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Estimating the convergence rate for eigenfrequencies of anisotropic plates with variable thickness

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Abstract Estimates of the differences between rescaled eigenvalues of the spectral problem for a thin anisotropic plate and eigenvalues of its two-dimensional models are obtained with bounds expressed in terms of the plate's thickness and attributes of the limit eigenvalue. ***To cite this article: S.A. Nazarov, C. R. Mecanique 330 (2002) 603–607.***

computational solid mechanics / thin anisotropic plate / two-dimensional problem

Estimation du taux de convergence des valeurs propres des plaques anisotropes, avec épaisseur variable

Résumé On propose d'obtenir une majoration des écarts entre les valeurs propres du problème spectral d'une plaque mince élastique anisotrope et les valeurs propres du problème modèle bi-dimensionnel, par des termes qui ne dépendent que de l'épaisseur de la plaque et de la valeur propre limite correspondante. ***Pour citer cet article : S.A. Nazarov, C. R. Mecanique 330 (2002) 603–607.***

mécanique des solides numériques / plaque mince anisotrope / modèle bi-dimensional

Version française abrégée

Il est démontré dans [1,12] (voir aussi [13,2,3]) que la valeur propre redimensionnée $h^{-2}\Lambda_k(h)$ du problème spectral (2) d'une plaque mince élastique anisotrope (1) et d'épaisseur variable $hH(y)$ converge quand $h \to +0$, vers le terme correspondant λ_k de la suite de valeurs propres (5) du modèle de plaque bi-dimensionnel. On remarque que les équations constituant (2) et (6) sont écrites sous forme matricielle, conformément aux relations (3), (7) et (8). L'expression de $\mathcal{A}$ dans (8) peut être trouvée dans [5–8,3,4]. De plus pour une plaque symétrique, (6) se réduit à un problème de Dirichlet relatif à une équation scalaire d'ordre quatre (cf. le modèle de Kirchhoff des plaques cylindriques isotropes dans [1,12,13,2]). Pour évaluer la différence $|h^{-2}\Lambda_k(h) - \lambda_k|$, on superpose la justification habituelle au procédé de réduction directe, introduit pour les domaines minces dans [11]. Ce procédé, développé dans [4], est destinè à remplacer les théorèmes de convergence (cf. [1,12,2,3]) et repose sur les estimations des normes pondérées des dérivées d'ordres suffisamment grands des vecteurs propres $u^k(h,x)$. Dans l'approche usuelle (réduction inverse), on prend une solution $\{\lambda_k, w_k\}$ de (6) comme solution approchée du problème (2) et on établit les Proposition 1 sur le comportement asymptotique *collectif* des valeurs propres. Le procédé de réduction directe consiste à construire une solution approchée (12) du problème limite (6) à partir d'une solution du problème de départ (2), qui permet d'établir le théorème sur les comportements asymptotiques *individuels* des valeurs

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propres. Les bornes des inégalités (10), (11), (15) et les restrictions de type (14), (16) sont exprimées en fonction du petit paramètre h, de la valeur propre λ_k, de sa multiplicité $\varkappa_k$ (voir (8)), et de la « distance relative » d_k définie en (13). Néanmoins les constantes c_i and C_i ne dépendent que de $H_\pm$, ω et A, γ. La condition (14) de validité de la première forme asymptotique $\Lambda_k(h) \sim h^2\lambda_k$ pour les basses fréquences propres, renvoie d'une certaine manière aux moyennes fréquences du spectre (4), relatif au modèle (17) d'oscillations longitudinales dans une plaque dont le comportement est décrit dans la Proposition 2. Cette dernière est obtenue à l'aide d'une réduction inverse alors que le procédé de réduction directe ne peut être appliqué que dans le cas d'une plaque (1), symétrique à la fois physiquement et géométriquement. Sous les mêmes conditions, on peut obtenir de même des développements asymptotiques analogues des vecteurs propres, individuels ou collectifs. Les résultats de cette note sont encore valables pour des plaques à bords encastrés et de formes assez quelconques, pour des coques de faible courbure, et pour des barres élastiques anisotropes de sections variables (cf. [14] et [4]). Pour une tige mince et sous certaines contraintes de symétrie, les conditions (14) et (16) peuvent être écrites sous la forme (19) qui contient uniquement la k-ième valeur propre mais cela semble très compliqué. Si h est plus grand qu'en (19) *on ne peut* fixer le nombre $\varkappa_k$ de valeurs propres $\Lambda_p(h)$ au voisinage du point $h^2\lambda_k$, et donc on doit faire appel à la notion de comportement asymptotique collectif des valeurs propres de la Proposition 1.

