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Crack front stability for a tunnel-crack propagating along its plane in mode $2+3$

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Note presented by Jean-Baptiste Leblond.

Abstract The authors recently studied stability of the straight configuration of the front of a tunnel-crack propagating along its plane in mixed mode $2+3$ conditions. Coincidence of the extrema of the supposedly sinusoidal crack front perturbation and the energy release rate was assumed, which imposed a certain phase difference between the perturbations of the two parts of the front. The aim of this paper is to relax this constraint. The system is time-dependent so that the stability issue is prone to problems of definition. It is notably shown that for some appropriate definition of stability, certain perturbations develop "unstably" if their wavelength is larger than some critical value depending on mean crack width, Poisson's ratio and mode mixity. ***To cite this article: V. Lazarus, J.-B. Leblond, C. R. Mecanique 330 (2002) 437–443.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

configurational stability / crack front / tunnel-crack / coplanar propagation / mixed mode

Stabilité du front d'une fissure en forme de fente infinie se propageant dans son plan en mode $2+3$

Résumé Les auteurs ont récemment étudié la stabilité de la forme rectiligne du front d'une fissure en forme de fente infinie se propageant dans son plan en conditions de mode mixte $2+3$. Les extrema de la perturbation du front, supposée sinusoïdale, et du taux de restitution d'énergie étaient supposés coïncider, ce qui imposait une certaine différence de phase entre les perturbations des deux parties du front. Le but de ce travail est de lever cette contrainte. Le système dépend du temps si bien que la question de la stabilité soulève des problèmes de définition. On montre notamment que pour une définition appropriée de la stabilité, certaines perturbations se développent de manière « instable » si leur longueur d'onde est supérieure à une valeur critique dépendant de la largeur moyenne de la fissure, du coefficient de Poisson et du rapport de mixité. ***Pour citer cet article : V. Lazarus, J.-B. Leblond, C. R. Mecanique 330 (2002) 437–443.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

stabilité configurationnelle / front de fissure / fente infinie / propagation coplanaire / mode mixte

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Considérons (Fig. 1) une fissure en forme de fente infinie de largeur $2a$ chargée en mode $2+3$ via des contraintes uniformes lointaines τ_p, τ_a. Perturbons, à l'intérieur du plan de fissure, les parties avant $(+)$ et arrière $(-)$ du front de quantités $\delta a(z^{\pm})$. Les variations des facteurs d'intensité de contraintes (FIC) sont alors données par (1, 2), où K_2 $(\equiv K_2^+ = K_2^-) = \tau_p\sqrt{\pi a}$ et $K_3^+ = -K_3^- = \tau_a\sqrt{\pi a}$ désignent les FIC avant perturbation, uniformes, et les $f_{\alpha\beta}$ et les $g_{\alpha\beta}$ des fonctions connues [1,2]. Pour une propagation coplanaire (cas d'une faille géologique) régie par le taux de restitution d'énergie local $\mathcal{G}(z^{\pm})$, on déduit de (1, 2) les variations $\delta\mathcal{G}(z^{\pm})$, ce qui permet de voir si une perturbation sinusoïdale va croître ou décroître lors de la propagation : problème de *stabilité configurationnelle* du front. Cette question a été examinée dans [1] en imposant à la différence de phase $\Delta\phi$ entre $\delta a(z^+)$ et $\delta a(z^-)$ une valeur assurant la coïncidence des extrema de $\delta a(z^+)$ et $\delta\mathcal{G}(z^+)$ d'une part, $\delta a(z^-)$ et $\delta\mathcal{G}(z^-)$ d'autre part. Elle est ici reprise en levant cette restriction.

La distance $a(z^{\pm},t)$ entre l'axe Oz et le point $(z^{\pm})$ du front est supposée de la forme (3) pour tout t, où $a(t)$, $\gamma^{\pm}(t)$ et k sont la demi-largeur moyenne de la fissure, des amplitudes complexes et un vecteur d'onde ; $\mathcal{G}(z^{\pm},t)$ est alors donné par (4)–(8), où $p \equiv p(t) \equiv ka(t) \equiv ka$ est un vecteur d'onde « réduit ». Pour une loi de propagation du type (9), adaptée à la propagation subcritique ou la fatigue, on en déduit l'équation différentielle $(10)_1$ ou $(10)_2$ sur le vecteur $\mathbf{\Gamma}$ des amplitudes complexes, équivalente aux équations (12) sur les modules $|\gamma^{\pm}|$ et les arguments $\phi^{\pm}$ de ces amplitudes.

