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Asymptotic analysis of nonlinear elastic plates

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Abstract We study in this Note the asymptotic analysis of nonlinear elastic plates with varying thickness. We suppose that the material moduli of the plates are anisotropic and nonhomogeneous, and the plates are submitted to body forces, to surfaces forces on the lower and upper faces and to pressure forces on the lateral boundary such that the displacements remains plane. ***To cite this article: D.A. Chacha, C. R. Mecanique 330 (2002) 581–586.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

solids and structures / plate / shallow shell / anisotropic / asymptotic analysis

Analyse asymptotique des plaques élastiques non linéaires

Résumé Nous étudions dans cette Note l'analyse asymptotique des plaques élastiques non linéaires d'épaisseur variable. Nous supposons que les constituants du matériau sont anisotropes et non homogènes. Les forces appliquées sont de type volumiques, surfaciques sur la face inférieure et supérieure ; sur le bord latéral on impose des forces de pression horizontales de sorte que les déplacements demeurent plans. ***Pour citer cet article : D.A. Chacha, C. R. Mecanique 330 (2002) 581–586.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Version française abrégée

On considère dans le cadre de l'élasticité non linéaire une plaque élastique tridimensionnelle non homogène et anisotrope dont l'épaisseur varie d'une façon non symétrique par rapport au plan moyen en général. On suppose que l'épaisseur dépend d'un petit paramètre ε qui sera destiné à tendre vers zéro. On s'intéresse dans ce travail à l'étude du comportement asymptotique des équations d'équilibre et des lois de comportement d'une telle plaque lorsque ε tend vers zéro, sachant que les forces appliquées sont de types : volumiques dans le domaine de la plaque, surfaciques sur les faces inférieure et supérieure ; sur le bord latéral on impose des forces de pressions tels que les déplacements soient plans. On utilise, d'une façon formelle, la méthode de développement asymptotique mixte, dite de Hellinger–Reissner, appliquée à la formulation variationnelle du problème posé dans un domaine de référence. En utilisant les mises à l'échelle (the scaling) des forces, des contraintes et du champ de déplacements en suivant Ciarlet et Destuynder [2], Ciarlet [1], ou on obtient au premier ordre significatif que le champ de déplacements est de type Kirchhoff–Love, ce qui est devenu classique pour la théorie des plaques en utilisant les méthodes

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asymptotiques, aussi la formulation variationnelle du problème bidimensionnel des plaques élastiques non linéaires, non homogènes et anisotropes d'épaisseur non constante. On obtient les cas particuliers suivants :

- Si l'épaisseur de la plaque est constante et la surface supérieure et inférieure sont planes et les forces volumiques et surfaciques sont verticales alors on obtient la formulation variationnelle du problème bidimensionnel des plaques élastiques, non linéaires, non homogènes et anisotropes de von Karman. Ainsi, on généralise le travail de [3,4] au cas des plaques de von Karman anisotropes et non homogènes.
- Si l'épaisseur de la plaque est constante, et les surfaces supérieure et inférieure sont courbées, et les forces volumiques et surfaciques sont verticales, alors on obtient la formulation variationnelle du problème bidimensionnel des coques élastiques peu profondes (shallow shell), non linéaires, non homogènes et anisotropes de Marguerre–von Karman. Ainsi, on généralise le travail de [5] au cas des coques peu profondes de Marguerre–von Karman anisotropes et non homogènes.

1. Setting of the problem

Before proceeding, we introduce some notations. We shall write $x = (x_1, x_2, x_3)$ for the current point in $\mathbb{R}^3$ space and $\overline{x} = (x_1, x_2)$ for that of $\mathbb{R}^2$ space. We shall underline and overline the vectors of physical entities of $\mathbb{R}^3$ and $\mathbb{R}^2$ space, respectively, using, for example, $\underline{u} = (u_1, u_2, u_3)$ and $\overline{u} = (u_1, u_2)$, respectively for displacement vector and $\underline{H}^1 = (H^1)^3$ and $\overline{H}^1 = (H^1)^2$ respectively, for Sobolev space, and double underline a second-order tensor like $\underline{\underline{\sigma}}$. Latin indices will usually range from 1 to 3 and Greek ones (except for ε) from 1 to 2. The convention of summation of repeated indices is applied. Moreover, the following symbols of differentiation will be used: $\partial_i^\varepsilon = \partial/\partial x_i^\varepsilon,\ \partial_i = \partial/\partial x_i,\ \partial_{ij}^2 = \partial^2/\partial x_i \partial x_j$.