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1. Spectral problems

Let $\omega \subset \mathbb{R}^2$ be a domain bounded by a simple smooth closed contour $\partial\omega$, and $H_\pm$ smooth functions in $y = (y_1, y_2) \in \overline{\omega} = \omega \cup \partial\omega$; $H = H_+ + H_- > 0$ in $\overline{\omega}$. For the thin inhomogeneous anisotropic plate

$$\Omega_h = \left\{x = (y, z) : y \in \omega,\ z \in \Upsilon_h = \left(-hH_-(y), hH_+(y)\right)\right\} \tag{1}$$

where $h \in (0, 1]$ is a small parameter, we consider the spectral boundary value problem

$$\begin{aligned} &D(-\nabla_x)^\top A D(\nabla_x)u = \Lambda(h)\gamma u \quad \text{in } \Omega_h \\ &D(n)^\top A D(\nabla_x)u = 0 \quad \text{on } \Sigma_h^\pm = \left\{x : y \in \omega,\ z = \pm hH_\pm(y)\right\} \\ &u = 0 \quad \text{on } \Gamma_h = \left\{x : y \in \partial\omega,\ z \in \Upsilon_h(y)\right\} \end{aligned} \tag{2}$$

Here the matrix form of the linear elasticity equations is used, i.e., $u = (u_1, u_2, u_3)^\top$ is a displacement column, A a positive definite symmetric matrix of size 6×6,

$$D(\nabla_x)^\top = \begin{pmatrix} \partial_1 & 0 & \alpha\partial_2 & \alpha\partial_z & 0 & 0 \\ 0 & \partial_2 & \alpha\partial_1 & 0 & \alpha\partial_z & 0 \\ 0 & 0 & 0 & \alpha\partial_1 & \alpha\partial_2 & \partial_z \end{pmatrix}, \quad \alpha = \frac{1}{\sqrt{2}} \tag{3}$$

Furthermore, $\top$ stands for transposition, n for the outward normal, and $\partial_i = \partial/\partial y_i$, $\partial_z = \partial/\partial z$. The stiffness matrix A and the density γ are assumed to depend smoothly on the rapid $\zeta = h^{-1}z$ and slow y variables. It is known that the eigenvalues of the problem (2) form the sequence

$$0 < \Lambda_1(h) \leqslant \Lambda_2(h) \leqslant \cdots \leqslant \Lambda_k(h) \leqslant \cdots \to +\infty \tag{4}$$

As proven in [1–4], the rescaled eigenvalue $h^{-2}\Lambda_k(h)$ tends to the eigenvalue λ_k in the sequence

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots \to +\infty \tag{5}$$

associated with the two-dimensional spectral problem of the plate theory

$$\mathcal{D}(-\nabla_y)^\top \mathcal{A}\mathcal{D}(\nabla_y)w = \lambda H\overline{\gamma}e_3w_3 \quad \text{in } \omega, \quad w = (w_1, w_2, w_3)^\top = 0, \quad \partial_\nu w_3 = 0 \quad \text{on } \partial\omega \tag{6}$$

where $\nabla_y = (\partial_1, \partial_2)^\top$, $e_3 = (0, 0, 1)^\top$, and ∂_ν is the derivative along the outward normal. Moreover, $\mathcal{D}(\nabla_y)^\top$ denotes the matrix $D(\nabla_y, 0)^\top$ from (3) with the third line changed for