La matrice $\mathbf{M}(p)/p$ dans (10) dépendant de $p \equiv p(t)$, le système dépend de t et la notion de stabilité est ambigüe. La « stabilité pour $t \to +\infty$ » ne peut signifier qu'une chose, à savoir que $\mathbf{\Gamma}$ demeure dans un voisinage borné de l'origine dans ces conditions, mais 2 définitions $(\mathcal{S}_1)$, $(\mathcal{S}_2)$ au moins de la « stabilité instantanée » sont possibles : voir (13). La condition de stabilité instantanée (14) au sens de $(\mathcal{S}_1)$ n'est satisfaite pour une perturbation symétrique $(\gamma^+ = \gamma^-)$ que si sa longueur d'onde $\lambda \equiv 2\pi/k$ est suffisamment petite, mais l'est toujours pour une perturbation antisymétrique $(\gamma^+ = -\gamma^-)$. Mais si on l'impose *pour toutes les perturbations de longueur d'onde donnée*, on obtient la condition (15), qui équivaut à $\lambda \leqslant \lambda_c \equiv 2\pi a/p_c$, où p_c est un nombre dépendant de ν et K_3^+/K_2 ; λ_c est ici la longueur d'onde « critique » de bifurcation pour laquelle il existe une configuration du front telle que $\delta\mathcal{G}(z^{\pm}) \equiv 0$ [1]. La conclusion qu'*il y a ainsi stabilité* (au sens ainsi défini) *pour les longueurs d'onde inférieures à la valeur critique de bifurcation* rejoint celles d'autres auteurs pour d'autres configurations, bien que ces dernières assurent la coïncidence automatique des extrema de $\delta a(z^{\pm})$ et $\delta\mathcal{G}(z^{\pm})$ contrairement à celle étudiée ici. Les conditions de stabilité instantanée (16) au sens de $(\mathcal{S}_2)$ sont équivalentes à (14) pour une perturbation symétrique ou antisymétrique ; mais elle ne sont jamais satisfaites pour *toutes* les perturbations de longueur d'onde donnée, étant toujours violées dans le cas où une seule partie du front est (initialement) perturbée $(\gamma^+ = 0$ ou $\gamma^- = 0)$. Enfin, il y a toujours stabilité pour $t \to +\infty$.

L'équation $(10)_2$ est finalement intégrée numériquement à titre d'exemple pour $K_3^+/K_2 = 1$, $\nu = 0{,}3$, $N = 3$, $\lambda/a(t=0) \equiv \lambda/a_0 = 1$; 7,40 ; 13,80, et les conditions initiales $\gamma^+ = \gamma_0 \in \mathbb{R}$, $\gamma^- = 0$: voir Figs. 2 et 3. Pour $\lambda/a_0 = 1$ et 7,40, il y a stabilité pour tout t : décroissance de $|\gamma^+|$ et faible croissance puis décroissance de $|\gamma^-|$; le déphasage $\Delta\phi \equiv \phi^- - \phi^+$ est grand. Pour $\lambda/a_0 = 13{,}80$, il y a d'abord instabilité : forte croissance de $|\gamma^+|$ et $|\gamma^-|$, puis stabilité pour $t \to +\infty$; le déphasage est moins important.

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1. Introduction

Let us consider (Fig. 1), within an infinite elastic isotropic body, a tunnel-crack of half-width a loaded in mixed mode $2+3$ conditions through remote plane (p) and antiplane (a) shear stresses τ_p, τ_a. The stress intensity factors (SIF) are then uniform along the fore $(+)$ and rear $(-)$ parts of the front and given, provided that the orientations of both parts are chosen as identical and the same as that of the Oz axis,

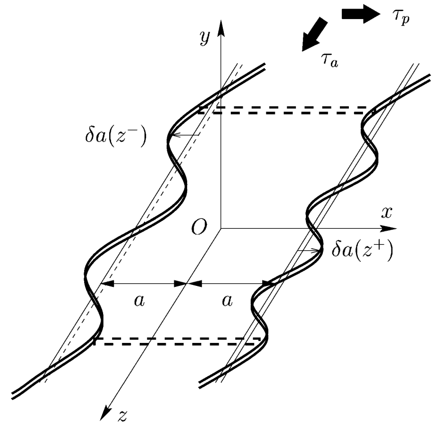

Figure 1. Tunnel-crack with a slightly perturbed front in mode $2+3$.

