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The elastic torus: anomalous stiffness of shells with mixed type

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Abstract We investigate the deformation of a thin elastic torus under axisymmetric surface loads. The strain concentrates near the top and bottom parallels and the inner and outer halves essentially undergo rigid-body translations in opposite directions. An analysis of the inner boundary layers is presented, which allows one to compute the effective stiffness of the torus for this loading. This stiffness is found anomalous compared to classical shells. These mechanical properties are interpreted using purely geometrical arguments. ***To cite this article: B. Audoly, Y. Pomeau, C. R. Mecanique 330 (2002) 425–432.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

solids and structures / elastic shells / boundary layers

Le tore élastique : raideur anormale des coques de type mixte

Résumé Nous étudions la déformation d'un tore élastique mince sous l'effet de forces de surface axisymétriques. La déformation est concentrée au voisinage des parallèles supérieur et inférieur et les moitiés externe et interne subissent essentiellement une translation uniforme l'une par rapport à l'autre. Une analyse de couche limite permet de calculer la rigidité effective du tore, qui suit une loi anormale. Toutes ces propriétés mécaniques sont interprétées par des arguments purement géométriques. ***Pour citer cet article : B. Audoly, Y. Pomeau, C. R. Mecanique 330 (2002) 425–432.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

solides et structures / coques élastiques / couches limite

In this Note, we investigate the deformation of a thin elastic torus with circular section under axisymmetric surface loads. Our results can be extended in a number of ways (to arbitrary sections, to arbitrary applied forces, and even to shells that have the topology of the torus but are not of revolution), as discussed below.

For thin elastic bodies, be they (curved) shells or (planar) plates, stretching and bending contribute to the elastic energy with different powers of the small (uniform) thickness h, namely h and h^3 respectively. As a result, very different mechanical behaviour is obtained depending on whether the middle surface of the shell admits nontrivial isometric deformations or not (note that the possibility of isometric deformations depends

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on boundary conditions imposed on the shell as well as on the geometry of the middle surface [1]). When such non trivial isometric deformations are present, the lowest energy state is reached by minimization of the bending energy among these isometric deformations, and penalization by the stretching energy is fully avoided. In the other case, the so-called inhibited case, the shell has no choice but to minimize the stretching energy, while the bending energy remains negligible. This is called the *membrane* problem (neglect of flexural effects). These different minimization problems lead to an effective mechanical stiffness k of the shell, defined as the ratio F/u of the typical applied forces F to the typical intensity of induced displacement field u, which scales as: $k \sim Eh$ in the inhibited case and $k \sim E\ h^3/L^2$ otherwise, where E is the Young's modulus of the shell, and L its typical (longitudinal) dimension (here, either diameter of the torus). Note that a toroidal shell is of mixed type (it contains both elliptical and hyperbolic regions), a class of shells for which little is known in general.

This general guideline can fail in some circumstances. If, for instance, the shell is inhibited and the boundary conditions are incompatible with the membrane formulation, a boundary layer will appear on the edge of the shell. This situation (very similar to that leading to viscous boundary layers near obstacles in high Reynolds number flows) has been dealt with in [2] and [3].

Another situation where the general theory of shells outlined above is not sufficient is when the applied forces are not sufficiently smooth for the minimization of the membrane energy to be a well-posed mathematical problem. This problem can be rather subtle, as even extremely regular force fields can turn out not to be smooth enough. This leads to boundary layers that are initiated where the applied forces is not smooth enough, and propagate along (or in some cases perpendicular to) asymptotic lines of the shell [4].

By studying the case of the torus, we shall point out a third type of boundary layer, different from the two types above: they are not due to edge effects, and their location is determined by geometrical properties of the surface rather than the point of application of the forces.

The present study was motivated by a remark in Gol'denveizer's monumental treatise [5] on the elasticity of shells. The membrane theory of shells does not yield a valid solution in the following simple situation: imagine a toroidal thin shell loaded with surface loads that are axisymmetric with a vanishing resulting force. The inner half, with negative Gaussian curvature, feels a force F directed along the axis of the torus, although the outer half, with positive Gaussian curvature, feels the opposite force $-F$. Membrane equations for thin shells fail to give a solution, because no shear can be transmitted through the two circles with zero Gaussian curvature, the top and bottom parallels (later referred to as "extreme parallels"). There is even a second, perhaps more subtle, difficulty why this theory fails, related to the absence of infinitesimal isometric deformations [6]. This will be discussed below.