$$\left(0, 0, 0, \alpha\partial_1^2, \alpha\partial_2^2, \partial_1\partial_2\right) \tag{7}$$

the positive definite 6×6-matrix $\mathcal{A}$ and the mean density $\overline{\gamma} > 0$ depend smoothly on $y \in \omega$,

$$\mathcal{A} = \int\limits_{\Upsilon_1} \begin{pmatrix} a & -\alpha\zeta a \\ -\alpha\zeta a & \alpha^2\zeta^2 a \end{pmatrix} \mathrm{d}\zeta, \qquad \overline{\gamma} = \frac{1}{H} \int\limits_{\Upsilon_1} \gamma \, \mathrm{d}\zeta$$
$$a(y,\zeta) = A_{(11)}(y,\zeta) - A_{(12)}(y,\zeta) A_{(22)}(y,\zeta)^{-1} A_{(21)}(y,\zeta) \tag{8}$$

where $A_{(ij)}$ imply 3×3-blocks of A. Formulae of the type (8) can be found in [5–8,4].

The aim of the paper is to establish estimates of the differences $\Lambda_k(h) - h^2\lambda_k$ with constants independent of both, the geometric parameter h and the eigenvalue number k. We also indicate the segments $(0, \boldsymbol{h}_k)$ for validity of these estimates and a range of the spectrum (4), where the primary asymptotics $\Lambda_k(h) \sim h^2\lambda_k$ remains plausible. The latter is consistent with observing middle- and high-frequency asymptotic series of eigenvalues in (4) (see, e.g., [9,10,3]). To compare the spectra, we supplement the usual justification scheme with a procedure of *direct reduction*, introduced for thin domains in [11], developed in [4], intended to replace proving convergence theorems (cf. [1–3,12]), and based on a scrupulous estimation of weighted norms of higher-order derivatives of the eigenvectors $u^k(h,x)$. In the sequel c_i, C_i and h_i stand for positive constants depending on $H_\pm$, ω and A, γ only, and λ_k stands for an eigenvalue of (6) of the multiplicity $\varkappa_k$,

$$\lambda_{q-1} < \lambda_q = \cdots = \lambda_{q+\varkappa_q-1} < \lambda_{q+\varkappa_q} \tag{9}$$

2. Inverse reduction

Asymptotic structures of elastic fields, revealed in [1–7,12,8] for different types of plates (see [5,14,4] for the general case), are used to derive the plate model (6) from the three-dimensional elasticity equations (2). At the same time, these structures can be applied in the *opposite* direction, namely, for recovering an approximate solution to the problem (2) in terms of a solution to the problem (6). As a result, the lemma on "almost eigenvalue" (see, e.g., [15]), equipped with simple algebraic calculations, furnishes the following assertion.

PROPOSITION 1. – *If, $h \leqslant \boldsymbol{h}_1 = h_1 n^{-2} \lambda_k^{-1/2}$ and $n \leqslant \varkappa_k$, the problem* (2) *has at least n eigenvalues satisfying the inequality*

$$\left| h^{-2}\Lambda_p(h) - \lambda_k \right| \leqslant C_1 h^{1/2} n \lambda_k^{5/4} \tag{10}$$

If $n = 1$, the restriction on h provides the formula $\Lambda_p(h) \sim h^2\lambda_k \leqslant c_1 = h_1^2$ which reflects the existence of the middle-frequency asymptotic form $\Lambda_q(h) \sim \mu_j$ (see Proposition 2). Thus, Proposition 1 covers the whole low-frequency range of the spectrum (4).