Figure 1. *Fissure en forme de fente infinie à front légèrement perturbé en mode* $2+3$.

by $K_2^+ = K_2^- \equiv K_2 = \tau_p\sqrt{\pi a}$, $K_3^+ = -K_3^- = \tau_a\sqrt{\pi a}$. Now slightly perturb the crack front, within the crack plane, by the amount $\delta a(z^\pm)$. Lazarus and Leblond [1,2], using Bueckner–Rice's theory of weight functions, have shown that the variations of the SIF are then given, to first order, by

$$\left\{\begin{aligned} \delta K_2(z^\pm) &= K_2\frac{\delta a(z^\pm)}{4a} - \frac{2}{2-\nu}K_3^\pm\frac{\mathrm{d}\delta a}{\mathrm{d}z}(z^\pm) \\ &\quad + PV\int_{-\infty}^{+\infty}\left[f_{22}\left(\frac{z'-z}{a}\right)K_2 + f_{23}\left(\frac{z'-z}{a}\right)K_3^\pm\right]\frac{\delta a(z'^\pm)-\delta a(z^\pm)}{(z'-z)^2}\,\mathrm{d}z' \\ &\quad + \int_{-\infty}^{+\infty}\left[g_{22}\left(\frac{z'-z}{a}\right)K_2 + g_{23}\left(\frac{z'-z}{a}\right)K_3^\mp\right]\frac{\delta a(z'^\mp)}{a^2}\,\mathrm{d}z' \end{aligned}\right. \tag{1}$$

$$\left\{\begin{aligned} \delta K_3(z^\pm) &= K_3^\pm\frac{\delta a(z^\pm)}{4a} + \frac{2(1-\nu)}{2-\nu}K_2\frac{\mathrm{d}\delta a}{\mathrm{d}z}(z^\pm) \\ &\quad + PV\int_{-\infty}^{+\infty}\left[-(1-\nu)f_{23}\left(\frac{z'-z}{a}\right)K_2 + f_{33}\left(\frac{z'-z}{a}\right)K_3^\pm\right]\frac{\delta a(z'^\pm)-\delta a(z^\pm)}{(z'-z)^2}\,\mathrm{d}z' \\ &\quad + \int_{-\infty}^{+\infty}\left[-(1-\nu)g_{23}\left(\frac{z'-z}{a}\right)K_2 + g_{33}\left(\frac{z'-z}{a}\right)K_3^\mp\right]\frac{\delta a(z'^\mp)}{a^2}\,\mathrm{d}z' \end{aligned}\right. \tag{2}$$

The $f_{\alpha\beta}$ and $g_{\alpha\beta}$ here denote known functions depending on Poisson's ratio ν.

Assume that in spite of mode mixity, propagation is coplanar; this is reasonable if it takes place along some very thin planar layer with low fracture toughness, such as a geological fault. The following stability problem arises: if the crack front is slightly perturbed, say sinusoidally, within the crack plane, will the perturbation decay or grow in time? Assume further that propagation is governed by the local energy release rate $\mathcal{G}(z^\pm)$ through some propagation law independent of mode combination; this is reasonable for coplanar propagation, since energy dissipation occurs through the same mechanisms (shear and friction) in modes 2 and 3. The problem can then be addressed by using Eqs. (1), (2) to compute the variations $\delta\mathcal{G}(z^\pm)$ on both parts of the front.

It turns out [1] that the extrema of $\delta a(z^+)$ and $\delta\mathcal{G}(z^+)$ do not coincide in general, and similarly for those of $\delta a(z^-)$ and $\delta\mathcal{G}(z^-)$. To simplify the discussion, Lazarus and Leblond enforced a certain value for the phase difference $\Delta\phi$ between the (sinusoidal) perturbations $\delta a(z^+)$ and $\delta a(z^-)$ ensuring such a coincidence. The issue was then easily addressed: stability or instability prevails according to whether the maxima of $\delta\mathcal{G}(z^\pm)$ correspond to the minima or maxima of $\delta a(z^\pm)$. The discussion was thus fully in line with those of Rice [3], Gao and Rice [4–6], Gao [7] and Lazarus and Leblond [8] for other crack configurations (semi-infinite crack, interior and exterior circular cracks, semi-infinite interface crack) for which coincidence of the extrema of $\delta a(z^\pm)$ and $\delta\mathcal{G}(z^\pm)$ was automatic. But *precisely because the tunnel-*

crack is the first configuration that has been discovered for which these extrema do not generally coincide, it appears quite desirable to then discuss the stability issue in full generality, without enforcing such a coincidence. Such is precisely the aim of this Note.