Gol'denveizer's remark raises the question of the stiffness of the elastic torus. He himself indicated that the inclusion of flexural terms in the equation of shells should yield back a solution. Surprisingly, no one seems to have undertaken this program far beyond this point. The object of the present Note is to show that one can fully characterize the mechanical behaviour of an elastic torus, and in particular compute its stiffness, by considering flexural effects in two internal elastic boundary layers only, located along the extreme parallels (we emphasize that these layers are *internal* boundary layers). Compared to membrane (i.e., stretching) effects, flexural effects appear at the next order in the equations of thin shells when expanded in powers of the thickness h. Flexural effects are therefore formally small. However, they do *not* remain small in a small boundary layer: near these extreme parallels, flexural effects regularize a divergence found when solving the membrane equation. As usual in this type of problem, only part of the flexural effects are relevant for this regularization, that yields ultimately the "universal" (parameterless) fifth order equation (8) below.

1. Equations for membranes of revolution

In the membrane approximation, the equilibrium equations for a shell of revolution take the following form, after Fourier decomposition of all the elastic quantities:

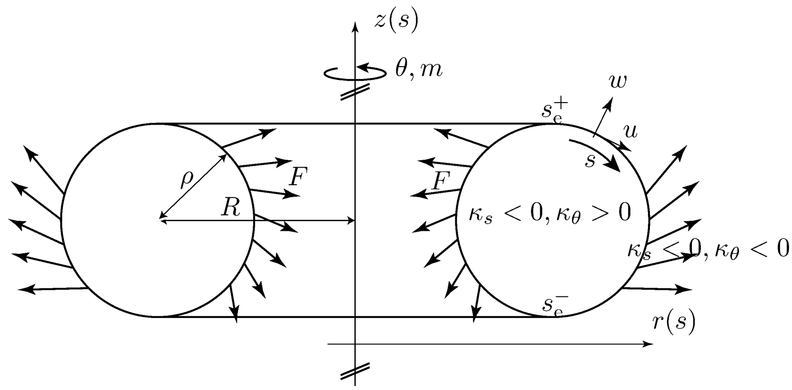

Figure 1. A toroidal shell loaded with axisymmetric surface loads.

$$\begin{cases} \dfrac{\mathrm{d}(r\sigma_{ss})}{\mathrm{d}s} + \imath m \sigma_{s\theta} - r'\sigma_{\theta\theta} + r F_s = 0 \\ \dfrac{\mathrm{d}(r\sigma_{s\theta})}{\mathrm{d}s} + \imath m \sigma_{\theta\theta} + r'\sigma_{s\theta} + r F_\theta = 0 \\ \kappa_s \sigma_{ss} + \kappa_\theta \sigma_{\theta\theta} + F_n = 0 \end{cases} \tag{1}$$

where $\imath = \sqrt{-1}$, m is the wavenumber in the θ direction (later m is set to zero), $\sigma_{ss}(s)$, $\sigma_{\theta\theta}(s)$ and $\sigma_{s\theta}(s)$ are the Fourier transformed components of the stress tensors, and the prime denotes derivation with respect to s, the curvilinear distance along the torus cross-section (see Fig. 1). The principal curvatures of the middle surface of the shell are given by: $\kappa_s = -z'r'' + z''r'$ and $\kappa_\theta = z'/r$, $(r(s), z(s))$ being a normal parameterization of the section. Bending forces of order h^3 have been neglected in these equations (membrane approximation). They can be restored by setting the surface force to $F = F^{\mathrm{f}} + F^{\mathrm{ext}}$ where F^{f} and F^{ext} respectively denote bending forces proportional to h^3 and true external forces. Instead of the complete expression of the bending forces, we shall need only one contribution, given in Eq. (7).