3. Direct reduction

Taking the solution $\{\Lambda_p(h), u^p(h,x)\}$ of the problem (2), we determine the approximate solution $\{\Lambda_p(h), W^p(h,y)\}$ to the problem (6),

$$W_j^p(h,y) = X_h(y) h^{\delta_{j,3} - 5/2} H(y)^{-1} \int_{\Upsilon_h(y)} u_j^p(h,y,z) \, \mathrm{d}z, \quad j = 1,2,3 \tag{11}$$

where $\delta_{j,k}$ is Kronecker's symbol and $X_h \in C_{\mathrm{o}}^\infty(\omega)$ a cut-off function, $X_h(y) = 1$ as $\mathrm{dist}\{y, \partial\omega\} > 2h$ and $X_h(y) = 0$ as $\mathrm{dist}\{y, \partial\omega\} < h$. Note that the normalization factors $h^{\delta_{j,3}-5/2}$ in (11) reflect the standard asymptotic ansatz in the theory of plates (see, e.g., [6,7,12,4]). After an accurate treatment of discrepancies left by (11) in the problem (6), the lemma on 'almost eigenvalue' extract from the sequence (5) the entry λ_k obeying the inequality

$$\left| h^2 \Lambda_p(h)^{-1} - \lambda_k^{-1} \right| \leqslant c_2 h^{1/2} \lambda_k^{-1/2} \tag{12}$$

Recalling (9), we put

$$\varepsilon_k = d_k(2\lambda_k)^{-1}, \quad d_k = \min\{\lambda_k\lambda_{k-1}^{-1} - 1, 1 - \lambda_k\lambda_{k+\varkappa_k}^{-1}\} \tag{13}$$

To any eigenvalue $\Lambda_j(h)^{-1} \in [h^{-2}\lambda_k^{-1} - h^{-2}\varepsilon_k, h^{-2}\lambda_k^{-1} + h^{-2}\varepsilon_k]$, under the condition

$$h \leqslant \boldsymbol{h}_2 = h_2\varkappa_k^{-2}(1 + d_k^{-1})^{-2}\lambda_k^{-1} \tag{14}$$

with $h_2 \leqslant (2c_2)^{-2}$, the lemma associates just the same λ_k. Moreover, certain calculations shows that the number of such eigenvalues $\Lambda_p(h)$ cannot exceed $\varkappa_k$. Since the hypothesis of Proposition 1 with $n = \varkappa_k$ is fulfilled due to (14), we arrive at the final assertion on low eigenfrequencies.

THEOREM. – *For the eigenvalues* $\Lambda_k(h), \ldots, \Lambda_{k+\varkappa_k-1}(h)$, *the restriction* (14) *provides the relation*

$$\left|h^{-2}\Lambda_p(h) - \lambda_k\right| \leqslant C_2 h^{1/2}\varkappa_k\lambda_k^{5/4} \tag{15}$$

There is no other eigenvalue in (4) *satisfying* (15). *Moreover, in the case*

$$h \leqslant 2h_2\varkappa_k^{-2}(1 + d_k^{-1})^{-2}(\lambda_k + \lambda_{k+\varkappa_k})^{-1} \tag{16}$$

the segment $[2^{-1}(\lambda_k + \lambda_{k-1}), 2^{-1}(\lambda_k + \lambda_{k+\varkappa_k})]$ *only contains eigenvalues mentioned above.*

4. Middle frequencies

The problem (2) is also related with the model for longitudinal oscillations of the plate (1),

$$\mathcal{D}'(-\nabla_y)^\top \mathcal{A}'\mathcal{D}'(\nabla_y)w' = \mu H\bar{\gamma}w' \quad \text{in } \omega, \qquad w' = 0 \quad \text{on } \partial\omega \tag{17}$$

Here $\mathcal{D}'(-\nabla_y)^\top$ and $\mathcal{A}'$ are the left upper 3×3-blocks of the matrices (3) and (8); $w' = (w_1, w_2)^\top$ stands for the longitudinal displacement column.

