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Partial homogenization

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Abstract The method of partial homogenization is proposed and justfied for equations with rapidly oscillating coefficients. Such equations simulate fields in composite materials. The proposed method keeps the initial equation in some thin boundary strip, homogenizes the equation in the resting part of the domain and prescribes the appropriate interface conditions for homogenized and non-homogenized parts. ***To cite this article: G.P. Panasenko, C. R. Mecanique 330 (2002) 667–672.***

computational solid mechanics / partial homogenization / partial asymptotic decomposition of domain

Homogénéisation partielle

Résumé La méthode de l'homogénéisation partielle est proposée et justifée pour des équations aux coefficients fortement oscillants. Ces équations modélisent des champs physiques dans les matériaux composites. La méthode proposée garde l'équation de départ dans un sous-domaine mince, elle prévoit l'homogénéisation d'ordre élevé dans la partie restante du domaine et elle prescrit les conditions de l'interface appropriées pour la partie homogéneisée et la partie non homogénéisée. ***Pour citer cet article : G.P. Panasenko, C. R. Mecanique 330 (2002) 667–672.***

mécanique des solides numérique / homogénéisation partielle / decomposition asymptotique partielle de domaine

1. Position of problem

The idea of partial homogenization on some subdomain was discussed in [1]: we replace the initial equation by the homogenized equation of high order on the main part of the domain and we keep the initial equation in some boundary strip. The main question is how to conjugate these two equations (probably of different order) to obtain the accuracy of a given power of ε.

Here below we consider the model problem in a layer with Dirichlet boundary condition, and we propose and justify a method of 'conjugation' of the high order homogenized equation with the initial equation.

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Consider the homogenization of a boundary value probem in a layer [2,3]:

$$\sum_{k,j=1}^{s} \frac{\partial}{\partial x_k}\left(A_{kj}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_j}\right) = f(x), \quad x_1 \in (0,d), \quad u_{\mathrm{e}}|_{x_1=0} = 0, \quad u_{\mathrm{e}}|_{x_1=d} = 0 \tag{1}$$

where $x = (x_1, x')$, $x' = (x_2, \ldots, x_s)$, $A_{kj}(\xi)$ are 1-periodic in $\xi \in \mathbb{R}^s$ functions, satisfying the following conditions:
(i) $\exists \kappa_0 > 0$, $\forall \xi \in \mathbb{R}^s$, $\forall \eta \in \mathbb{R}^s$, $\eta = (\eta_1, \ldots, \eta_s)$, $\sum_{k,j=1}^s A_{kj}(\xi)\eta_j\eta_i \geqslant \kappa_0 \sum_{i=1}^s \eta_i^2$;
(ii) $\forall \xi \in \mathbb{R}^s$, $k, j \in \{1, \ldots, s\}$, $A_{kj}(\xi) = A_{jk}(\xi)$,
f is a C^∞-smooth function, T-periodic in $x_2, \ldots, x_s$, T in the sense of multiple of ε.

Assume that A_{kj} are piecewise smooth functions in sense [3]. Then there exist a unique solution to this problem (see [3]) and it is a solution of the following variational formulation.

Let $H^1_{0,\mathrm{per}'}$ be a space that is completion of the space of C^∞-smooth T-periodic in $x_2, \ldots, x_s$ functions vanishing if $x_1 = 0$ or $x_1 = d$ with respect to the norm $H^1((0,d) \times (0,T)^{s-1})$; then u_ε is sought as the function of $H^1_{0,\mathrm{per}'}$ satisfying

$$J(u,\varphi) = \int_{\Omega_{d,T}} \left\{ \sum_{k,j=1}^{s} A_{kj}\left(\frac{x}{\varepsilon}\right)\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_k} + f\varphi \right\} \mathrm{d}x = 0, \quad \forall \varphi \in H^1_{0,\mathrm{per}'} \tag{2}$$

where $\Omega_{d,T} = (0,d) \times (0,T)^{s-1}$.