These equations of mechanical equilibrium are complemented by a constitutive law relating stresses to strains in terms of Young's modulus E and of the Poisson ratio ν, derived from Hooke's law:

$$\sigma_{ss} = \frac{2Eh}{1-\nu^2}(\varepsilon_{ss} + \nu\varepsilon_{\theta\theta}), \quad \sigma_{\theta\theta} = \frac{2Eh}{1-\nu^2}(\varepsilon_{\theta\theta} + \nu\varepsilon_{ss}), \quad \sigma_{s\theta} = \frac{Eh}{1+\nu}\varepsilon_{s\theta} \tag{2}$$

and by the definition of 2D strains in terms of displacements:

$$\varepsilon_{ss} = u' - \kappa_s w, \quad \varepsilon_{\theta\theta} = \frac{\imath m v}{r} + \frac{r'u}{r} - \kappa_\theta w, \quad \varepsilon_{s\theta} = \frac{\imath m u}{r} + r\frac{\mathrm{d}}{\mathrm{d}s}\frac{v}{r} \tag{3}$$

where $\{u(s), v(s), w(s)\}$ is the Fourier component with index m of the 3D displacement vector expressed in the local frame $\{e_s, e_\theta, e_n\}$. Substituting F^{ext} for F in (1) yields the equilibrium equations for shells in the membrane approximation, while (2) and (3) then yield the displacement field.

The principal curvature along meridians, κ_s, can be recognized as the 2D curvature of the meridian drawn in a plane. Because the section is assumed circular, κ_s never vanishes. The Gaussian curvature $K = \kappa_s\kappa_\theta$ does change its sign, however, because the surface is of mixed type (it is hyperbolic in the interior part of the torus, elliptic in the exterior part). As a result, the second principal curvature κ_θ vanishes along the extreme parallels. This vanishing has dramatic consequences for the mechanical behaviour of the torus, as we shall see. Along these extreme parallels $s = s_{\mathrm{e}}^{\pm}$, $r'(s_{\mathrm{e}}^{\pm}) = \pm 1$, $z'(s_{\mathrm{e}}^{\pm}) = 0$ and $r''(s_{\mathrm{e}}^{\pm}) = 0$, hence indeed $\kappa_\theta(s_{\mathrm{e}}^{\pm}) = 0$ by the formula above. This follows from a direct geometrical argument: the curvature vector of any parallel circles points to its center on the axis of revolution and is therefore tangent to the surface in the case of extreme parallels, while the principal curvatures are defined by projection of this curvature vector on the local *normal* e_n to the surface.

2. Case of axisymmetric forces ($m = 0$)

In this Note, we consider the simplest axisymmetric case ($m = 0$). Extensions are discussed at the end. When one specifies the equations above with $m = 0$, the force F_θ, the shear $\sigma_{s\theta}$ and the orthoradial displacement v satisfy two equations that are independent from any other elastic quantities. This problem is regular in the small thickness limit, and can be easily solved. It will not be further considered here, and we shall heretofore assume $F_\theta \equiv 0$.

Eliminating σ_{ss} and $\sigma_{s\theta}$ from the equilibrium equations, one obtains for $m = 0$:

$$\frac{\kappa_\theta}{\kappa_s}\sigma'_{\theta\theta}(s) + \left[\frac{2r'}{r} + \kappa_\theta\left(\frac{1}{\kappa_s}\right)'\right]\sigma_{\theta\theta}(s) = -\frac{r'}{r}\frac{F_n}{\kappa_s} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{F_n}{\kappa_s}\right) + F_s \tag{4}$$

Similarly, the geometric equations (3) can be rewritten in terms of the normal displacement, $w(s)$, only:

$$\kappa_\theta w'(s) - \frac{r''\kappa_\theta}{r'}w(s) = \frac{r'\varepsilon_{ss}}{r} - \varepsilon'_{\theta\theta} - \left(\frac{r'}{r} - \frac{r''}{r'}\right)\varepsilon_{\theta\theta} \tag{5}$$