Let ω is a bounded domain of $\mathbb{R}^2$ with smooth boundary γ. We define the functions:

- $\phi_\alpha : \overline{x} \in \overline{\omega} \to \phi_\alpha(\overline{x}) \in \mathbb{R}^3$ $(\alpha = 1, 2)$, two functions which will be serve to describe the lower and upper surfaces of the plates;
- $h : \overline{x} \in \overline{\omega} \to h(\overline{x}) \in \mathbb{R}_+^*$ with $h(\overline{x}) = \frac{1}{2}(\phi_2 - \phi_1)(\overline{x})$;
- $S : \overline{x} \in \overline{\omega} \to S(\overline{x}) \in \mathbb{R}^3$ with $S(\overline{x}) = \frac{1}{2}(\phi_1 + \phi_2)(\overline{x})$.

We suppose in the following that:

$$h(\overline{x}) \geqslant h_0 > 0, \quad \forall \overline{x} \in \overline{\omega}, \quad \phi_\alpha(\overline{x}) \in W^{2,\infty}(\omega), \quad \alpha = 1, 2 \tag{1}$$

Let ε be a small parameter $(0 \leqslant \varepsilon < 1)$ tending to zero, determining the lenght scale of the thickness. The three-dimensional geometry of the plate is defined by

$$\Omega^\varepsilon = \left\{x^\varepsilon \in \mathbb{R}^3;\ \overline{x}^\varepsilon = \left(x_1^\varepsilon, x_2^\varepsilon\right) \in \omega,\ \varepsilon\phi_1\left(\overline{x}^\varepsilon\right) < x_3^\varepsilon < \varepsilon\phi_2\left(\overline{x}^\varepsilon\right)\right\} \tag{2}$$

We shall denote by Γ_1^ε and Γ_2^ε the lower and the upper faces of the plate and by Γ_0^ε the lateral ones:

$$\Gamma_\alpha^\varepsilon = \left\{x^\varepsilon \in \mathbb{R}^3;\ \overline{x}^\varepsilon \in \omega,\ x_3^\varepsilon = \varepsilon\phi_\alpha\left(\overline{x}^\varepsilon\right)\right\} \quad (\alpha = 1, 2) \tag{3}$$

$$\Gamma_0^\varepsilon = \left\{x^\varepsilon \in \mathbb{R}^3;\ \overline{x}^\varepsilon \in \gamma,\ \varepsilon\phi_1\left(\overline{x}^\varepsilon\right) < x_3^\varepsilon < \varepsilon\phi_2\left(\overline{x}^\varepsilon\right)\right\} \tag{4}$$

Our goal in the sequel is the study of the asymptotic analysis of a nonlinearly elastic anisotropic non-homogeneous plates of variable thickness occupying the set $\overline{\Omega}^\varepsilon$. We suppose that the elastic moduli (a_{ijkl}^ε) of the plates satisfy the following conditions:

$$\begin{cases} a_{ijkl}^\varepsilon(x^\varepsilon) \in L^\infty(\Omega^\varepsilon) \\ a_{ijkl}^\varepsilon = a_{jikl}^\varepsilon = a_{klij}^\varepsilon = a_{klji}^\varepsilon \\ \exists c > 0, \quad a_{ijkl}^\varepsilon \tau_{kl}\tau_{ij} \geqslant c\tau_{ij}\tau_{ij}, \quad \forall \tau_{ij} = \tau_{ji} \end{cases} \tag{5}$$

The plate is subjected to three kinds of applied forces:

- volumic forces throughout Ω^ε, of density $\underline{f}^\varepsilon$;
- surface forces on the upper and lower faces $\Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon$, of density $\underline{g}^\varepsilon$;
- surface forces on the lateral face Γ_0^ε, for which only the resultant $\overline{p}^\varepsilon$ along the boundary γ of ω is know. Notice that the functions p_α^ε are defined only on γ. We suppose that the forces acting on the plate satisfy:

$$\underline{f}^\varepsilon \in \underline{L}^2(\Omega^\varepsilon), \quad \underline{g}^\varepsilon \in \underline{L}^2(\Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon), \quad \overline{p}^\varepsilon \in \overline{L}^2(\gamma) \tag{6}$$

The boundary conditions involving the displacement $\underline{u}^\varepsilon$ are

$$u_1^\varepsilon,\ u_2^\varepsilon,\ \text{independent of } x_3, \quad u_3^\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon \tag{7}$$

We can see [1] (p. 72) for more details on the boundary conditions (7).