2. Structure of the asymptotic expansion of the solution

The asymptotic expansion of the solution to problem (1) was constructed in [3]. It has a form

$$(u_\varepsilon^a)^{(K)} = u_{\mathrm{BLO}}^{(K)}\left(x, \frac{x}{\varepsilon}\right) + u_{\mathrm{BLd}}^{(K)}\left(x, \frac{x_1 - d, x'}{\varepsilon}\right) + \sum_{l=0}^{K+1} \varepsilon^l \sum_{i=(i_1,\ldots,i_l),\ i_j \in \{1,\ldots,s\}} N_i\left(\frac{x}{\varepsilon}\right) D^i v_\varepsilon^{(K)}(x) \tag{3}$$

$$v_\varepsilon^{(K)}(x) = \sum_{j=0}^{K+1} \varepsilon^j v_j(x) \tag{4}$$

where the ‘boundary layers’ $u_{\mathrm{BLO}}^{(K)}$ and $u_{\mathrm{BLd}}^{(K)}$ are exponentially decaying functions such that for all $x \in [0,d] \times \mathbb{R}^{s-1}$, $|u_{\mathrm{BLO}}^{(K)}(x,\xi)|$, $|u_{\mathrm{BLd}}^{(K)}(x,\xi)| \leqslant C_1 \mathrm{e}^{-C_2|\xi_1|}$, $C_1, C_2 > 0$, C_1, C_2 do not depend on ε; $N_i(\xi)$ are 1-periodic functions, solutions of the chain of cell problems:

$$L_{\xi\xi} N_{i_1} = -\sum_{k=1}^{s} \frac{\partial}{\partial \xi_k} A_{ki_1}, \quad \bigl(i_1 \in \{1, \ldots, s\}\bigr) \tag{5}$$

$$\begin{aligned} L_{\xi\xi} N_i = {} & -\sum_{k=1}^{s} \frac{\partial}{\partial \xi_k}(A_{ki_1} N_{i_2\ldots i_l}) - \sum_{k=1}^{s} A_{i_1 k}\frac{\partial N_{i_2\ldots i_l}}{\partial \xi_k} - A_{i_1 i_2} N_{i_3\ldots i_l} \\ & + \left\langle \sum_{k=1}^{s} A_{i_1 k}\frac{\partial N_{i_2\ldots i_l}}{\partial \xi_k} + A_{i_1 i_2} N_{i_3\ldots i_l} \right\rangle, \quad i = (i_1, \ldots, i_l);\ l \geqslant 2, i_j \in \{1, \ldots, s\} \end{aligned} \tag{6}$$

here $\langle \cdot \rangle = \int_{(0,1)^s} \mathrm{d}\xi$, $L_{\xi\xi} = \sum_{k,j=1}^s \frac{\partial}{\partial \xi_k}(A_{kj}(\xi)\frac{\partial}{\partial \xi_j})$; $\langle N_i \rangle = 0$ for $i \neq \emptyset$, $D^i = D^{(i_1,\ldots,i_l)} = \partial^l / \partial x_{i_1} \cdots \partial x_{i_l}$.

These cell problems are posed in variational formulation in H^1_{per}, that is the completion of the space of C^∞-regular 1-periodic functions with respect to the norm $H^1((0,1)^s)$. Here $N_\emptyset = 1$, $(i_3 \ldots i_l) = \emptyset$ if $l = 2$. This chain of problems is solved consequently for $l = 1, 2, \ldots, K+1$.

Functions v_j are defined in other chain of problems (cf. [3]) but we do not need this information for further study here.

3. Partial asymptotic decomposition of the domain

Let us describe the method of asymptotic partial decomposition of domain for problem (1), (2).