2. Evolution equations for the perturbations of the two parts of the front

It is assumed that at each instant, the distance $a(z^{\pm},t)$ between the middle axis Oz of the crack and the point $(z^{\pm})$ of the crack front is of the form

$$a\big(z^{\pm},t\big)=a(t)+\delta a\big(z^{\pm},t\big),\quad \delta a\big(z^{\pm},t\big)=\mathrm{Re}\big[\gamma^{\pm}(t)\,\mathrm{e}^{\mathrm{i}kz}\big],\quad \big|\gamma^{\pm}(t)\big|\ll a(t) \tag{3}$$

where $a(t)$ is the average half-width of the crack, $\gamma^{+}(t)$, $\gamma^{-}(t)$ complex amplitudes and k some *time-independent*, positive wavevector. Using Eqs. (1) and (2), one then gets the following expression of $\mathcal{G}(z^{\pm},t)$, where E is Young's modulus:

$$\mathcal{G}\big(z^{\pm},t\big)=\mathcal{G}(t)+\delta\mathcal{G}\big(z^{\pm},t\big) \tag{4}$$

$$\mathcal{G}(t)=\frac{1-\nu^2}{E}K_2^2(t)+\frac{1+\nu}{E}\big(K_3^+\big)^2(t),\quad K_2(t)=\tau_p\sqrt{\pi a(t)},\quad K_3^+(t)=\tau_a\sqrt{\pi a(t)} \tag{5}$$

$$\begin{cases}\delta\mathcal{G}\big(z^{\pm},t\big)=2\dfrac{1-\nu^2}{E}\dfrac{K_2^2(t)}{a(t)}\big\{\big[F\,\mathrm{Re}\,\gamma^{\pm}(t)+G\,\mathrm{Re}\,\gamma^{\mp}(t)\pm H\,\mathrm{Im}\,\gamma^{\mp}(t)\big]\cos kz\\ \quad+\big[-F\,\mathrm{Im}\,\gamma^{\pm}(t)-G\,\mathrm{Im}\,\gamma^{\mp}(t)\pm H\,\mathrm{Re}\,\gamma^{\mp}(t)\big]\sin\,kz\big\}\end{cases} \tag{6}$$

Quantities F, G, H here are given (disregarding the argument t for shortness) by the following expressions, where p is a "reduced", dimensionless wavevector:

$$F\equiv\bar{f}_{22}(p)+\frac{1}{1-\nu}\frac{(K_3^+)^2}{K_2^2}\bar{f}_{33}(p),\quad G\equiv\hat{g}_{22}(p)-\frac{1}{1-\nu}\frac{(K_3^+)^2}{K_2^2}\hat{g}_{33}(p),\quad H\equiv-2i\frac{K_3^+}{K_2}\hat{g}_{23}(p) \tag{7}$$

$$\begin{cases}\bar{f}_{\alpha\beta}(p)\equiv\dfrac14+2\displaystyle\int_0^{+\infty}f_{\alpha\beta}(u)\frac{\cos pu-1}{u^2}\,\mathrm{d}u,\quad(\alpha,\beta)=(2,2),(3,3)\\ \hat{g}_{\alpha\beta}(p)\equiv2\displaystyle\int_0^{+\infty}g_{\alpha\beta}(u)\cos pu\,\mathrm{d}u,\quad(\alpha,\beta)=(2,2),(3,3),\qquad p\equiv ka\\ \hat{g}_{23}(p)\equiv2i\displaystyle\int_0^{+\infty}g_{23}(u)\sin pu\,\mathrm{d}u\end{cases} \tag{8}$$