The problem then consists in studying the asymptotic behavior ($\varepsilon \downarrow 0$) of the problem (P^ε):

$$(P^\varepsilon) \quad \begin{cases} -\partial_j^\varepsilon(\sigma_{ij}^\varepsilon + \sigma_{kj}^\varepsilon \partial_k^\varepsilon u_i^\varepsilon) = f_i^\varepsilon \quad \text{in } \Omega^\varepsilon \\ (\sigma_{ij}^\varepsilon + \sigma_{kj}^\varepsilon \partial_k^\varepsilon u_i^\varepsilon) n_j^\varepsilon = g_i^\varepsilon \quad \text{on } \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon \\ \frac{1}{\varepsilon h}\int_{\varepsilon\phi_1}^{\varepsilon\phi_2} (\sigma_{\alpha\beta}^\varepsilon + \sigma_{k\beta}^\varepsilon \partial_k^\varepsilon u_\alpha^\varepsilon)\nu_\beta^\varepsilon \,\mathrm{d}x_3^\varepsilon = p_\alpha^\varepsilon \quad \text{on } \gamma \\ u_\alpha^\varepsilon \text{ independent of } x_3^\varepsilon \text{ on } \Gamma_0^\varepsilon \\ u_3^\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon \end{cases} \tag{8}$$

where

$$\begin{cases} \sigma_{ij}^\varepsilon = a_{ijkl}^\varepsilon E_{kl}^\varepsilon(\underline{u}^\varepsilon) \text{ the stress tensor} \\ E_{ij}^\varepsilon(\underline{u}^\varepsilon) = \frac{1}{2}[\partial_i^\varepsilon u_j^\varepsilon + \partial_j^\varepsilon u_i^\varepsilon + \partial_i^\varepsilon u_m^\varepsilon \partial_j^\varepsilon u_m^\varepsilon] \text{ the nonlinear strain tensor} \end{cases} \tag{9}$$

$\underline{n}^\varepsilon$ is the exterior unit normal to $\Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon$ and $\overline{\nu}^\varepsilon$ is the exterior one to γ. This problem is similar to that proposed in [1] (p. 71) but for nonhomogeneous anisotropic plates with varying thickness.

The variational formulation of the problem (P^ε) is

$$\begin{cases} \text{Find } (\underline{u}^\varepsilon, \underline{\underline{\sigma}}^\varepsilon) \in V^\varepsilon \times \Sigma^\varepsilon \text{ such that} \\ \int_{\Omega^\varepsilon} (\sigma_{ij}^\varepsilon + \sigma_{kj}^\varepsilon \partial_k^\varepsilon u_i^\varepsilon)\partial_j^\varepsilon v_i^\varepsilon \,\mathrm{d}x^\varepsilon = \int_{\Omega^\varepsilon} \underline{f}^\varepsilon \underline{v}^\varepsilon \,\mathrm{d}x^\varepsilon + \int_{\Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon} \underline{g}^\varepsilon \underline{v}^\varepsilon \,\mathrm{d}\Gamma^\varepsilon + \frac{1}{2}\int_\gamma \left\{\int_{\varepsilon\phi_1}^{\varepsilon\phi_2} v_\alpha^\varepsilon \,\mathrm{d}x_3^\varepsilon\right\} p_\alpha^\varepsilon \,\mathrm{d}\gamma, \quad \forall \underline{v}^\varepsilon \in V^\varepsilon \\ \int_{\Omega^\varepsilon} (A\sigma^\varepsilon)_{ij}\tau_{ij}^\varepsilon \,\mathrm{d}x^\varepsilon - \int_{\Omega^\varepsilon} \tau_{ij}^\varepsilon e_{ij}^\varepsilon(\underline{u}^\varepsilon)\,\mathrm{d}x^\varepsilon - \frac{1}{2}\int_{\Omega^\varepsilon} \tau_{ij}^\varepsilon \partial_i^\varepsilon u_l^\varepsilon \partial_j^\varepsilon u_l^\varepsilon \,\mathrm{d}x^\varepsilon = 0, \quad \forall \underline{\underline{\tau}}^\varepsilon \in \Sigma^\varepsilon \end{cases} \tag{10}$$

where

$$\begin{cases} V^\varepsilon = \{\underline{v}^\varepsilon \in \underline{W}^{1,4}(\Omega^\varepsilon);\ v_3^\varepsilon|_{\Gamma_0^\varepsilon} = 0 \text{ and } v_\alpha^\varepsilon|_{\Gamma_0^\varepsilon} is \text{ independent of } x_3^\varepsilon\} \\ \Sigma^\varepsilon = \{\tau^\varepsilon = (\tau_{ij}^\varepsilon);\ \tau_{ij}^\varepsilon = \tau_{ji}^\varepsilon,\ \tau_{ij}^\varepsilon \in L^2(\Omega^\varepsilon)\} \end{cases} \tag{11}$$

