}}{\partial \kappa}\lambda + \frac{\partial \widehat{\boldsymbol{S}}}{\partial \boldsymbol{\alpha}} \cdot \boldsymbol{\beta}\right\} > 0 \tag{20}$$

If, instead of $\boldsymbol{\rho}$, λ, $\boldsymbol{\beta}$, another set of functions, $\boldsymbol{\rho}^*$, λ^*, $\boldsymbol{\beta}^*$ having the same sense are chosen, with a corresponding (positive) multiplier π^*, it is obvious that $\pi^*\boldsymbol{\rho}^* = \pi\boldsymbol{\rho}$, $\pi^*\lambda^* = \pi\lambda$, $\pi^*\boldsymbol{\beta}^* = \pi\boldsymbol{\beta}$, and the results (19) and (20) would hold for π^*, $\boldsymbol{\rho}^*$, λ^*, $\boldsymbol{\beta}^*$.

4. Conditions of Nguyen and Bui

Recognizing the inadequacy of the classical loading criteria, Nguyen and Bui [1] proposed an alternative set of conditions in stress space. Thus, instead of $(16)_{1,2,3}$, suppose in the present section that the rates of the variables Z are of the form

$$\dot{\boldsymbol{E}}_p = \gamma\boldsymbol{\rho}, \quad \dot{\kappa} = \gamma\lambda, \quad \dot{\boldsymbol{\alpha}} = \gamma\boldsymbol{\beta} \tag{21}$$

where $\gamma \geqslant 0$ is an undetermined multiplier having a dimension of $1/\text{time}$. Here, one does not pre-suppose that γ is linear in the strain-rate – this property will follow later. There is no loss in generality in selecting the functions $\boldsymbol{\rho}$, λ, $\boldsymbol{\beta}$ in $(21)_{1,2,3}$ to be the same as those in $(16)_{1,2,3}$. If $f < 0$ (and hence also $g < 0$), γ is assumed to vanish, and $(21)_{1,2,3}$ reduce to $(7)_{1,2,3}$.

Nguyen and Bui [1] define "elastic unloading" by the conditions

$$f = 0, \qquad \dot{f} < 0 \tag{22}$$

and they assume that γ then vanishes. The conditions $(22)_{1,2}$ are equivalent to the strain-space conditions

$$g = 0, \qquad \dot{g} < 0 \tag{23}$$

Further, with $\gamma = 0$, it follows from $(21)_{1,2,3}$, (3), (4), and $(22)_2$ that

$$\dot{f} = \widehat{f} = \widehat{g} = \dot{g} < 0 \tag{24}$$

Thus, "elastic unloading" in the sense of Nguyen and Bui [1] is equivalent to "unloading from an elastic–plastic state" in the sense of Naghdi and Trapp [2].

Nguyen and Bui [1] define "plastic loading" by the conditions

$$f = 0, \qquad \dot{f} = 0 \tag{25}$$

[It would be preferable here to regard $(25)_{1,2}$ as following from the consistency condition (c) of Section 2, and to define "plastic loading" by the condition $\gamma > 0$ (see (29) below, and also the development in Section 5).] From $(25)_{1,2}$, $(5)_1$, and $(21)_{1,2,3}$, it follows that

$$\widehat{f} + \gamma\left(\frac{\partial f}{\partial \boldsymbol{E}_p} \cdot \boldsymbol{\rho} + \frac{\partial f}{\partial \kappa}\lambda + \frac{\partial f}{\partial \boldsymbol{\alpha}} \cdot \boldsymbol{\beta}\right) = 0 \tag{26}$$

The conditions $(25)_{1,2}$ are equivalent to

$$g = 0, \qquad \dot{g} = 0 \tag{27}$$

Equations $(27)_{1,2}$, $(5)_2$, and $(21)_{1,2,3}$ imply that

$$\widehat{g} + \gamma\left(\frac{\partial g}{\partial \boldsymbol{E}_p} \cdot \boldsymbol{\rho} + \frac{\partial g}{\partial \kappa}\lambda + \frac{\partial g}{\partial \boldsymbol{\alpha}} \cdot \boldsymbol{\beta}\right) = 0 \tag{28}$$

It is important to note that at this stage of the development, the conditions that have been obtained in strain space have exactly the same form as those in stress space. Consequently, the loading conditions (NT 1,2,3) cannot possibly be deduced yet, because it is known that these are not equivalent to stress-space conditions having the same form as (11), (12), and (13) (see [3,17]). Moreover, one cannot solve for the undetermined multiplier γ without making a further assumption. What assumption should be made? It is known from special cases that the terms in parentheses in (26) can vanish (e.g., when f is independent of $\boldsymbol{E}_p$ and $\boldsymbol{\alpha}$, and λ is zero). Consequently, (26) cannot be used to solve for γ, in general. But, turning to (28), one recognizes the terms that have appeared previously in the inequality in (19) of the strain-space formulation. In the context of the development of Nguyen and Bui [1], let us therefore assume that the inequality in (19) holds. Equivalently, the inequality in (20) is satisfied. Upon appropriate specialization, the latter inequality reduces to the inequality (4) in [1]; see [6].

It follows immediately from (28) and the inequality in (19) that

$$\gamma = -\widehat{g} \Big/ \left(\frac{\partial g}{\partial \boldsymbol{E}_p} \cdot \boldsymbol{\rho} + \frac{\partial g}{\partial \kappa}\lambda + \frac{\partial g}{\partial \boldsymbol{\alpha}} \cdot \boldsymbol{\beta} \right) \tag{29}$$

If $\gamma = 0$, (28) implies that $\widehat{g} = 0$. We then have "neutral loading from an elastic–plastic state" in the sense of Naghdi and Trapp [2]. If $\gamma > 0$, it follows from the inequality in (19) and (29) that $\widehat{g} > 0$. We then have "loading from an elastic–plastic state" in the sense of Naghdi and Trapp [2]. In this case, it follows from (19) and (29) that

$$\gamma = \pi \widehat{g} \ (>0) \tag{30}$$

Thus, γ is linear in the strain rate and $(21)_{1,2,3}$ must take on the form $(16)_{1,2,3}$.