Consider the subspace $H_{\varepsilon,\delta,\mathrm{dec}}$ of the space $H^1_{0,\mathrm{per}'}$ that consists of all functions u of $H^1_{0,\mathrm{per}'}$ having for all $x_1 \in [\delta, d-\delta]$ the following presentation in the form of the Bakhvalov ansatz [3]:

$$u(x) = \sum_{l=0}^{K+1} \varepsilon^l \sum_{i=(i_1\ldots i_l),\ i_j\in\{1,\ldots,s\}} N_i\left(\frac{x}{\varepsilon}\right) D^i v \tag{7}$$

where $v \in H^{K+2}_{T,\mathrm{per}}([\delta, d-\delta]\times\mathbb{R}^{s-1})$, i.e., v belongs to the completion of the space of T-periodic in $x_2,\ldots,x_s$, C^∞-regular functions with respect to the norm $H^{K+2}([\delta, d-\delta]\times[0,T]^{s-1})$. So every $u \in H_{\varepsilon,\delta,\mathrm{dec}}$ is related by (7) to some $v \in H^{K+2}_{T,\mathrm{per}}([\delta, d-\delta]\times\mathbb{R}^{s-1})$, so that we can consider a couple (u, v).

Let $\delta = \widehat{K}\varepsilon|\ln\varepsilon|$. Consider the partially decomposed variational problem

$$J(u,\varphi) = 0, \quad \forall\varphi \in H_{\varepsilon,\delta,\mathrm{dec}}$$

If u_d is its solution then the main theorem of [4] proves that for any $K \in (0,\infty)$, there exist $\widehat{K}$ such that if $\delta = \widehat{K}\varepsilon|\ln\varepsilon|$ then the estimate holds

$$\|u_\varepsilon - u_d\|_{H^1([0,d]\times[0,T]^{s-1})} = \mathrm{O}\left(\varepsilon^{K+1}\right) \tag{8}$$

4. Partial homogenization

Let us consider the modified formulation of partially decomposed problem:

minimize the functional

$$W(u,v) = \int_{[0,\delta]\times[0,T]^{s-1}\cup[d-\delta,d]\times[0,T]^{s-1}} \left\{\sum_{k,j=1}^{s} A_{kj}\left(\frac{x}{\varepsilon}\right)\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_k} + 2fu\right\}\mathrm{d}x$$
$$+ \int_{[\delta,d-\delta]\times[0,T]^{s-1}} \left\{\sum_{l,m=1}^{K+2} \varepsilon^{l+m-2} \sum_{\bar{i}=(\bar{i}_1\ldots\bar{i}_l),\ \bar{i}_r\in\{1,\ldots,s\}} \ \sum_{\bar{\bar{i}}=(\bar{\bar{i}}_1\ldots\bar{\bar{i}}_m),\ \bar{\bar{i}}_r\in\{1,\ldots,s\}} \tilde{h}_{\bar{i}\bar{\bar{i}}} D^{\bar{i}} v D^{\bar{\bar{i}}} v + 2fv\right\}\mathrm{d}x \tag{9}$$

where $u \in H_{\varepsilon,\delta,\mathrm{dec}}$ and is related to v by (7) and

$$\tilde{h}_{\bar{i}\bar{\bar{i}}} = \sum_{k,j=1}^{s} \left\langle A_{kj}\left(\frac{\partial}{\partial\xi_j}N_{\bar{\bar{i}}} + \delta_{j\bar{\bar{i}}_1}N_{\bar{\bar{i}}_2\ldots\bar{\bar{i}}_m}\right)\left(\frac{\partial}{\partial\xi_k}N_{\bar{i}} + \delta_{k\bar{i}_1}N_{\bar{i}_2\ldots\bar{i}_l}\right)\right\rangle \tag{10}$$

defined by the solutions of the cell problems (5), (6). In (10) the derivatives $\partial/\partial\xi_k$ must be dropped if $|\bar{i}| = K+2$, and $\partial/\partial\xi_j$ must be dropped if $|\bar{\bar{i}}| = K+2$ [5].