The mapping $A : S^3 \to S^3$, S^3 is the space of symmetric tensors of order 3, is defined by: $(A\underline{\underline{X}})_{ij} = b_{ijkl}X_{kl}$, $\forall \underline{\underline{X}} = (X_{ij}) \in S^3$, the functions b_{ijkl} are the compliances. The mapping $A^{-1} : S^3 \to S^3$ is defined by: $(A^{-1}\underline{\underline{Y}})_{ij} = a_{ijkl}Y_{kl}$, $\forall \underline{\underline{Y}} = (Y_{ij}) \in S^3$, the functions a_{ijkl} are the elastic stifnesses. They enjoy the symmetry properties (5).

2. Variational formulation of (P^ε) on reference domain

We define a problem equivalent to problem (P^ε) but now posed over a domain which does not depend on ε. Accordingly, we let

$$\Omega = \omega \times]-1, +1[, \quad \Gamma_0 = \partial\omega \times]-1, +1[, \quad \Gamma_1 = \omega \times \{-1\}, \quad \Gamma_2 = \omega \times \{+1\} \tag{12}$$

We define a C^1-diffeomorphism F^ε:

$$F^\varepsilon : x = (x_1, x_2, x_3) \in \overline{\Omega} \to x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) \in \overline{\Omega}^\varepsilon, \quad \text{where } x_\alpha^\varepsilon = x_\alpha \text{ and } x_3^\varepsilon = \varepsilon(S + x_3 h)$$

With the spaces V^ε, Σ^ε of (11), we associate the spaces

$$\begin{cases} V=\{\underline{v}\in \underline{W}^{1,4}(\Omega),\ v_3|_{\Gamma_0}=0 \text{ and } v_\alpha|_{\Gamma_0} \text{ is independent of } x_3\} \\ \Sigma=\{\underline{\underline{\tau}}=(\tau_{ij});\ \tau_{ij}=\tau_{ji},\ \tau_{ij}\in L^2(\Omega)\} \end{cases} \tag{13}$$

We follow Ciarlet and Destuynder [2], and claim that the displacements, stresses, as well as the interior and exterior forces scale as follows:

$$\begin{cases} u_\alpha^\varepsilon(x^\varepsilon)=\varepsilon^2 u_\alpha(x;\varepsilon), \quad u_3^\varepsilon(x^\varepsilon)=\varepsilon u_3(x;\varepsilon) \\ v_\alpha^\varepsilon(x^\varepsilon)=\varepsilon^2 v_\alpha(x), \quad v_3^\varepsilon(x^\varepsilon)=\varepsilon v_3(x) \\ \sigma_{\alpha\beta}^\varepsilon(x^\varepsilon)=\varepsilon^2\sigma_{\alpha\beta}(x;\varepsilon), \quad \sigma_{\alpha 3}^\varepsilon(x^\varepsilon)=\varepsilon^3\sigma_{\alpha 3}(x;\varepsilon), \quad \sigma_{33}^\varepsilon(x^\varepsilon)=\varepsilon^4\sigma_{33}(x;\varepsilon) \\ \tau_{\alpha\beta}^\varepsilon(x^\varepsilon)=\varepsilon^2\tau_{\alpha\beta}(x), \quad \tau_{\alpha 3}^\varepsilon(x^\varepsilon)=\varepsilon^3\tau_{\alpha 3}(x), \quad \tau_{33}^\varepsilon(x^\varepsilon)=\varepsilon^4\tau_{33}(x) \end{cases} \tag{14}$$

$$\begin{cases} f_\alpha^\varepsilon(x^\varepsilon)=\varepsilon^2 f_\alpha(x), \quad f_3^\varepsilon(x^\varepsilon)=\varepsilon^3 f_\alpha(x) \\ g_\alpha^\varepsilon(x^\varepsilon)=\varepsilon^3 g_\alpha(x), \quad g_3^\varepsilon(x^\varepsilon)=\varepsilon^4 g_3(x) \\ p_\alpha^\varepsilon(\overline{x}^\varepsilon)=\varepsilon^2 p_\alpha(\overline{x}) \end{cases} \tag{15}$$