In order to now derive evolution equations for a, γ^{+}, γ^{-}, we introduce some propagation law in the form (typical for subcritical crack growth or fatigue)

$$\frac{\partial}{\partial t}a\big(z^{\pm},t\big)=V\big(\mathcal{G}\big(z^{\pm},t\big)\big),\quad V(\mathcal{G})\equiv C\mathcal{G}^N \tag{9}$$

where C and N are material constants. Inserting (4)–(6) into (9), expanding $V(\mathcal{G}(z^{\pm},t))$ to first order, using (3) and identifying terms independent of z, proportional to $\cos kz$ and $\sin kz$, one gets the expressions of $\mathrm{d}a/\mathrm{d}t$, $\mathrm{d}(\mathrm{Re}\,\gamma^{\pm})/\mathrm{d}t$, $\mathrm{d}(\mathrm{Im}\,\gamma^{\pm})/\mathrm{d}t$. Eliminating $\mathrm{d}t$ and recombining the equations, one finally obtains

$$\frac{\mathrm{d}\boldsymbol{\Gamma}}{\mathrm{d}a}=\frac{2N}{D}\frac{1}{a}\mathbf{M}(p)\cdot\boldsymbol{\Gamma}\quad\Longleftrightarrow\quad\frac{\mathrm{d}\boldsymbol{\Gamma}}{\mathrm{d}p}=\frac{2N}{D}\frac{1}{p}\mathbf{M}(p)\cdot\boldsymbol{\Gamma} \tag{10}$$

$$\boldsymbol{\Gamma}\equiv\begin{pmatrix}\gamma^+\\ \gamma^-\end{pmatrix},\quad \mathbf{M}(p)\equiv\begin{bmatrix}F & G-\mathrm{i}H\\ G+\mathrm{i}H & F\end{bmatrix},\quad D\equiv1+\frac{1}{1-\nu}\frac{(K_3^+)^2}{K_2^2} \tag{11}$$

Because the matrices $\mathbf{M}(p_1)$ and $\mathbf{M}(p_2)$, for $p_1 \neq p_2$, do not generally commute, it is impossible in general to provide an analytical solution to the differential equation $(10)_2$ in the form of an exponential of the integral of a matrix. Exceptions however occur for pure mode 2 or 3. Indeed $\mathbf{M}(p)/D$ then reduces to $\begin{bmatrix} \bar{f}_{22}(p) & \hat{g}_{22}(p) \\ \hat{g}_{22}(p) & \bar{f}_{22}(p) \end{bmatrix}$ or $\begin{bmatrix} \bar{f}_{33}(p) & -\hat{g}_{33}(p) \\ -\hat{g}_{33}(p) & \bar{f}_{33}(p) \end{bmatrix}$ so that $\mathbf{M}(p_1)/D$ and $\mathbf{M}(p_2)/D$ commute and Eq. $(10)_2$ admits an analytical solution analogous to that for mode 1 (see [9]).

Equivalent differential equations on the moduli $|\gamma^{\pm}|$ and arguments $\phi^{\pm}$ of the complex amplitudes can be obtained by setting $\gamma^{\pm} \equiv |\gamma^{\pm}|\mathrm{e}^{\mathrm{i}\phi^{\pm}}$ in $(10)_1$:

$$\begin{cases} \dfrac{\mathrm{d}|\gamma^{\pm}|}{\mathrm{d}a} = \dfrac{2N}{D}\dfrac{1}{a}\left[F\left|\gamma^{\pm}\right| + (G\cos\Delta\phi + H\sin\Delta\phi)\left|\gamma^{\mp}\right|\right], \\ \dfrac{\mathrm{d}\phi^{\pm}}{\mathrm{d}a} = \pm\dfrac{2N}{D}\dfrac{1}{a}\,(G\sin\Delta\phi - H\cos\Delta\phi)\dfrac{|\gamma^{\mp}|}{|\gamma^{\pm}|} \end{cases} \qquad \Delta\phi \equiv \phi^{-} - \phi^{+} \tag{12}$$

One notably recovers from there the condition on $\Delta\phi$ (not enforced here) ensuring coincidence of the extrema of $\delta a(z^{\pm})$ and $\delta\mathcal{G}(z^{\pm})$. Indeed this condition implies coincidence of the extrema of $\delta a(z^{\pm}, t)$ and $\delta a(z^{\pm}, t + \mathrm{d}t)$, i.e. $\mathrm{d}\phi^{\pm}/\mathrm{d}a = 0 \Leftrightarrow \tan\Delta\phi = H/G$ by Eq. $(12)_2$, which is precisely Lazarus and Leblond's [1] result.

