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Mathematical modeling of an array of underground waste containers

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Abstract We consider a mathematical model describing the behavior of an underground waste repository, once the containers start to leak. Due to the high contrast of the characteristic lengths, numerical simulations on a such model are unrealistic. After renormalization, a small parameter ε appears and the global model is obtained when ε tends to zero, by means of homogenization and boundary layers methods. The asymptotic model obtained could be used as a global repository model for large field numerical simulations. ***To cite this article: A. Bourgeat et al., C. R. Mecanique 330 (2002) 371–376.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

computational solid mechanics / underground waste repository / global model / homogenization / boundary layers

Modélisation mathématique d'un réseau souterrain de stockage de déchets

Résumé Nous considérons un modèle mathématique de stockage souterrain de déchets lors de la rupture des parois des conteneurs. A cause de la grande disparité entre les longueurs caractéristiques du modèle, la résolution numérique d'un tel modèle détaillé n'est pas envisageable. Aprés avoir dégagé un petit paramètre ε liant ces longueurs, on étudie le comportement du modèle renormalisé lorsque ε tend vers zéro. Par des techniques d'homogénéisation et de couche limite, on obtient un modèle asymptotique qui peut être ensuite utilisé comme un modèle global de stockage pour des simulations numériques à l'échelle d'une région. ***Pour citer cet article : A. Bourgeat et al., C. R. Mecanique 330 (2002) 371–376.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

mécanique des solides numérique / stokage souterrain de dechéts / modèle globale / homogénéisation / couches limites

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On considère le comportement d'un stockage souterrain de déchets dans le cas de la rupture des conteneurs. Il s'agit d'un réseau constitué d'un grand nombre de modules de stockage à l'intérieur d'une couche géologique de faible perméabilité (par exemple de l'argile) incluse entre deux couches plus grandes avec des perméabilités beaucoup plus élevées. La fuite du polluant dure un temps très court $]0, t_m[$ comparé à l'ordre de grandeur (plusieurs millions d'années) de la diffusion et de la convection de celui-ci dans les couches souterraines. Ici, pour simplifier, le stockage consiste en un ensemble de modules, reposant sur une surface Σ et on représente la fuite de polluant par un trou avec une densité de flux de polluant entrant dans le domaine. De plus, sans perte de généralité, on suppose le champ convectif $\mathbf{v}$ donné. En accord avec le cas test [1], la taille typique l d'un module est d'une centaine de mètres pour l'épaisseur, d'un kilomètre pour sa longueur et de cinq mètres pour sa hauteur. La distance entre deux modules est de l'ordre d'une centaine de mètres et la couche de faible perméabilité dans laquelle est inclus le dispositif a respectivement une hauteur h de cent cinquantes mètres et une longueur L de trois mille mètres. Considérant comme petit paramètre ε le rapport entre la largeur l d'un seul module et la longueur L d'une couche géologique, alors un module a une hauteur de ε^2, et est inclus dans une couche d'épaisseur ε. L'étude du comportement du modèle renormalisé, quand ε tend vers 0, donne un modèle asymptotique, via des méthodes d'homogénéisation et de couche limite.