LEMMA 1. – *Let us put $\psi(x)=\varphi\circ F^\varepsilon(x)$ for all φ defined in Ω^ε. So we have*:

$$\int_{\Omega^\varepsilon}\varphi\,\mathrm{d}x^\varepsilon=\varepsilon\int_\Omega h\psi\,\mathrm{d}x, \quad \int_{\Gamma_1^\varepsilon\cup\Gamma_2^\varepsilon}\varphi\,\mathrm{d}\Gamma^\varepsilon=\int_{\Gamma_1\cup\Gamma_2}\Phi^\varepsilon\psi\,\mathrm{d}\Gamma, \quad \int_{\Gamma_0^\varepsilon}\varphi\,\mathrm{d}t^\varepsilon=\varepsilon\int_{\Gamma_0}h\psi\,\mathrm{d}t$$

where $\Phi^\varepsilon=\Phi_\alpha^\varepsilon=\sqrt{1+\varepsilon^2[(\partial_1\phi_\alpha)^2+(\partial_2\phi_\alpha)^2]}$ on Γ_α. Moreover, we have

$$\begin{cases} (\partial_\alpha\varphi)^\varepsilon=\partial_\alpha\psi-\dfrac{1}{h}\left[\partial_\alpha(S+x_3h)\right]\partial_3\psi \\ (\partial_3\varphi)^\varepsilon=\dfrac{1}{\varepsilon h}\partial_3\psi \end{cases}$$

PROPOSITION 2. – *Let $(\underline{u}(\varepsilon),\underline{\underline{\sigma}}(\varepsilon))\in V\times\Sigma$ construct from one solution $(\underline{u}^\varepsilon,\underline{\underline{\sigma}}^\varepsilon)\in V^\varepsilon\times\Sigma^\varepsilon$ of problem* (10) *via the scaling formulas* (14), (15). *Then $(\underline{u}(\varepsilon),\underline{\underline{\sigma}}(\varepsilon))$ solve the following problem*:

$$\begin{cases} \forall\underline{\underline{\tau}}\in\Sigma:\ a_0(\underline{\underline{\sigma}},\underline{\underline{\tau}})+\varepsilon a_1(\underline{\underline{\sigma}},\underline{\underline{\tau}})+\varepsilon^2 a_2(\underline{\underline{\sigma}},\underline{\underline{\tau}})+\varepsilon^3 a_3(\underline{\underline{\sigma}},\underline{\underline{\tau}})+\varepsilon^4 a_4(\underline{\underline{\sigma}},\underline{\underline{\tau}}) \\ \qquad +A(\underline{\underline{\tau}},\underline{u})+\dfrac{1}{2}B(\underline{\underline{\tau}},\underline{u},\underline{u})+\varepsilon^2\dfrac{1}{2}C(\underline{\underline{\tau}},\underline{u},\underline{u})=0 \end{cases} \tag{16}$$

$$\forall\underline{v}\in V: A(\underline{\underline{\sigma}},\underline{v})+B(\underline{\underline{\sigma}},\underline{u},\underline{v})+\varepsilon^2 C(\underline{\underline{\sigma}},\underline{u},\underline{v})=F(\underline{v}) \tag{17}$$

where $\forall\underline{\underline{\sigma}},\underline{\underline{\tau}}\in\Sigma$ and $\forall\underline{u},\underline{v}\in V$,

$$a_0(\underline{\underline{\sigma}},\underline{\underline{\tau}})=\int_\Omega h b_{\alpha\beta\gamma\delta}\sigma_{\gamma\delta}\tau_{\alpha\beta}\,\mathrm{d}x, \quad a_1(\underline{\underline{\sigma}},\underline{\underline{\tau}})=2\int_\Omega h[b_{\alpha\beta\gamma 3}\sigma_{\gamma 3}\tau_{\alpha\beta}+b_{\alpha 3\gamma\delta}\sigma_{\gamma\delta}\tau_{\alpha 3}]\,\mathrm{d}x \tag{18}$$

$$a_2(\underline{\underline{\sigma}},\underline{\underline{\tau}})=\int_\Omega h[b_{\alpha\beta 33}\sigma_{33}\tau_{\alpha\beta}+4b_{\alpha 3\gamma 3}\sigma_{\gamma 3}\tau_{\alpha 3}+b_{33\gamma\delta}\sigma_{\gamma\delta}\tau_{33}]\,\mathrm{d}x \tag{19}$$

$$a_3(\underline{\underline{\sigma}},\underline{\underline{\tau}})=2\int_\Omega h[b_{\alpha 333}\sigma_{33}\tau_{\alpha 3}+b_{33\gamma 3}\sigma_{\gamma 3}\tau_{33}]\,\mathrm{d}x, \quad a_4(\underline{\underline{\sigma}},\underline{\underline{\tau}})=\int_\Omega h b_{3333}\sigma_{33}\tau_{33}\,\mathrm{d}x \tag{20}$$