PROPOSITION 2. – *Let* μ_k *be an eigenvalue of multiplicity* $\varkappa_k$ *for the problem* (17). *If* $h \leqslant \boldsymbol{h}_3 = h_3\varkappa_k^{-2}\mu_k^{-1}$, *the problem* (2) *has at least* $\varkappa_k$ *eigenvalues satisfying the inequality*

$$\left|\Lambda_p(h) - \mu_k\right| \leqslant C_3 h^{1/2}\varkappa_k\mu_k^{3/2} \tag{18}$$

Looking quite analogous to Proposition 1, the latter assertion is derived by applying the inverse reduction. According to theorem, the total multiplicity of the spectrum (4) on the interval $(0, \mu_k)$ increases indefinitely as $h \to +0$ and hence, for an infinitesimal sequence $\{h_{(j)}\}$, the number of eigenvalues $\Lambda_m(h)$, which verify (18) at $h = h_{(j)}$, is larger than $\varkappa_k$. This makes impossible to conclude accomplished results for the middle-frequency range of the spectrum, except for plates possessing the geometric and physical symmetry (cf. [2]).

5. Discussion

For simplicity, we here suppose that the lateral side of the plate (1) is cylindrical. Nevertheless, Korn's inequalities with anisotropic distribution of weights (see [5,16,4]) provide a justification of the usual asymptotic forms for a plate with rather arbitrary shaped edge (cf. [14,4]).

Proposition 1 does not warrant that the $C_1 h^{5/2}\varkappa_k\lambda_k^{5/4}$-neighborhood of the point $h^2\lambda_k$ contains exactly eigenvalues $\Lambda_k(h), \ldots, \Lambda_{k+\varkappa_k-1}(h)$ and, therefore, it reveals the *collective* asymptotics of eigenvalues. Based on a harder restriction than in Proposition 1, the theorem makes these asymptotics *individual*. If λ_{k-1} is close to λ_k, it is worth to apply collective asymptotics because the relative distance d_k and the bound in (14) can become unacceptably small. Clearly, proper asymptotic representations for the eigenvectors $u^k(h, x), \ldots, u^{k+\varkappa_k-1}(h, x)$ appear only in the case of the individual asymptotics of the corresponding eigenvalues, i.e., under the condition (14). For the sake of brevity, we here skip asymptotic formulas for $u^p(h, x)$ (cf.[1–4,13]).

An assertion, which repeats the theorem word by word, holds true for a thin anisotropic rod with a variable cross-section (see [4]). Certain symmetry assumptions reduce the one-dimensional model of the rod to the Dirichlet problems for two ordinary differential equations of fourth order. Eigenvalues λ_k of the corresponding spectral problems are known to behave like $k^4(\mathbf{c}_0 + o(1))$ as $k \to +\infty$. This fact furnishes the following *heuristical* calculations related to the simple eigenvalue λ_k. Since $d_k = O(k^{-1})$ according to (13), the restriction (14) turns into

$$h \leqslant \mathbf{c}_2 k^{-6} \tag{19}$$

while the bounds in (15) and (10) get the order $h^{1/2}k^5$. Hence, the individual asymptotics occurs for the eigenvalues $\Lambda_k(h) \leqslant \mathbf{C}_2 h^{3/4}$, i.e. on a rather narrow part of the low-frequency range. Providing the collective asymptotics of $\Lambda_k(h)$, Proposition 1 requires much softer restriction $h \leqslant \mathbf{c}_1 k^{-2}$ than (19). We point out that, for the plate model (6), the eigenvalues in (5) are 'rambling' and, therefore, a behavior of d_k as $k \to +\infty$ is not available.

As shown in [13,2], the boundary layers at the lateral side Γ_h *only mediately* influence the asymptotics of eigenvalues. Owing to these augmented asymptotic approximations, the bound in (10) may be modified. However, the boundary layer structures refer to higher-order derivatives of w^k and consequently increasing the exponent of h by $1/2$ brings additional factor $\lambda_k^{1/4}$ into the bound. We emphasize that it remains an open question to reflect the boundary layer phenomenon in the direct reduction procedure which creates in (14) the hardest restriction on h.

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