Both equations become singular when κ_θ vanishes, i.e. along the extreme parallels. This prevents the determination of the mechanical state of the shell using the membrane equations, as noted by Gol'denveizer, unless F_s and F_n satisfy a nongeneric condition. Flügge moreover noticed that the geometric problem (3) itself is singular near the same extreme parallels in the presence of regular stresses [6]. This second source of singularity has to do with a fundamental result from the geometry of surfaces: the extreme parallels are rigidifying curves for the problem of infinitesimal isometric deformations of the torus. Due to purely geometrical constraints, these extreme parallels can only deform in a very particular way when the torus is forbidden to undergo any tangential stretching [7]. This contrasts with arbitrary curves drawn along the torus, which can deform freely under the same constraints. This geometrical property of the extreme parallels shows up as a singular point in Eqs. (3) or (5). The equations for infinitesimal isometric deformations (also called infinitesimal bendings) are indeed nothing but equations (3) for vanishing 2D strains (vanishing right-hand side).

3. Boundary layer analysis near the extreme parallels

In this section, bending effects are considered in the vicinity of the extreme parallels $s = s_\mathrm{e}^+$ or s_e^-. Equation (4) above is indeed singular when $\kappa_\theta(s) = 0$, i.e. for $s = s_\mathrm{e}^\pm$ (we remind that κ_s never vanishes). In the vicinity of these extreme parallels, one can find the asymptotic form of the stress $\sigma_{\theta\theta}$ predicted by the membrane theory (outer solution), as well as that of w as deduced from (5):

$$\sigma_{\theta\theta}(s) \propto \frac{1}{(s - s_\mathrm{e}^\pm)^2}, \qquad w(s) \propto \frac{1}{(s - s_\mathrm{e}^\pm)^3} \tag{6}$$

This is because the homogeneous part of Eq. (4) for $\sigma_{\theta\theta}$ takes the asymptotic form $r'/r[s\sigma'_{\theta\theta}(s) + 2\sigma_{\theta\theta}(s)] = 0$ for $s \sim s_\mathrm{e}^\pm$.

These divergences of the stress and displacement are not acceptable physically, and the solution given by Eq. (4) is only valid away from the extreme parallels. Near these parallels, the stress is regularized by mechanisms beyond the membrane approximation: flexural effects, or eventually nonlinear elastic response or large deformation effects. Here, we focus on flexural effects which will certainly be dominant over nonlinear effects for sufficiently small applied forces.

The flexural force will be regularizing because it contains higher order derivatives of the displacement compared to stretching forces, although with an additional factor h^2. As shown by inspection of the full shell equations, the flexural force takes a simple form in the boundary layer (other contributions are negligible):

$$F_n^{\mathrm{f}} = -\frac{2Eh^3}{3(1-\nu^2)} \frac{\mathrm{d}^4 w(s)}{\mathrm{d}s^4} \tag{7}$$

Here, one recognizes the flexural term present in the Föppl–von Kármán equations for plates. This is not surprising because the surface is almost flat near the extreme parallels.

In the boundary layer, the displacement w is related to 2D strains by (3) $\varepsilon_{\theta\theta}(s) = -\kappa_\theta w(s)$, while the latter are related to $\sigma_{\theta\theta}$ via the constitutive relations of the material: $\varepsilon'_{\theta\theta}(s) = -\kappa_\theta w'(s) = \sigma_{\theta\theta}(s)/2Eh$. As a result, the regularizing flexural force F_n^{f} above can be expressed in terms of $\sigma_{\theta\theta}$. When plugged into the equilibrium condition (4), this leads to the following boundary layer equation:

$$2f(x) + xf'(x) + \frac{\mathrm{d}^4}{\mathrm{d}x^4}\left(\frac{f'(x)}{x}\right) = 0 \tag{8}$$

for the *rescaled* stress

$$\sigma_{\theta\theta}(s) = \frac{f(x)}{\varepsilon^2} \quad \text{where } x = \frac{s}{\varepsilon},\ \varepsilon = \left(\frac{h}{|\kappa'_s(s_{\mathrm{e}}^{\pm})|}\right)^{1/3} \frac{1}{(3(1-\nu^2))^{1/6}} \tag{9}$$