On considère, pour cette note, une géométrie simplifiée en supposant $\Omega =]-1/2, 1/2[^n$ (voir Fig. 1). Les modules sont supposés rectangulaires : $M = \prod_{i=1}^{n-1}]-m_i, m_i[,\ 1/2 > m_i > 0$. On définit un module normalisé par $\mathcal{M}_\varepsilon = M \times]-\varepsilon, \varepsilon]$; $M = \prod_{i=1}^{n-1}]-m_i, m_i[,\ 1/2 > m_i > 0$. Par répétition périodique de $\mathcal{M}_\varepsilon$, on définit $B_\varepsilon = \bigcup_{\alpha \in J(\varepsilon)} \varepsilon \mathcal{M}_\varepsilon^\alpha$, où $\mathcal{M}_\varepsilon^\alpha = \alpha + \mathcal{M}_\varepsilon$ et $J(\varepsilon) = \{\alpha \in Z^{n-1};\ \varepsilon \mathcal{M}_\varepsilon^\alpha \cap \Omega \neq \emptyset\}$. Soit enfin $\Omega_\varepsilon^T = \{\Omega \backslash B_\varepsilon\} \times]0, T[$ et $\Gamma_\varepsilon^T = \partial B_\varepsilon \times]0, T[$. L'évolution de la concentration φ_ε dans Ω_ε^T est régie par l'équation de convection diffusion (2) avec des coefficients de diffusion variant suivant les couches, un terme source Φ au bord des modules Γ_ε^T (3) et des conditions aux limites (4) sur le bord du domaine $S = \partial\Omega = S_1 \cup S_2$. La solution φ_ε du problème (2)–(4), aprés prolongement à tout $\Omega \times]0, T[$ suivant des méthodes classiques [6] converge et on a :

THÉORÈME 0.1. – *Le prolongement φ_ε à tout $\Omega \times]0, T[$ de la solution du problème* (2)–(4) *converge vers φ qui est l'unique solution du problème* (5)–(7) : *$\varphi_\varepsilon \rightharpoonup \varphi$ faible* dans $L^\infty(0, T; L^2(\Omega))$.*

La limite faible ci-dessus décrit le comportement en temps long dans le cas où le flux Φ n'est pas trop important, mais si on veut connaitre plus précisemment le comportement de la concentration aux temps lointains, on a besoin d'un développement asymptotique plus précis car on s'attend à conserver des variations rapides de la concentration. On introduit alors dans une région proche des conteneurs, la variable rapide $y = x/\varepsilon$, et on utilise une méthode de raccordement asymptotique (cf. [4] par exemple). Pour cela, on scinde Ω en trois parties Ω_ε^+, G_ε et Ω_ε^- séparées par des interfaces Σ_ε^+ et Σ_ε^- comme décrit dans (9). Dans $\Omega_\varepsilon^\pm$ on approxime φ_ε par φ_ε^0 satisfaisant (5)–(6) et dans G_ε, on cherche un développement asymptotique de la forme (10). On a alors l'estimation d'erreur :

THÉORÈME 0.2. – *Il existe une constante $C > 0$ indépendante de ε, telle que, pour tout $d \geqslant \frac{3}{2}$:*

$$|\varphi_\varepsilon - H_\varepsilon|_{L^2(0,T;H^1(\widetilde{\Omega}_\varepsilon))} \leqslant C\big(\varepsilon \log(1/\varepsilon)\big)^{3/2} \tag{1}$$

où $H_\varepsilon(x, t)$ est le développement avec raccord (14) *et $\widetilde{\Omega}_\varepsilon = \Omega \backslash (\Sigma_\varepsilon^+ \cup \Sigma_\varepsilon^-)$.*

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1. Introduction

The goal of this paper is to give a mathematical model describing the global behavior of an underground waste repository, once the containers start to leak. The purpose of such a global model is to be used for the

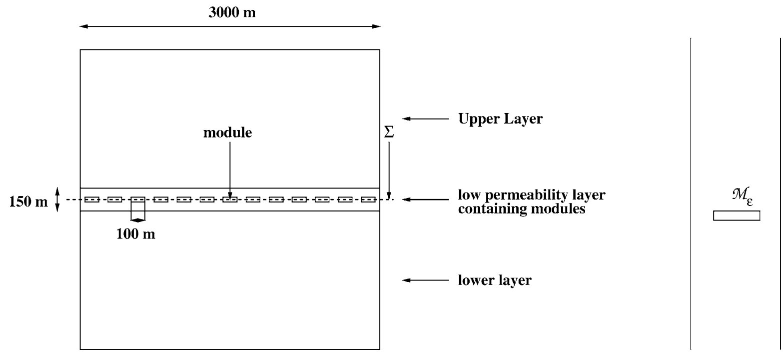

Figure 1. The soil model. Left: three layers of soil containing the modules; right: strip $\mathcal{G}_\varepsilon$.