Let us denote

$$G_{\mathrm{BL}} = [0,\delta]\times[0,T]^{s-1}\cup[d-\delta,d]\times[0,T]^{s-1}, \qquad G_I = [\delta, d-\delta]\times[0,T]^{s-1}$$

$$a(u,v;\varphi,w)=\int_{G_{\mathrm{BL}}}\sum_{k,j=1}^{s}A_{kj}\left(\frac{x}{\varepsilon}\right)\frac{\partial u}{\partial x_j}\frac{\partial\varphi}{\partial x_k}\,\mathrm{d}x + \int_{G_I}\sum_{l,m=1}^{K+2}\varepsilon^{l+m-2}\sum_{\bar{i}=(\bar{i}_1,\dots,\bar{i}_l),\ \bar{i}_r\in\{1,\dots,s\}}\ \sum_{\bar{\bar{i}}=(\bar{\bar{i}}_1\dots\bar{\bar{i}}_m),\ \bar{\bar{i}}_r\in\{1,\dots,s\}}\tilde{h}_{\bar{i}\bar{\bar{i}}}D^{\bar{i}}vD^{\bar{\bar{i}}}w\,\mathrm{d}x$$

$$b(\varphi,w)=\int_{G_{\mathrm{BL}}}f\varphi\,\mathrm{d}x+\int_{G_I}fw\,\mathrm{d}x$$

then

$$W(u,v)=a(u,v;u,v)+2b(u,v)$$

The variational formulation of problem (9) is as follows:
find $\bar{u}_d\in H_{\varepsilon,\delta,\mathrm{dec}}$ related to the function v by (7) such that $\forall\varphi\in H_{\varepsilon,\delta,\mathrm{dec}}$ related to w by

$$\varphi(x)=\sum_{l=0}^{K+1}\varepsilon^l\sum_{i=(i_1,\dots,i_l),\ i_j\in\{1,\dots,s\}}N_i\left(\frac{x}{\varepsilon}\right)D^iw$$

the following identity holds

$$a(\bar{u}_d,v;\varphi,w)+b(\varphi,w)=0 \tag{11}$$

We will call this problem partially homogenized.

Let $u^{(K)}$, $v^{(K)}$ be the pair of asymptotic solutions of K-th order (3), (4). Then defining the norm $\|\cdot\|_a=\sqrt{a(\cdot,\cdot;\cdot,\cdot)}$, we have the estimate

$$\left\|u^{(K)}-\bar{u}_d\right\|_a=\mathrm{O}\left(\varepsilon^{K+1}\right) \tag{12}$$

Thus

$$\begin{cases}\left\|u^{(K)}-u_\varepsilon\right\|_{H^1((0,d)\times(0,T)^{s-1})}=\mathrm{O}\left(\varepsilon^{K+1}\right)\\ \left\|u^{(K)}-\bar{u}_d\right\|_a=\mathrm{O}\left(\varepsilon^{K+1}\right)\end{cases} \tag{13}$$

5. Example of partial homogenization for $K=0$

Consider an example when f depends on x_1 only. Then the function $v^{(K)}$ depends on x_1 only [3] and therefore we search for the solution of the partially decomposed problem (9) in the form (7), where v also depends only on x_1. (It means that in the definition of the subspace $H_{\varepsilon,\delta,\mathrm{dec}}$ the function v depends only on x_1, i.e., $\nabla_{x'}v=0$.) It simplifies formulation (11) because it contains only the derivatives of v with respect to x_1. Consider the simpliest case $K=0$. We have then

$$a(u,v;\varphi,w)=\int_{G_{\mathrm{BL}}}\sum_{k,j=1}^{s}A_{kj}\left(\frac{x}{\varepsilon}\right)\frac{\partial u}{\partial x_j}\frac{\partial\varphi}{\partial x_k}\,\mathrm{d}x+\int_{G_I}\sum_{l,m=1}^{2}\varepsilon^{l+m-2}\tilde{\tilde{h}}_{lm}\frac{\partial^l v}{\partial x_1^l}\frac{\partial^m w}{\partial x_1^m}\,\mathrm{d}x \tag{14}$$

where $\tilde{\tilde{h}}_{lm}=\tilde{h}_{1\dots1,1\dots1}$, here the subscript consists of the two sets of units; the first set contains l units, the second set contains m units.