3. The stability problem

Since the matrix $\mathbf{M}(p)/p$ in (10) depends on p ($\equiv ka$, $a \equiv a(t)$), the system is time-dependent and the notion of stability raises problems of definition. One may distinguish between 'instantaneous stability' and 'stability for t (or a, or p) $\to +\infty$'. Instantaneous stability means that the pair $(|\gamma^{+}|, |\gamma^{-}|)$ 'tends to come back to $(0,0)$ at time t', whereas stability for $t \to +\infty$ means that it remains in a finite neighbourhood of that point in such conditions. We mainly focus here on instantaneous stability (just like other authors quoted above), for which 2 possible definitions (among others) are:

$$\begin{cases} (\mathcal{S}_1) \quad \dfrac{\mathrm{d}\|\mathbf{\Gamma}\|}{\mathrm{d}a} \leqslant 0, \quad \|\mathbf{\Gamma}\| \equiv \left({}^t\overline{\mathbf{\Gamma}} \cdot \mathbf{\Gamma}\right)^{1/2} = \left(\left|\gamma^{+}\right|^2 + \left|\gamma^{-}\right|^2\right)^{1/2} \\ (\mathcal{S}_2) \quad \dfrac{\mathrm{d}|\gamma^{+}|}{\mathrm{d}a} \leqslant 0, \quad \dfrac{\mathrm{d}|\gamma^{-}|}{\mathrm{d}a} \leqslant 0 \end{cases} \tag{13}$$

3.1. Instantaneous stability in the sense of $(\mathcal{S}_1)$

Eq. $(10)_1$ immediately implies that $\mathrm{d}(\|\mathbf{\Gamma}\|^2)/\mathrm{d}a = (4N/Da)\,{}^t\overline{\mathbf{\Gamma}} \cdot \mathbf{M}(p) \cdot \mathbf{\Gamma}$. Therefore condition $(13)_1$ becomes

$${}^t\overline{\mathbf{\Gamma}} \cdot \mathbf{M}(p) \cdot \mathbf{\Gamma} = F\left(\left|\gamma^{+}\right|^2 + \left|\gamma^{-}\right|^2\right) + 2(G\cos\Delta\phi + H\sin\Delta\phi)\left|\gamma^{+}\right|\left|\gamma^{-}\right| \leqslant 0 \tag{14}$$

and depends on ν, K_3^{+}/K_2, a, k, $|\gamma^{\pm}|$ and $\Delta\phi$.

Two special cases are of particular interest:

- Symmetric perturbation: $|\gamma^{+}| = |\gamma^{-}|$, $\Delta\phi = 0$. Condition (14) then reduces to $F + G \leqslant 0$. Using Lazarus and Leblond's [2] results, it may be checked that this is true for $p \geqslant p_s \Leftrightarrow \lambda \equiv 2\pi/k = 2\pi a/p \leqslant \lambda_s \equiv 2\pi a/p_s$, where p_s is a dimensionless number depending only on ν and K_3^{+}/K_2. Thus symmetric perturbations of wavelength λ greater than λ_s grow unstably.
- Antisymmetric perturbation: $|\gamma^{+}| = |\gamma^{-}|$, $\Delta\phi = \pi$. Condition (14) reduces to $F - G \leqslant 0$, and Lazarus and Leblond's [2] results imply that this is always true. Thus antisymmetric perturbations always decay in a stable manner.

Also, some more stringent, appealing condition can be obtained by demanding instantaneous stability *for all possible perturbations of fixed, given wavelength*. This condition reads ${}^t\overline{\mathbf{\Gamma}} \cdot \mathbf{M}(p) \cdot \mathbf{\Gamma} \leqslant 0$, $\forall\mathbf{\Gamma}$, which means that the eigenvalues $F \pm \sqrt{G^2 + H^2}$ of the Hermitian matrix $\mathbf{M}(p)$ must be non-positive; this is equivalent to