Figure 1. *Le modèle. Gauche* : *les* 3 *couches géologiques* ; *droite* : *la bande* $\mathcal{G}_\varepsilon$.

full field simulations used in safety assessments. The physical situation can be described as an array made of high number of leaking modules inside a thin low permeable layer (e.g., clay), included between two bigger layers with higher permeability (e.g., limestone or marl). The pollutant is transported both by the convection produced by the water flowing slowly (creeping flow) through the rocks and by the diffusion coming from the dilution in the water. The leaking lasts over a period of time $]0, t_m[$, that is small compared to the millions of years over which convection and diffusion are active. In a real repository, there is a pressure drop producing the flow crossing a large number of disposal modules where each module includes several containers. Herein, for simplicity, the repository consists of a set of modules lying on a hypersurface Σ and we represent the leaking of a disposal module by a localized density source inside the domain or by a hole in the domain with a given flux on its boundary. Moreover, without lost of generality, we assume the convection velocity field to be given. According to the test case [1], the typical size of a module is a hundred of meters for the width, a kilometer for the length and five meters for the height. The distance between two modules is also of order 100 meters and the low permeable layer (the clay layer), in which the repository is embedded, has respectively a height and a length of order 150 and 3000 meters. Since there is a large number of modules, each of them with a small size compared to the layers size (see Fig. 1), direct numerical simulations of the full field, based on a *microscopic* model taking in account all the details, is unrealistic. The ratio between the width of a single module l and the layer length L, is of order $1/30$, and can be considered as a small parameter, ε, in the *microscopic* model. Now, the modules have a height of order ε^2, and are imbedded in a layer of thickness ε. The study of the renormalized model behavior, as ε tends to 0, by means of the homogenization method and boundary layers, gives an asymptotic model which could be used as a global repository model for numerical simulations.

We use similar methods to those applied for modelling the flow through a sieve, such as in [2,3] or [4]; detailed proofs are in [5].

2. Setting the problem

For the purpose of this note, assuming the above renormalization, we simplify the geometry by assuming that $\Omega =]-1/2, 1/2[^n$ (see Fig. 1) and by assuming that the thin modules have all the same shape. More precisely, we define a normalized module $\mathcal{M}_\varepsilon = M \times]-\varepsilon, \varepsilon]$; $M = \prod_{i=1}^{n-1}]-m_i, m_i[$, $1/2 > m_i > 0$. By periodic translation of $\mathcal{M}_\varepsilon$, we define $B_\varepsilon = \bigcup_{\alpha \in J(\varepsilon)} \varepsilon \mathcal{M}_\varepsilon^\alpha$, where $\mathcal{M}_\varepsilon^\alpha = \alpha + \mathcal{M}_\varepsilon$ and $J(\varepsilon) = \{\alpha \in Z^{n-1};\ \varepsilon \mathcal{M}_\varepsilon^\alpha \cap \Omega \neq \emptyset\}$. We also assume that the small parameter $\varepsilon \ll 1$, is such that $\varepsilon = 1/m$, $m \in \mathbf{N}$. Defining first the median hyperplane $\Sigma =]-1/2, 1/2[^{n-1} \times \{0\}$, we define the modules boundary $\Gamma_\varepsilon = \partial B_\varepsilon$ and $\Gamma_\varepsilon^T = \Gamma_\varepsilon \times]0, T[$. The α-th module boundary is denoted $\Gamma_\alpha^\varepsilon = \partial \mathcal{M}_\varepsilon^\alpha$. Finally, we denote $\Omega^T = \Omega \times]0, T[$, $\Omega_\varepsilon^T = \Omega_\varepsilon \times]0, T[$ where $\Omega_\varepsilon = \Omega \backslash B_\varepsilon$ is the porous media around the modulus.