The small quantity ε is the width of the boundary layer. It has indeed the dimension of a length. In the case of a torus,

$$\varepsilon = \frac{(hR\rho)^{1/3}}{(3(1-\nu^2))^{1/6}} \tag{10}$$

where R is the mean radius of the torus, while ρ is the radius of the section. For the inner boundary layer to be small compared to ρ, one should have $\rho \gg \sqrt{hR}$. This shows that the limit case of a cylinder ($R \to \infty$, ρ constant) is not described by the present theory – this is to be expected because the extreme parallels loose their special geometrical meaning in this limit.

We emphasize that this boundary layer equation is universal and free of any parameter. In particular, it does not depend on the aspect ratio ρ/R of the torus (as long as $\rho \gg \sqrt{hR}$). Its integration is outlined below.

WKB analysis of Eq. (8) yields the following asymptotic behaviour:

$$f(x) \approx \frac{\mathrm{e}^{\sqrt{2}|x|^{3/2}/3}}{x}\left(A^{\pm}\cos\frac{\sqrt{2}|x|^{3/2}}{3} + B^{\pm}\sin\frac{\sqrt{2}|x|^{3/2}}{3}\right) \quad \text{for } x \to \pm\infty \tag{11}$$

Such an exponential divergence, if present, cannot be matched with the outer solution (6), and must be discarded. This leads to four constraints $A^+ = B^+ = 0$ for $s \to +\infty$ and $A^- = B^- = 0$ for $s \to -\infty$, to be imposed on the solution of (8). Because Eq. (8) is fifth order, only *one* acceptable boundary layer solution $f(x)$ grows algebraically at plus and minus infinity, up to an arbitrary multiplicative constant.

It turns out that the generic solution of (8) can be expressed in terms of generalized hypergeometric functions, and the condition for algebraic growth at infinities: $A^{\pm} = B^{\pm} = 0$ can be worked out analytically. This leads to an explicit form of $f(x)$:

$$f(x) = \phi\left(-\frac{x^2}{6^{4/3}}\right) \quad \text{where } \phi(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(\frac{1}{6}+\frac{k}{3})\Gamma(\frac{1}{2}+\frac{k}{3})\Gamma(\frac{2}{3}+\frac{k}{3})\Gamma(1+\frac{k}{3})} \tag{12}$$

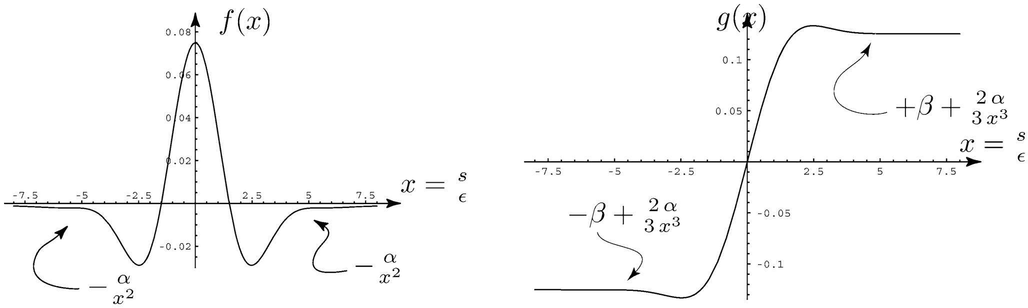

Figure 2. Integration of the boundary layer Eq. (8): plot of the rescaled stress component $f(x) = \varepsilon^2 \sigma_{\theta\theta}(x/\varepsilon)$ (left) and rescaled vertical displacement $g(x) \propto w(s)$. The stress described by $f(x)$ can be matched directly with the outer solution diverging in $1/s^2$ as in Eq. (6). Matching of the normal displacement $w(s)$ described by $g(x)$ requires a large rigid-body displacement of the outer half-torus with respect to the inner one (β is a numerical constant different from zero).

where Γ stands for the Gamma function. This function ϕ can eventually be expressed as sum and products of Bessel functions. The function $f(x)$ is plotted in Fig. 2 left.