$$F + \sqrt{G^2 + H^2} \leqslant 0 \tag{15}$$

This condition is equivalent to $p \geqslant p_c \Leftrightarrow \lambda = 2\pi a/p \leqslant \lambda_c \equiv 2\pi a/p_c$, where p_c is another number depending on ν and K_3^+/K_2 [1]; λ_c is in fact the 'critical', bifurcation wavelength, for which there is a perturbation of the front, corresponding to some suitable value of $\Delta\phi$, with a *zero* $\delta\mathcal{G}(z^{\pm})$ (uniform $\mathcal{G}$ along the front). The finding that *instantaneous stability* (in the sense just defined) *thus prevails for wavelengths smaller than the bifurcation wavelength* is fully in line with the conclusions of other authors quoted above, even though the configurations they envisaged imposed automatic coincidence of the extrema of $\delta a(z^{\pm})$ and $\delta\mathcal{G}(z^{\pm})$ contrarily to that studied here.

3.2. Instantaneous stability in the sense of $(\mathcal{S}_2)$

By Eq. $(12)_1$, conditions $(13)_2$ can be written

$$F\left|\gamma^{\pm}\right| + (G\cos\Delta\phi + H\sin\Delta\phi)\left|\gamma^{\mp}\right| \leqslant 0 \tag{16}$$

and again depend on ν, K_3^+/K_2, a, k, $|\gamma^{\pm}|$ and $\Delta\phi$.

For symmetric or antisymmetric perturbations, conditions (16) reduce to the same condition as for definition $(\mathcal{S}_1)$, and the same conclusions apply.

However, with definition $(\mathcal{S}_2)$, instantaneous stability versus all perturbations of given wavelength is never achieved, because it never occurs when, for instance, the sole fore part of the front is perturbed $(\gamma^- = 0)$. To see that, note that Eq. $(12)_2$ for $\mathrm{d}\phi^-/\mathrm{d}a$ then makes sense only if $G\sin\Delta\phi - H\cos\Delta\phi = 0$, since $|\gamma^-|$ appears as a denominator here; this implies that $\tan\Delta\phi = H/G \Rightarrow (\cos\Delta\phi, \sin\Delta\phi) = \pm(G,H)/\sqrt{G^2+H^2} \Rightarrow \mathrm{d}|\gamma^-|/\mathrm{d}a = \pm(2N/Da)\sqrt{G^2+H^2}|\gamma^+|$ by Eq. $(12)_1$. Now it follows from Lazarus and Leblond's [2] results that G and H never vanish simultaneously, and hence that $G^2 + H^2 \neq 0$. Therefore the sign – here is ruled out, because it would imply $\mathrm{d}|\gamma^-|/\mathrm{d}a < 0$ which is impossible with γ^- initially nil. Hence the sign is necessarily $+$, and this implies $\mathrm{d}|\gamma^-|/\mathrm{d}a > 0$ so that condition $(13)_2$ is violated, as announced. Incidentally, this reasoning has also shown that if γ^- is initially nil, the initial value of the phase ϕ^- (corresponding to the first increments of growth) is *not* arbitrary, but determined by $(\cos\Delta\phi, \sin\Delta\phi) = (G,H)/\sqrt{G^2+H^2}$.

3.3. Stability for $t \to +\infty$

For $t \to +\infty$, $p \to +\infty \Rightarrow p \gg p_c$ so that the eigenvalues of $\mathbf{M}(p)$ become smaller than $-\alpha$, $\alpha > 0$. Then ${}^t\overline{\mathbf{\Gamma}} \cdot \mathbf{M}(p) \cdot \mathbf{\Gamma} \leqslant -\alpha\|\mathbf{\Gamma}\|^2 \Rightarrow \mathrm{d}(\|\mathbf{\Gamma}\|^2)/\mathrm{d}p = (4N/Dp){}^t\overline{\mathbf{\Gamma}} \cdot \mathbf{M}(p) \cdot \mathbf{\Gamma} \leqslant -\beta\|\mathbf{\Gamma}\|^2/p$, $\beta \equiv 4N\alpha/D$. Thus $\mathrm{d}(\|\mathbf{\Gamma}\|^2 p^{\beta})/\mathrm{d}p \leqslant 0$, so that the function $\|\mathbf{\Gamma}\|^2 p^{\beta}$ is non-increasing, which implies that $\|\mathbf{\Gamma}\|^2 \leqslant Cst. \times p^{-\beta} \Rightarrow \lim_{p\to+\infty}\|\mathbf{\Gamma}\| = 0$. Thus *stability always prevails for* $t \to +\infty$. As can be seen, the key point here is that for $t \to +\infty$, $p \gg p_c$ or equivalently $\lambda \ll \lambda_c$ (because λ is constant whereas λ_c increases proportionally to a).