Let $\Phi \in L^\infty([0, T])$ be the function describing the time behaviour of a module. In the real life situation, as mentioned before, it has a compact support $[0, t_m] \subset]0, T[$. Let $\lambda = (\log 2)/\tau > 0$, with τ being the half life time of the radioactive element, and let $\varphi_0 \in H^1(\Omega_\varepsilon)$ be the initial pollutant concentration in the soil.

The diffusion tensor $\mathbf{A} \in L^{\infty}(\mathbf{R}; \mathbf{R}^{n\times n})$ is a positive definite matrix function. Since the soil's layers have different properties, we assume that

$$\mathbf{A}(y_n) = \begin{cases} \mathbf{A}^1, & |y_n| < h, \\ \mathbf{A}^2, & |y_n| > h, \end{cases} \quad \mathbf{v}(x, y_n, t) = \begin{cases} \mathbf{v}^1(x,t), & |y_n| < h, \\ \mathbf{v}^2(x,t), & |y_n| > h, \end{cases} \quad \omega(y_n) = \begin{cases} \omega^1 & |y_n| < h \\ \omega^2 & |y_n| > h \end{cases}$$

Now we write the diffusion matrix in the form $\mathbf{A}^{\varepsilon}(x_n) = \mathbf{A}(x_n/\varepsilon)$ and the porosity as $\omega^{\varepsilon}(x_n) = \omega(x_n/\varepsilon)$. We assume that the convection velocity is $\mathbf{v}^{\varepsilon}(x,t) = \mathbf{v}(x, x_n/\varepsilon, t)$ is horizontal (i.e., $\mathbf{v}_n^1 = \mathbf{v}_n^2 = 0$) with $\mathrm{div}_{x'}\mathbf{v}^i = 0$, $i = 1, 2$.

With the above notations, the process is governed by the following convection–diffusion equation:

$$\omega^{\varepsilon}\frac{\partial \varphi_{\varepsilon}}{\partial t} - \mathrm{div}\big(\mathbf{A}^{\varepsilon}\nabla\varphi_{\varepsilon}\big) + \big(\mathbf{v}^{\varepsilon}\cdot\nabla\big)\varphi_{\varepsilon} + \lambda\omega^{\varepsilon}\varphi_{\varepsilon} = 0 \quad \text{in } \Omega_{\varepsilon}^{T} \tag{2}$$

$$\mathbf{n}\cdot\big(\mathbf{A}^{\varepsilon}\nabla\varphi_{\varepsilon} - \mathbf{v}^{\varepsilon}\varphi_{\varepsilon}\big) = \Phi(t) \quad \text{on } \Gamma_{\varepsilon}^{T}, \quad \varphi_{\varepsilon}(0,x) = \varphi_0(x), \quad x \in \Omega_{\varepsilon} \tag{3}$$

With some of typical boundary conditions on the exterior boundary $S = \partial\Omega = S_1 \cup S_2$, where S_i are disjoint and connected parts of S:

$$\varphi_{\varepsilon} = 0 \quad \text{on } S_1, \quad \mathbf{n}\cdot\big\{\mathbf{A}^{\varepsilon}\nabla\varphi_{\varepsilon} - \mathbf{v}^{\varepsilon}\varphi_{\varepsilon}\big\} = 0 \quad \text{on } S_2 \tag{4}$$

3. Weak convergence

Our solution φ_{ε} is defined on a variable domain Ω_{ε}^{T}. To use the weak convergence methods, we extend φ_{ε} to whole domain Ω^{T} preserving the a priori estimates. We extend φ_{ε}, by means of the classical method [6], and we denote that extension by the same symbol. Using the a priori estimates and the convergence $\int_{\Gamma_{\varepsilon}}\psi \to 2|M|\int_{\Sigma}\psi(x',0)\,\mathrm{d}x'$, $\forall\psi \in C_0(\Omega)$, we obtain the following theorem:

THEOREM 3.1. – *$\varphi_{\varepsilon} \rightharpoonup \varphi$ weak* in $L^{\infty}(0,T; L^2(\Omega))$, where φ is the unique solution of the problem*

$$\omega^2\frac{\partial\varphi}{\partial t} - \mathrm{div}\big(\mathbf{A}^2\nabla\varphi\big) + \big(\mathbf{v}^2\cdot\nabla\big)\varphi + \lambda\omega^2\varphi = 0 \quad \textit{in } \widetilde{\Omega}^{T} = (\Omega\backslash\Sigma)\times\,]0,T[\tag{5}$$

$$\varphi = 0 \quad \textit{on } S_1, \quad \mathbf{n}\cdot\big(\mathbf{A}^2\nabla\varphi - \mathbf{v}^2\varphi\big) = 0 \quad \textit{on } S_2, \quad \varphi(x,0) = \varphi_0(x), \quad x \in \widetilde{\Omega} = \Omega\backslash\Sigma \tag{6}$$

$$[\varphi] = 0, \quad \big[\mathbf{e}_n\cdot\big\{\mathbf{A}^2\nabla\varphi - \mathbf{v}^2\varphi\big\}\big] = -2\Phi|M| \quad \textit{on } \Sigma \tag{7}$$

$[w](x') = w(x',0+) - w(x',0-)$, $x' = (x_1,\ldots,x_{n-1})$ denotes the jump over Σ and $2|M|$ corresponds to the limit of the area $|\partial\mathcal{M}_{\varepsilon}|$ when $\varepsilon \to 0$.

4. Asymptotic expansion

In the sequel, we assume the summation from 1 to n over the repeating index. The above weak limit describes only the global long time mean behaviour of the process if the flux amplitude Φ is not too large. But, if either we need more accurate information on the oscillations in the vicinity of Σ, or the flux amplitude is not small, more precise asymptotics are needed.

To avoid cumbersome computations of lateral boundary layer (not significantly influencing the result), we simplify the boundary conditions. We impose the Dirichlet condition on the bottom S^-, Neumann condition on the top S^+ of the domain, $S^{\pm} = \,]-1/2, 1/2[^{n-1}\times\{\pm 1/2\}$ and periodicity condition on the lateral sides:

$$\varphi_{\varepsilon} = 0 \quad \text{on } S^-, \quad \mathbf{n}\cdot\big(\mathbf{A}^{\varepsilon}\nabla\varphi_{\varepsilon} + \mathbf{v}^{\varepsilon}\varphi_{\varepsilon}\big) = 0 \quad \text{on } S^+, \quad \varphi_{\varepsilon} \text{ is 1-periodic in } x' \tag{8}$$

We expect some fast changes of solution in vicinity of the containers. Thus we introduce in that region, the fast variable $y = x/\varepsilon$. Far from the sources we expect φ_{ε}, the solution of (2), (3) and (8), to behave almost like the weak limit φ, defined in (5)–(7) plus the additional conditions (8). That suggests the use of matched asymptotic expansions like in [4].

For this, we split the domain in three parts (Ω_ε^+, Ω_ε^-, G_ε) separated by Σ_ε^+, Σ_ε^-:

$$\Omega_\varepsilon^+ =]-1/2, 1/2[^{n-1} \times \big]d\varepsilon\log(1/\varepsilon), 1/2\big[, \quad \Omega_\varepsilon^- =]-1/2, 1/2[^{n-1} \times]-1/2, -d\varepsilon\log(1/\varepsilon)[$$
$$\Sigma_\varepsilon^\pm =]-1/2, 1/2[^{n-1} \times \big\{\pm d\varepsilon\log(1/\varepsilon)\big\}, \quad G_\varepsilon =]-1/2, 1/2[^{n-1} \times \big]-d\varepsilon\log(1/\varepsilon), d\varepsilon\log(1/\varepsilon)\big[$$
$$\widetilde{\Omega}_\varepsilon = \Omega \backslash \big(\Sigma_\varepsilon^+ \cup \Sigma_\varepsilon^-\big) \tag{9}$$