$$A(\underline{\underline{\tau}},\underline{u})=-\int_\Omega h\tau_{ij}H_{ij}(\underline{u}), \quad B(\underline{\underline{\tau}},\underline{u},\underline{v})=-\int_\Omega h\tau_{ij}I_{i3}(\underline{u})I_{j3}(\underline{v})\,\mathrm{d}x$$

$$C(\underline{\underline{\tau}},\underline{u},\underline{v})=-\int_\Omega h\tau_{ij}I_{i\lambda}(\underline{u})I_{j\lambda}(\underline{v})\,\mathrm{d}x \tag{21}$$

$$F(\underline{v})=-\int_\Omega h\underline{f}\,\underline{v}\,\mathrm{d}x-\int_{\Gamma_1\cup\Gamma_2}\Phi^\varepsilon\underline{g}\,\underline{v}\,\mathrm{d}\Gamma-\frac{1}{2}\int_{\Gamma_0}hp_\alpha v_\alpha\,\mathrm{d}t \tag{22}$$

$$H_{ij}(\underline{v}) = \begin{cases} H_{\alpha\beta}(\underline{v}) = e_{\alpha\beta}(\underline{v}) - \frac{1}{2h}[\Phi_\alpha \partial_3 v_\beta + \Phi_\beta \partial_3 v_\alpha] \\ H_{\alpha 3}(\underline{v}) = H_{3\alpha}(\underline{v}) = \frac{1}{2}[\frac{1}{h}\partial_3 v_\alpha + \partial_\alpha v_3 - \frac{\Phi_\alpha}{h}\partial_3 v_3] \\ H_{33}(\underline{v}) = \dfrac{1}{h}\partial_3 v_3 \end{cases} \tag{23}$$

$$I_{ij}(\underline{v}) = \begin{cases} I_{\alpha j}(\underline{v}) = \partial_\alpha v_j - \frac{1}{h}\Phi_\alpha \partial_3 v_j, \\ I_{3j}(\underline{v}) = \frac{1}{h}\partial_3 v_j, \end{cases} \qquad \Phi_\alpha = \partial_\alpha S + x_3 \partial_\alpha h \tag{24}$$

3. Asymptotic analysis

We postulate that the scaled displacement and stress $(\underline{u}(\varepsilon), \underline{\underline{\sigma}}(\varepsilon))$ can be written as

$$\big(\underline{u}(\varepsilon), \underline{\underline{\sigma}}(\varepsilon)\big) = \big(\underline{u}^0, \underline{\underline{\sigma}}^0\big) + \varepsilon\big(\underline{u}^1, \underline{\underline{\sigma}}^1\big) + \varepsilon^2\big(\underline{u}^2, \underline{\underline{\sigma}}^2\big) + \cdots \tag{25}$$

Substituting expansion (25) into (16), (17) and equating the terms of the same order with respect to ε. We obtain at order ε^0, ε^1 and ε^2 respectively:

$$\big(P^0\big) \quad \begin{cases} \forall \underline{\underline{\tau}} \in \Sigma : a_0(\underline{\underline{\sigma}}^0, \underline{\underline{\tau}}) + A(\underline{\underline{\tau}}, \underline{u}^0) + \frac{1}{2}B(\underline{\underline{\tau}}, \underline{u}^0, \underline{u}^0) = 0 \\ \forall \underline{v} \in V : A(\underline{\underline{\sigma}}^0, \underline{v}) + \frac{1}{2}B(\underline{\underline{\sigma}}^0, \underline{u}^0, \underline{v}) = F(\underline{v}) \end{cases}$$

$$\big(P^1\big) \quad \begin{cases} \forall \underline{\underline{\tau}} \in \Sigma : a_0(\underline{\underline{\sigma}}^1, \underline{\underline{\tau}}) + a_1(\underline{\underline{\sigma}}^0, \underline{\underline{\tau}}) + A(\underline{\underline{\tau}}, \underline{u}^1) + \frac{1}{2}B(\underline{\underline{\tau}}, \underline{u}^0, \underline{u}^1) + \frac{1}{2}B(\underline{\underline{\tau}}, \underline{u}^1, \underline{u}^0) = 0 \\ \forall \underline{v} \in V : A(\underline{\underline{\sigma}}^1, \underline{v}) + B(\underline{\underline{\sigma}}^1, \underline{u}^0, \underline{v}) + B(\underline{\underline{\sigma}}^0, \underline{u}^1, \underline{v}) = 0 \end{cases}$$