4. Numerical example

Since for pure mode 2 or 3, the differential equation $(10)_2$ is amenable to an analytic solution analogous to that for mode 1 studied by Leblond et al. [9], we only consider here a typical mixed-mode case with $K_3^+/K_2 = 1$. The material constants used are $\nu = 0.3$ and $N = 3$. The initial conditions are $\gamma^+ = \gamma_0 \in \mathbb{R}$, $\gamma^- = 0$: the sole fore part of the front is initially perturbed. Three wavelengths are envisaged, $\lambda = a_0$, $7.40a_0$ and $13.80a_0$ where $a_0 \equiv a(t=0)$; the second one is equal to the initial bifurcation wavelength $\lambda_{c0} \equiv 2\pi a_0/p_c$. The normalized moduli $|\gamma^{\pm}|/\gamma_0$ and arguments $\phi^{\pm}$ of the complex amplitudes, deduced

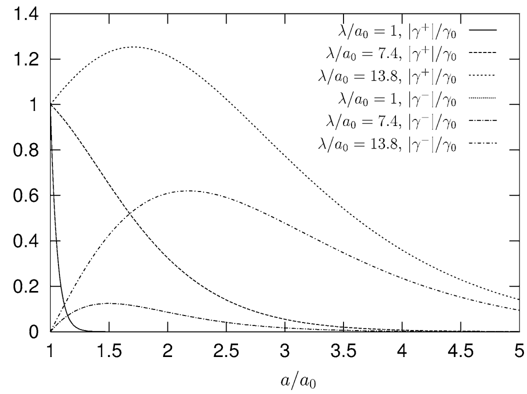

Figure 2. Normalized perturbation amplitudes versus normalized half-width of the crack.

Figure 2. *Amplitudes de perturbation normalisées en fonction de la demi-largeur normalisée de la fissure.*

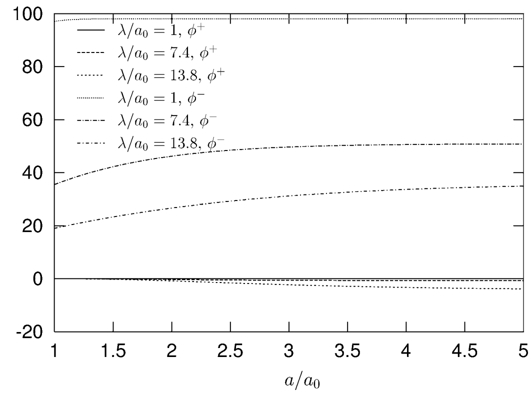

Figure 3. Perturbation phases (in degrees) versus normalized half-width of the crack.

Figure 3. *Phases de la perturbation* (*en degrés*) *en fonction de la demi-largeur normalisée de la fissure.*

from numerical integration of $(10)_2$, are shown versus the normalized half-width of the crack a/a_0 in Figs. 2 and 3.

With regard to amplitudes, for $\lambda = a_0$, since $\lambda < \lambda_{c0}$ and λ_c increases with a, λ remains forever smaller than λ_c, so that stability always prevails: $|\gamma^+|$ quickly decreases and $|\gamma^-|$ remains negligible. For $\lambda = 7.40a_0$, $|\gamma^+|$ still decreases but much less quickly and $|\gamma^-|$ slightly increases before decaying. (Note that although $\lambda = \lambda_{c0}$ in this case, the perturbation is *not* initially identical to the bifurcation mode which has $|\gamma^+| = |\gamma^-|$: see [1].) For $\lambda = 13.80a_0$, initial instability conspicuously appears through some notable increase of $|\gamma^+|$ and $|\gamma^-|$, but stability ultimately prevails, as anticipated. With regard to phases, ϕ^+ never significantly departs from its initial value of 0. In contrast, ϕ^- (the initial value of which can be noted to be perfectly well-defined even though γ^- is initially zero, as anticipated) takes large values strongly depending on λ; for $\lambda = a_0$, it is even greater than $\pi/2$: the configuration of the front is closer to an antisymmetric one than to a symmetric one (but $|\gamma^-|/\gamma_0$ then remains very small).

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