where the constant $d > 0$ will be determined in order to minimize the approximation error. In the outer domain $\Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$, we approximate φ_ε by φ_ε^0 (outer approximation) that mimics the behaviour of φ, i.e. satisfies Eq. (5) and boundary conditions (8). In the inner layer G_ε we approximate φ_ε by an asymptotic expansion (inner approximation), in the form:

$$\varphi_\varepsilon(x,t) \approx \varphi_\varepsilon^0(x,t) + \varepsilon\left[\chi_\varepsilon^k\left(\frac{x}{\varepsilon}\right)\frac{\partial\varphi_\varepsilon^0}{\partial x_k}(x,t) + w_\varepsilon\left(\frac{x}{\varepsilon}\right)\Phi(t)\right] + \cdots \tag{10}$$

The function φ_ε^0 now has two jumps instead of one. In fact, an accurate approximation of the real situation would have two jumps of the flux; one above and another one below the array of modules. Considering only the weak limit, like in the Theorem 3.1 above, leads to merging those two jumps into only one. Namely φ_ε^0 is defined by

$$\omega^\varepsilon\frac{\partial\varphi_\varepsilon^0}{\partial t} - \operatorname{div}\big(\mathbf{A}^\varepsilon\nabla\varphi_\varepsilon^0\big) + \big(\mathbf{v}^\varepsilon\cdot\nabla\big)\varphi_\varepsilon^0 + \lambda\omega^\varepsilon\varphi_\varepsilon^0 = 0 \quad \text{in } \widetilde{\Omega}_\varepsilon^T = \widetilde{\Omega}_\varepsilon \times]0,T[\tag{11}$$
$$\varphi_\varepsilon^0(x,0) = \varphi_0(x), \quad x \in \widetilde{\Omega}_\varepsilon, \quad \varphi_\varepsilon^0 = 0 \quad \text{on } S^+, \quad \mathbf{n}\cdot\big(\mathbf{A}^2\nabla\varphi_\varepsilon^0 + \mathbf{v}^2\varphi_\varepsilon^0\big) = 0 \quad \text{on } S^-$$
$$\big[\varphi_\varepsilon^0\big] = 0, \quad \big[\mathbf{e_n}\cdot\big(\mathbf{A}^2\nabla\varphi_\varepsilon^0 - \mathbf{v}\varphi_\varepsilon^0\big)\big] = -\frac{1}{2}\Phi|\partial\mathcal{M}\varepsilon| \quad \text{on } \Sigma_\varepsilon^\pm, \quad \varphi_\varepsilon^0 \text{ is 1-periodic in } x'$$

Functions $\chi_\varepsilon^k(y)$ $(k = 1,\dots,n)$ and $w_\varepsilon(y)$ in (10) are solutions of stationary diffusion type auxiliary problems posed in the infinite strip $\mathcal{G}_\varepsilon = \{y \in (]-1/2, 1/2[^{n-1}\times\mathbf{R})\backslash\mathcal{M}_\varepsilon\}$ (see Fig. 1). Denoting $y' = (y_1,\dots,y_{n-1})$, the first two auxiliary problems read:

$$\begin{cases} -\operatorname{div}\big(\mathbf{A}\nabla\chi_\varepsilon^k\big) = 0 & \text{in } \mathcal{G}_\varepsilon, \\ \mathbf{n}\cdot\mathbf{A}\nabla\big(\chi_\varepsilon^k + y_k\big) = 0 & \text{on } \partial\mathcal{M}_\varepsilon, \\ \chi_\varepsilon^k \text{ is 1-periodic in } y', \\ \lim\limits_{y_n\to\infty}\nabla\chi_\varepsilon^k = 0, \end{cases} \qquad \begin{cases} -\operatorname{div}(\mathbf{A}\nabla w_\varepsilon) = 0 & \text{in } \mathcal{G}_\varepsilon \\ \mathbf{n}\cdot\mathbf{A}\nabla w_\varepsilon = 1 & \text{on } \partial\mathcal{M}_\varepsilon \\ w_\varepsilon \text{ is 1-periodic in } y' \\ \lim\limits_{y_n\to\pm\infty}\mathbf{A}\nabla w_\varepsilon(y) = \mp\frac{1}{2}|\partial\mathcal{M}_\varepsilon|\mathbf{e}_n \end{cases} \tag{12}$$

Solvability of the two problems (12) is classical and the solutions exponentially decay as $|y_n| \to \infty$ (see, e.g., [7] or [8]). Due to the symmetry of the domain $\mathcal{G}_\varepsilon$, χ_ε^k is odd, i.e., $\chi_\varepsilon^k(y) = \chi_\varepsilon^k(-y)$. In general we have two stabilization constants $c_\pm^k$ at $\pm\infty$, but since χ_ε^k is odd, obviously $c_+^k = c_-^k \equiv c^k$, and because χ_ε^k is defined up to a constant we may chose $c^k = 0$. The second problem in (12) admits a unique solution, possessing the following asymptotic behaviour, for large $|y_n|$:

$$w_\varepsilon(y) \approx -\big(\mathbf{A}_{nn}^1\big)^{-1}|y_n|\frac{1}{2}|\partial\mathcal{M}_\varepsilon| + \text{exponentially decaying term} \tag{13}$$

To justify the expansion, we can prove the error estimate (see [5] for the proof).

THEOREM 4.1. – *Let*

$$H_\varepsilon(x) = \begin{cases} \varphi_\varepsilon^0(x), & \textit{in } \Omega_\varepsilon^\pm \\ \varphi_\varepsilon^0(x) + \varepsilon\left[\chi_\varepsilon^k\left(\frac{x}{\varepsilon}\right)\frac{\partial\varphi_\varepsilon^0}{\partial x_k}(x) + w_\varepsilon\left(\frac{x}{\varepsilon}\right)\Phi(t)\right] & \textit{in } G_\varepsilon \end{cases} \tag{14}$$

Then $|\varphi_\varepsilon - H_\varepsilon|_{L^2(0,T;H^1(\widetilde{\Omega}_\varepsilon))} \leqslant C(\varepsilon\log(1/\varepsilon))^{3/2}$. *Furthermore*

$$|\varphi_\varepsilon - \varphi|_{L^2(0,T;H^1(\Omega_\varepsilon))} \leqslant C\sqrt{\varepsilon\,\log(1/\varepsilon)}, \qquad \big|\varphi_\varepsilon - \varphi_\varepsilon^0\big|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leqslant C\big(\varepsilon\,\log(1/\varepsilon)\big)^{3/2} \tag{15}$$

5. Conclusion

Usually, the leaking occurs during very short time but has a very large amplitude. It has been observed by direct simulations that during this relatively short period of time, the spatial oscillations of the pollutant concentration are still appearing at the global scale. The expansion (10) clearly points out two different behaviours according the time range we are considering. During a relatively short time, the amplitude of Φ is large and the highly oscillating term $\varepsilon w_\varepsilon(x/\varepsilon)\Phi$ dominates the behaviour of the solution φ_ε (despite of ε multiplying it). Moreover, the typical diffusion coefficient **A** in a low permeable layer (clay) is small, compared to the one in the rest of the domain (limestone) and the diffusion is not able to smear out the oscillations at the beginning of the process. But, after a short period of time, Φ almost vanishes and the first term φ_ε^0 becomes the most important, leading the diffusion to overcome the oscillations.

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