The vertical displacement w in the boundary layer can be determined using the relation $\kappa_\theta w'(s) = -\sigma'_{\theta\theta}(s)/(2Eh)$ derived above, which is valid at first order in ε in the boundary layer. This leads to:

$$w'(s) = -\frac{1}{2Eh\kappa'_\theta(s_\mathrm{e}^\pm)\varepsilon^2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{f(s/\varepsilon)}{s - s_\mathrm{e}^\pm}\right) \tag{13}$$

hence by integration

$$w(s) = w_0^\pm + \frac{g(s/\varepsilon)}{2Eh\kappa'_\theta(s_\mathrm{e}^\pm)\varepsilon^3} \quad \text{where } g(x) = -\int_0^x \frac{f'(x')}{x'}\,\mathrm{d}x' \tag{14}$$

The new function $g(x)$ is a numerical function of order unity. The coefficient $1/\kappa'_\theta(s_\mathrm{e}^\pm)$ takes the value $\pm R\rho$ in the case of a torus with circular section.

The asymptotic form for $f(x)$ and $g(x)$ reads:

$$f(x) \approx -\frac{\alpha}{x^2} + \cdots \quad \text{and} \quad g(x) \approx \beta\,\mathrm{sgn}(x) + \frac{2\alpha}{3x^3} + \cdots \quad \text{for } x \to \pm\infty \tag{15}$$

where the constants have the numerical values $\alpha = 0.079$, $\beta = 0.125$. This allows one to derive the asymptotic form of the inner solution, up to an overall arbitrary factor determined later to match the outer solution:

$$\sigma_{\theta\theta}(s) = -\frac{\alpha}{s^2}, \quad w(s) = w_0^\pm + \frac{\beta\,\mathrm{sgn}(s - s_\mathrm{e}^\pm)}{\varepsilon^3}\frac{R\rho}{2Eh} + \frac{\alpha R\rho}{3Ehs^3} \tag{16}$$

4. Matching

It remains to match the outer solution given by the membrane theory in Eq. (6) with the above boundary layer solution (16).

In the presence of an integration constant $\beta \neq 0$ in $g(x)$, one cannot match directly the $1/s^3$ behaviour for w. This constant implies that there is a *step* of magnitude

$$\Delta = w\big((s_\mathrm{e}^\pm)^+\big) - w\big((s_\mathrm{e}^\pm)^-\big) = \frac{\beta R\rho}{Eh\varepsilon^3} \tag{17}$$

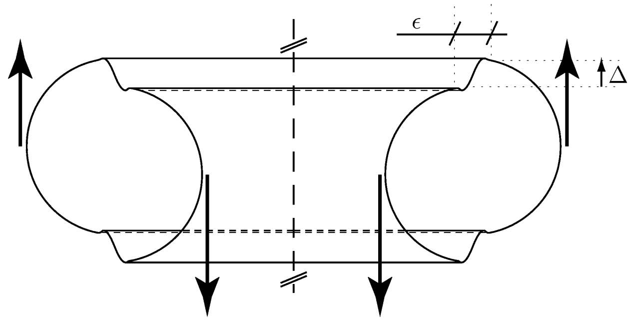

Figure 3. Deformation of an elastic torus under axisymmetric forces applied along the equators. Shear forces cannot be balanced by stretching of the middle surface near the top and bottom parallels but by flexural effects which are weaker. This leads to an elastic boundary layers of width $\varepsilon \propto h^{1/3}$ where the elastic energy concentrates. These boundary layer coincide with the 'rigifying' lines of the isometric deformations problem.

in the vertical displacement when one goes across the boundary layers. This large step (note the presence of ε^3 in the denominator) amounts to a rigid-body translation of the outer half-torus with respect to the inner one, caused by shear forces applied on the section (see Fig. 3).

Besides this constant β, the inner (boundary layer) solution and the outer (membrane) solutions can be matched without difficulty, as $\sigma_{\theta\theta}(s)$ varies as $1/s^2$ and $w(s)$ as $1/s^3$ in both the outer and inner expansions.