$$\big(P^2\big) \quad \begin{cases} \forall \underline{\underline{\tau}} \in \Sigma : a_0(\underline{\underline{\sigma}}^2, \underline{\underline{\tau}}) + a_1(\underline{\underline{\sigma}}^1, \underline{\underline{\tau}}) + a_2(\underline{\underline{\sigma}}^0, \underline{\underline{\tau}}) \\ \quad + A(\underline{\underline{\tau}}, \underline{u}^2) + \frac{1}{2}B(\underline{\underline{\tau}}, \underline{u}^1, \underline{u}^1) + \frac{1}{2}B(\underline{\underline{\tau}}, \underline{u}^0, \underline{u}^2) + \frac{1}{2}B(\underline{\underline{\tau}}, \underline{u}^2, \underline{u}^0) + \frac{1}{2}C(\underline{\underline{\tau}}, \underline{u}^0, \underline{u}^0) = 0 \\ \forall \underline{v} \in V : A(\underline{\underline{\sigma}}^2, \underline{v}) + B(\underline{\underline{\sigma}}^2, \underline{u}^0, \underline{v}) + B(\underline{\underline{\sigma}}^1, \underline{u}^1, \underline{v}) + C(\underline{\underline{\sigma}}^0, \underline{u}^0, \underline{v}) = 0 \end{cases}$$

PROPOSITION 3. – *Assume that* $(\underline{u}^0, \underline{\underline{\sigma}}^0)$ *solution of problem* (P^0) *belonging to* $V \times \Sigma$ *and* $\partial_3 u_3^0 \in C^0(\overline{\Omega})$. *Then* $\underline{u}^0$ *is a (scaled) Kirchhoff–Love displacement field, i.e., there exist function* $\underline{\xi}$ *defined on the middle surface* ω *of the reference plate such that*:

$$\begin{aligned} u_3^0(x_1, x_2, x_3) &= \xi_3(x_1, x_2) \quad \forall x \in \overline{\Omega} \\ u_\alpha^0(x_1, x_2, x_3) &= \xi_\alpha(x_1, x_2) - x_3 h \partial_\alpha \xi_3(x_1, x_2) \quad \forall x \in \overline{\Omega} \end{aligned} \tag{26}$$

The function $\underline{\xi}$ is solution of the variational problem

$$\text{Find } \underline{\xi} \in \mathbf{H} = \overline{H}^1(\omega) \times H_0^2(\omega) \text{ such that } c(\underline{\xi}, \underline{\zeta}) = F(\underline{\zeta}),\ \forall \underline{\zeta} \in \mathbf{H}, \text{ where}$$

$$c(\underline{\xi}, \underline{\zeta}) = \int_\omega h C_{\alpha\beta\gamma\delta}^{\mu\nu}(\overline{x}) \Pi_{\gamma\delta}^{\nu}(\underline{\xi}) \Pi_{\alpha\beta}^{\mu}(\underline{\zeta})\, \mathrm{d}\overline{x} + \int_\omega h C_{\alpha\beta\gamma\delta}^{1\nu}(\overline{x}) \Pi_{\gamma\delta}^{\nu}(\underline{\xi}) \partial_\alpha \xi_3 \partial_\beta \zeta_3\, \mathrm{d}\overline{x}$$

$$F(\underline{\zeta}) = \int_\omega h \left(\int_{-1}^{+1} \underline{f}\, \mathrm{d}x_3 \right) \underline{\zeta}\, \mathrm{d}\overline{x} - \int_\omega h^2 \left(\int_{-1}^{+1} x_3 f_\alpha\, \mathrm{d}x_3 \right) \partial_\alpha \zeta_3\, \mathrm{d}\overline{x}$$
$$+ \int_\omega \big(\underline{g}^1 + \underline{g}^2\big) \underline{\zeta}\, \mathrm{d}\overline{x} + \int_\omega h \big(g_\alpha^1 - g_\alpha^2\big) \partial_\alpha \zeta_3\, \mathrm{d}\overline{x} + \int_\gamma h p_\alpha \zeta_\alpha\, \mathrm{d}\gamma$$

$$C_{\alpha\beta\gamma\delta}^{\mu\nu}(\overline{x}) = \int_{-1}^{+1} t^{\mu+\nu-2} c_{\alpha\beta\gamma\delta}(x_1, x_2, t)\, \mathrm{d}t$$

$$c_{\alpha\beta\gamma\delta}(x) = a_{\alpha\beta\gamma\delta}(x) - a_{\alpha\beta i 3}(x) d_{ij}(x) a_{j3\gamma\delta}(x)$$

where $(c_{\alpha\beta\gamma\delta})$ is the inverse of $(b_{\alpha\beta\gamma\delta})$ and $d = (d_{ij})$ is the inverse of (a_{i3j3}) matrix.