$$\tilde{\tilde{h}}_{11}=\widehat{A}_{11}=\sum_{j=1}^{s}\left\langle A_{1j}\frac{\partial(N_1+\xi_1)}{\partial\xi_j}\right\rangle$$

$$\tilde{\tilde{h}}_{12}=\tilde{\tilde{h}}_{21}=\sum_{j=1}^{s}\left\langle A_{ij}\frac{\partial(N_1+\xi_1)}{\partial\xi_j}N_1\right\rangle;\qquad \tilde{\tilde{h}}_{22}=\left\langle A_{11}N_1^2\right\rangle$$

Integrating identity (14) by parts and taking into account that w is an arbitrary function of the space $H^2([\delta, d-\delta])$ (as a function of x_1), we see that v is a solution to the equation of the fourth order (if $\tilde{\tilde{h}}_{22} \neq 0$)

$$\varepsilon^2 \tilde{\tilde{h}}_{22} \frac{\partial^4 v}{\partial {x_1}^4} - \tilde{\tilde{h}}_{11} \frac{\partial^2 v}{\partial x_1^2} = -f(x_1), \quad x_1 \in [\delta, d-\delta] \tag{15}$$

with the interface conditions following from the equations

$$\int_{\{x_1=\delta, x_2, \ldots, x_s \in [0,T]\}} \left\{ \tilde{\tilde{h}}_{11} \frac{\partial v}{\partial x_1} w(\delta) + \left(\varepsilon^2 \tilde{\tilde{h}}_{22} \frac{\partial^2 v}{\partial x_1^2} + \varepsilon \tilde{\tilde{h}}_{12} \frac{\partial v}{\partial x_1} \right) \frac{\partial w}{\partial x_1}(\delta) - \varepsilon^2 \tilde{\tilde{h}}_{22} \frac{\partial^3 v}{\partial x_1^3} w(\delta) \right\} \mathrm{d}x'$$
$$= \int_{\{x_1=\delta, x_2, \ldots, x_s \in [0,T]\}} \sum_{j=1}^{s} A_{ij} \frac{\partial \bar{u}_d}{\partial x_j} \left(w(\delta) + \varepsilon N_1 \left(\frac{x}{\varepsilon} \right) \frac{\partial w}{\partial x_1}(\delta) \right) \mathrm{d}x'$$

and

$$\bar{u}_d(\delta, x_2, \ldots, x_s) = v(\delta) + \varepsilon N_1 \left(\frac{x}{\varepsilon} \right) \frac{\partial v}{\partial x_1}(\delta);$$

i.e., for $x_1 = \delta$ denoting $S_\delta = \{x_1 = \delta, x_2, \ldots, x_s \in [0,T]\}$, we get

$$\begin{cases} \tilde{\tilde{h}}_{11} \dfrac{\partial v}{\partial x_1} - \varepsilon^2 \tilde{\tilde{h}}_{22} \dfrac{\partial^3 v}{\partial x_1^3} = \dfrac{1}{T^{s-1}} \displaystyle\int_{S_\delta} \sum_{j=1}^{s} A_{1j} \dfrac{\partial \bar{u}_d}{\partial x_j} \, \mathrm{d}x' \\ \varepsilon^2 \tilde{\tilde{h}}_{22} \dfrac{\partial^2 v}{\partial x_1^2} = \dfrac{1}{T^{s-1}} \displaystyle\int_{S_\delta} \sum_{j=1}^{s} A_{1j} \dfrac{\partial \bar{u}_d}{\partial x_j} N_1 \left(\dfrac{x}{\varepsilon} \right) \mathrm{d}x' \\ \bar{u}_d(\delta, x_2, \ldots, x_s) = v(\delta) + \varepsilon N_1 \left(\dfrac{\delta}{\varepsilon}, \dfrac{x_2}{\varepsilon}, \ldots ; \dfrac{x_s}{\varepsilon} \right) \dfrac{\partial v}{\partial x_1}(\delta) \end{cases} \tag{16}$$