On the other hand, if the strain-space formulation is taken as primary, the inequality in (19) holds automatically, and (30) may be regarded as defining γ. The conditions $(13)_{1,2}$ then imply that $f = 0$ and $\gamma > 0$, and $(16)_{1,2,3}$ imply $(21)_{1,2,3}$.

Remark. – Note that (26), (30), and (19) furnish

$$\frac{\widehat{f}}{\widehat{g}} = \left(\frac{\partial f}{\partial \boldsymbol{E}_p} \cdot \boldsymbol{\rho} + \frac{\partial f}{\partial \kappa}\lambda + \frac{\partial f}{\partial \boldsymbol{\alpha}} \cdot \boldsymbol{\beta} \right) \Big/ \left(\frac{\partial g}{\partial \boldsymbol{E}_p} \cdot \boldsymbol{\rho} + \frac{\partial g}{\partial \kappa}\lambda + \frac{\partial g}{\partial \boldsymbol{\alpha}} \cdot \boldsymbol{\beta} \right) \tag{31}$$

Casey and Naghdi [3] employed the quotient $\widehat{f}/\widehat{g}$ to characterize hardening, softening, and perfectly plastic behavior, according as $\widehat{f}/\widehat{g}$ is positive, negative, or zero. While the yield surface in strain space is moving outwards, the yield surface in stress space moves outwards, inwards, or is stationary, according as hardening, softening, or perfectly plastic behavior is occurring. Nguyen and Bui [1] characterized strain hardening in terms of the sign of $-(\partial f/\partial \boldsymbol{E}_p \cdot \boldsymbol{\rho} + \partial f/\partial \kappa \lambda + \partial f/\partial \boldsymbol{\alpha} \cdot \boldsymbol{\beta})$. The characterizations of strain-hardening given by Casey and Naghdi [3] and Nguyen and Bui [1] are equivalent to one another.

5. Conditions of Kuhn–Tucker type

Motivated by the manner in which inequality constraints are treated in optimization theory, a number of authors write loading conditions in Kuhn–Tucker form [12–15]. Thus, consider the conditions

$$f \leqslant 0, \quad \gamma \geqslant 0, \quad f\gamma = 0 \tag{32}$$

and suppose that the rates of the variables Z are given by $(21)_{1,2,3}$. By virtue of (1), $(32)_{1,2,3}$ are equivalent to the following conditions, expressed in terms of the yield function g:

$$g \leqslant 0, \quad \gamma \geqslant 0, \quad g\gamma = 0 \tag{33}$$

If $f < 0$, it follows from $(32)_3$ that $\gamma = 0$ (equivalently, $g < 0$ and $(33)_3$ imply that $\gamma = 0$), and $(21)_{1,2,3}$ then reduce to $(7)_{1,2,3}$. If $\gamma > 0$, $(32)_3$ implies that $f = 0$ and $(33)_3$ implies that $g = 0$. Ortiz and Popov [12] and Simo and Ortiz [13] regard the condition "$\gamma > 0$" as characterizing "plastic flow". The consistency condition (c) is enforced, and hence when $\gamma > 0$, both (9) and (10) hold. Eqs. (26) and (28) follow immediately from (9), (10) and $(21)_{1,2,3}$, with γ now positive.

At this stage of the argument, just as in Section 4, the Kuhn–Tucker conditions have exactly the same form in stress space as they do in strain space. And, here again, we cannot solve for γ without a further assumption. If the inequality in (19) is assumed to hold, γ is then given by (29), and further, satisfies (30).

Thus, once the inequality in (19) is adopted, the conditions $(32)_3$ and $\gamma > 0$ are equivalent to the conditions $(13)_{1,2}$ of the strain-space formulation. In the context of infinitesimal plasticity, Simo and Hughes [15, Section 2.2] assume an equality which is an appropriately specialized form of the inequality in (20).

If both f (or g) and γ vanish, all three conditions in (32), and also in (33), are satisfied, and $(21)_{1,2,3}$ reduce to $(7)_{1,2,3}$. Furthermore, (8) holds. Two separate cases are included here, namely (NT 1) and (NT 2). Conversely, if (NT 1) holds, then $f = 0$, and further, $(7)_{1,2,3}$ and $(21)_{1,2,3}$ lead to $\gamma = 0$. Similarly, if (NT 2) holds, then $f = 0$ and $\gamma = 0$.

Remark. – Suppose that, in the strain-space formulation of Section 3, the relations $(21)_{1,2,3}$ were adopted instead of $(16)_{1,2,3}$. Then, during "loading from an elastic–plastic state", instead of (18), we would obtain

$$1 + \frac{\gamma}{\widehat{g}}\left(\frac{\partial g}{\partial \boldsymbol{E}_p} \cdot \boldsymbol{\rho} + \frac{\partial g}{\partial \kappa}\lambda + \frac{\partial g}{\partial \boldsymbol{\alpha}} \cdot \boldsymbol{\beta}\right) = 0 \tag{34}$$

Consequently, neither γ nor the terms in parentheses can vanish. Therefore, $\gamma > 0$ and the inequality in (19) must again hold. Also, now defining π through the equation in (19), from (35) we have $\gamma = \pi\widehat{g}$, and $(21)_{1,2,3}$ revert to the form $(16)_{1,2,3}$.

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