$$\Pi^1_{\alpha\beta}(\underline{\eta}) = \left(e_{\alpha\beta}(\underline{\eta}) + \frac{1}{2}\partial_\alpha\eta_3\partial_\beta\eta_3\right) + \frac{1}{2}(\partial_\alpha S\partial_\beta\eta_3 + \partial_\beta S\partial_\alpha\eta_3), \qquad \Pi^2_{\alpha\beta}(\underline{\eta}) = -h\partial_{\alpha\beta}\eta_3$$

and the stresses $\sigma^0_{\alpha\beta}$ are given by

$$\sigma^0_{\alpha\beta} = c_{\alpha\beta\gamma\delta}(x)\left[H_{\gamma\delta}(\underline{u}^0) + \frac{1}{2}I_{\gamma 3}(\underline{u}^0)I_{\delta 3}(\underline{u}^0)\right] = c_{\alpha\beta\gamma\delta}(x)\left[\Pi^1_{\gamma\delta}(\underline{\xi}) + x_3\Pi^2_{\gamma\delta}(\underline{\xi})\right] \tag{27}$$

Remarks. –

- If the plate is isotropic and homogeneous with variable thickness we have

$$c_{\alpha\beta\gamma\theta}(x) = \left(\frac{2\mu\lambda}{\lambda+2\mu}\right)\delta_{\alpha\beta}\delta_{\gamma\theta} + \mu(\delta_{\alpha\gamma}\delta_{\beta\theta} + \delta_{\alpha\theta}\delta_{\beta\gamma})$$

  where λ and μ are the Lamé constants.
- If the plate is isotropic and homogeneous with constant thickness $(\phi_2 = +1, \phi_1 = -1)$, we have

$$\sigma^0_{\alpha\beta} = c_{\alpha\beta\gamma\theta}(x)\left[\Upsilon^1_{\gamma\theta}(\underline{\zeta}) + x_3\Upsilon^2_{\gamma\theta}(\underline{\zeta})\right] = \frac{1}{2}n^0_{\alpha\beta} + \frac{3}{2}x_3 m^0_{\alpha\beta}$$

  where

$$\Upsilon^1_{\gamma\theta}(\underline{\zeta}) = e_{\gamma\theta}(\underline{\zeta}) + \frac{1}{2}\partial_\gamma\zeta_3\partial_\theta\zeta_3, \qquad \Upsilon^2_{\gamma\theta}(\underline{\zeta}) = -\partial_{\gamma\theta}\zeta_3$$

$$n^0_{\alpha\beta} = \int_{-1}^{+1}\sigma^0_{\alpha\beta}\,\mathrm{d}x_3 = 2c_{\alpha\beta\gamma\theta}\Upsilon^1_{\gamma\theta}(\underline{\zeta}) = \left(\frac{4\mu\lambda}{\lambda+2\mu}\right)\Upsilon^1_{\theta\theta}(\underline{\zeta})\delta_{\alpha\beta} + 4\mu\Upsilon^1_{\alpha\beta}(\underline{\zeta})$$

$$m^0_{\alpha\beta} = \int_{-1}^{+1}x_3\sigma^0_{\alpha\beta}\,\mathrm{d}x_3 = \frac{2}{3}c_{\alpha\beta\gamma\theta}\Upsilon^2_{\gamma\theta}(\underline{\zeta}) = \frac{4\mu\lambda}{3(\lambda+2\mu)}\Upsilon^2_{\theta\theta}(\underline{\zeta})\delta_{\alpha\beta} + \frac{4}{3}\mu\Upsilon^2_{\alpha\beta}(\underline{\zeta})$$

  Moreover, if we have $f_\alpha = g_\alpha = 0$, we obtain the variational formulation of the von Karman plates problem, which is the same result as proposed in [1,3,4].
- If the lower and upper faces of the plate are of the same geometrical form $(\phi_2 = S+1, \phi_1 = S-1)$, this is a shallow shell, we suppose that is homogeneous and isotropic and $f_\alpha = g_\alpha = 0$. Thus we obtain the variational formulation of the Marguerre–von Karman shallow shell problem which is the same as proposed in [1,5].

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