We have the same interface conditions on the surface $S_{d-\delta} = \{x_1 = d - \delta, x_2, \ldots, x_s \in [0,T]\}$; δ in these conditions have to be replaced by $d - \delta$. In G_{BL} we keep Eq. (1) for $\bar{u}_d$.

So, (1) in G_{BL} for $\bar{u}_d$, (15) in G_I for v, and interface conditions (16) on S_δ and on $S_{d-\delta}$ constitute the differential version of the partially homogenized problem (11) (see Fig. 1) for $K = 0$. Estimates (13) give the error of order $\mathrm{O}(\varepsilon)$.

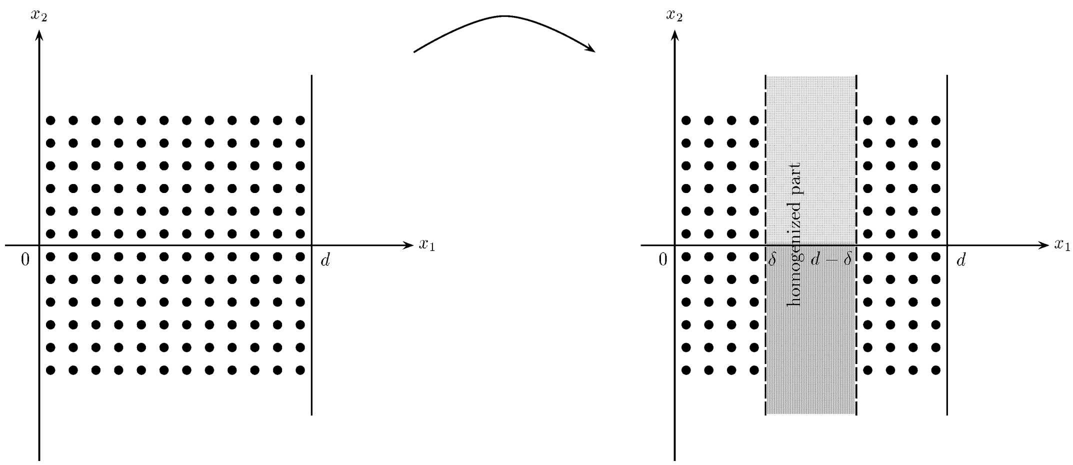

Figure 1. Partial homogenization.

Figure 1. *Homogénéisation partielle.*

References

[1] G.P. Panasenko, Method of asymptotic partial decomposition of domain, Math. Models Methods Appl. Sci. 8 (1) (1998) 139–156.
[2] G.P. Panasenko, High order asymptotic of solutions of problems on the contact of periodic structures, Mat. Sb. 110 (152) (4) (1979) 505–538. English translation: Math. USSR-Sb. 38 (4) (1981) 465–494.
[3] N.S. Bakhvalov, G.P. Panasenko, Homogenization: Averaging Processes in Periodic Media, Moscow, Nauka, 1984. English translation: Kluwer Academic, Dordrecht, 1989.
[4] G.P. Panasenko, Asymptotic partial decomposition of variational problems, C. R. Acad. Sci. Paris, Série IIb 327 (1999) 1185–1190.
[5] V.P. Smyshlyaev, K.D. Cherednichenko, On derivation of "strain gradient" effects in the overall behaviour of periodic heterogeneous media, J. Mech. Phys. Solids 48 (2000) 1325–1357.