5. Explicit solution with forces applied at the equators

In order to show a 'practical' application of these equations, we explicitly solve the problem of a torus submitted to vertical forces applied along the equators, as in Fig. 3. This is described by a force field

$$F_s = F\left(\frac{\delta(s - s_\mathrm{o})}{2\pi(R+\rho)} + \frac{\delta(s - s_\mathrm{i})}{2\pi(R-\rho)}\right), \quad F_n = 0 \tag{18}$$

where F is the magnitude of the resultant force on either half torus, and s_o and s_i denote the abscissa of the outermost and innermost parallels (equators). Note that the two normals are in opposite directions $e_s(s_\mathrm{o}) = -e_s(s_\mathrm{i})$, hence the same sign in front of the delta contributions.

The strain $\sigma_{\theta\theta}(s)$ can now be obtained by integrating the membrane equations (4), and by using the results above to go across the internal boundary layers:

$$\sigma_{\theta\theta}(s) = \mp\frac{F}{4\pi}\frac{\kappa_s(s)}{z'(s)^2} \quad \text{for } \varepsilon \ll \left|s - s_\mathrm{e}^\pm\right| < \frac{\rho\pi}{2}, \qquad \sigma_{\theta\theta}(s) = \mp\frac{f(s/\varepsilon)}{4\pi\alpha}\frac{F\rho}{\varepsilon^2} \quad \text{for } s - s_\mathrm{e}^\pm = \mathcal{O}(\varepsilon) \tag{19}$$

where the expression on the left is the outer solution determined by matching the obvious solution of the homogeneous equation (4) with the singularity of the force at s_i and s_o. The equation on the right is the inner solution. $\sigma_{ss} = -\kappa_\theta\sigma_{\theta\theta}/\kappa_s$ can then be found by Eq. (1), while $\sigma_{s\theta}$ satisfies a (regular) autonomous equation not considered here.

In this expression, $\sigma_{\theta\theta}(s)$ is single valued thanks to the overall mechanical equilibrium of the torus. Besides this, the function w (not given here) could be made single valued thanks to an appropriate choice of the homogeneous solution in the equation for $\sigma_{\theta\theta}$. In particular, the vertical step Δ is identical across the upper and lower boundary layers. This leads to the picture in Fig. 3, where the outer part of the torus undergoes a rigid body translation of magnitude Δ at dominant order in ε, that is $1/\varepsilon^3$. This surprising result can be interpreted by noticing that the torus is very compliant near the extreme parallels: there its stiffness relies on bending rather than on stretching.

The offset of the outer torus with respect to the inner one, Δ, and the overall stiffness of the shell, k, defined as $k = F/\Delta$, are given by:

$$\Delta = \frac{F}{k}, \quad k = \frac{Eh^2}{\rho} \frac{4\pi\alpha}{\beta\sqrt{3(1-\nu^2)}}, \ \alpha = 0.079 \tag{20}$$

From this expression, the torus is seen to display a stiffness that is intermediate between the two well known types of classical shells (Eh for inhibited shells, and Eh^3 for isometrically deformable ones). Finally, one should note that a torus in vibration is similar to an acoustic resonator: the kinetic energy is stored in the outer region (Δ), while the elastic energy is stored in the inner boundary layers.

The domain of validity of the present solution is limited by nonlinear effects. This puts an upper bound on the force F: $F_{\max} \sim Eh^{7/3}R^{1/3}/\rho^{2/3}$. Far above $F_{\max}$, there is a range of forces F where the membrane theory is regularized by large displacements effects near the extreme parallels instead of flexion, while the strains remain small everywhere. This regime will be considered in future work [8].

In this Note, we have considered the deformation of a thin elastic torus under axisymmetric surface loads. We have shown that the deformations concentrate near two internal boundary layers located along the top and bottom parallels. The geometry considered here is the simplest one that exhibits such boundary layers, but these layers are expected to remain essentially unchanged when a number of generalization are performed: arbitrary (i.e. not circular) section, and even surfaces that are not of revolution but merely incorporate “rigidifying curves” in the sense of [7], and forces that have are not axisymmetric ($m \geqslant 1$).

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