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Uniformly valid approximation for singular perturbation problems and matching principle

Jacques Mauss [a], Jean Cousteix [b,c]

[a] **Institut de mécanique des fluides de Toulouse UMR-CNRS et Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse cedex, France**

[b] **Département modèles pour l'aérodynamique et l'énergétique, ONERA, 2, avenue Édouard Belin, BP 4025, 31055 Toulouse cedex 4, France**

[c] **École nationale supérieure de l'aéronautique et de l'espace, 10, avenue Édouard Belin, 31055 Toulouse cedex, France**

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Abstract After a brief reminder of the notion of asymptotic expansion, a counter-example of the Van Dyke matching principle is solved thanks to a modified form of this principle. This leads to a composite approximation to a given order. The proposed method of successive complementary expansions reverses the analysis by starting with a supposed form of the uniformly valid approximation. This method does not require any matching principle which, in fact, is a by-product. The method is illustrated with the very often studied one-dimensional model of Stokes–Oseen for the circular cylinder. ***To cite this article: J. Mauss, J. Cousteix, C. R. Mecanique 330 (2002) 697–702.***

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Développements asymptotiques uniformément valables pour des problèmes de perturbations singulières et principe de raccordement

Résumé Après un bref rappel de la notion de développement asymptotique, un contre-exemple du principe de raccordement de Van Dyke est levé grâce à une forme modifiée de ce principe. Ceci conduit à une approximation composite à un ordre donné. La méthode des développements successifs complémentaires proposée renverse l'analyse en partant d'une forme supposée d'une approximation uniformément valable. Cette méthode ne fait appel à aucun principe de raccordement. En fait, un tel principe apparaît comme un résultat. La méthode est illustrée avec le modèle unidimensionnel souvent étudié de Stokes–Oseen pour le cylindre circulaire. ***Pour citer cet article : J. Mauss, J. Cousteix, C. R. Mecanique 330 (2002) 697–702.***

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Les définitions d'un développement asymptotique (équation (1)) et de l'opérateur de développement $\mathrm{E}^{(n)}$ à l'ordre $\delta^{(n)}$ (équation (2)) sont d'abord rappelées ainsi que la notion de développement local pour un problème de perturbation singulière (équation (3)). Le raccordement entre les approximations obtenues dans

E-mail addresses: mauss@imft.fr (J. Mauss); Jean.Cousteix@onecert.fr (J. Cousteix).

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des régions significatives différentes est alors discuté. On appelle E_0 et E_1 les opérateurs de développement définis au même ordre δ et relatifs aux régions considérées. Le principe de raccordement de Kaplun et Lagerstrom fait appel aux variables intermédiaires avec l'opérateur E_ν (équation (5)) ; le principe de Van Dyke est donné par l'équation (6). Les contre-exemples connus ont conduit à proposer un principe de raccordement modifié suivant l'équation (8). Une approximation uniformément valable est alors donnée par l'équation (9).

La méthode des développements successifs complémentaires (MDSC) est décrite dans la Section 4. Le point de départ est de supposer l'existence d'une approximation uniformément valable (AUV) donnée par l'équation (10) sous une forme non régulière. La variable x est appropriée à l'étude de la région externe et la variable X est appropriée à l'étude de la région interne (couche limite). Une forme régulière de l'AUV est donnée par l'équation (11). Une propriété essentielle de cette approximation est de se présenter comme la somme de deux développements asymptotiques réguliers (équations (12)). Les fonctions Φ_0 et Φ_1 sont régulières ; l'analyse du comportement de Φ_0 au voisinage de $x = 0$ et celui de Φ_1 pour $X \to \infty$ conduit aux relations (19). On en déduit finalement les relations (20), (21) qui montrent que le principe de raccordement modifié est obtenu. Ainsi, ce principe est en fait un résultat de la méthode proposée pour construire une AUV.

La Section 5 étudie la solution de l'équation (25) avec les conditions aux limites (26). Les approximations obtenues au premier et au second ordre avec la MDSC sont données par les équations (28) et (30) ; elles sont en excellent accord avec la solution numérique de l'équation complète même pour des valeurs élevées de ε (Fig. 1). En outre, les difficultés liées à la présence de logarithmes dans la méthode des développements asymptotiques raccordés n'apparaissent pas dans la mesure où des developpements non réguliers sont acceptés.

1. Introduction

In mathematical physics, approximation methods are very useful for analysing the evolution of the solution to a problem controlled by a small parameter. The solution to regular perturbation problems is straightforward and a uniformly valid approximation is easily obtained from a series expansion. When the perturbation is singular, the straightforward perturbation solution is not uniformly valid in the domain of definition of the problem and two questions arise. The first one is to determine the structure of the solution, i.e., the definition of significant domains with appropriate variables and with an appropriate expansion of the solution. The second question is the construction of a uniformly valid approximation. To answer these questions, the method of matched asymptotic expansions (MMAE) is a powerful tool for certain classes of problems. A typical example is a second order differential equation where the second derivative of the unknown function is multiplied by a small parameter.

A new method – called Successive Composite Expansions Method (SCEM) – leading to the construction of uniformly valid approximations (UVA) without using an asymptotic principle is proposed. It is shown that a matching principle is contained in this method. An example of application of the SCEM to a singularly perturbed second order differential equation is presented.

2. Definitions

The behaviour of a function $\Phi(x, \varepsilon)$ defined on a domain $D \times [0, \varepsilon_0]$ is investigated when $\varepsilon \to 0$. An asymptotic expansion is defined by:

$$\Phi_{an}(x, \varepsilon) = \sum_{i=1}^{n} \delta^{(i)}(\varepsilon)\Phi^{(i)}(x, \varepsilon) \quad \text{with } \Phi(x, \varepsilon) = \Phi_{an}(x, \varepsilon) + o\left[\delta^{(n)}(\varepsilon)\right] \tag{1}$$

where Φ_{an} is a uniformly valid approximation (UVA) of Φ to the order $\delta^{(n)}$. The set $\delta^{(i)}(\varepsilon)$ is an asymptotic sequence of order functions or gauge functions. A gauge function is a suitable representative order function chosen in each equivalence class of order functions.

If a regular asymptotic expansion can be constructed, the expansion operator $\mathrm{E}^{(n)}$ to the order $\delta^{(n)}$ is defined according to Eckhaus [1] by:

$$\mathrm{E}^{(n)}\Phi = \sum_{i=1}^{n} \delta^{(i)}(\varepsilon)\Phi^{(i)}(x) \quad \text{with } \Phi(x,\varepsilon) = \mathrm{E}^{(n)}\Phi + \mathrm{o}\left[\delta^{(n)}(\varepsilon)\right] \tag{2}$$

Expansion (2) is said *regular* because $\Phi^{(i)}$ is function of x only. Simply by contrast, expansion (1) is said *nonregular*.

Now, for the sake of simplicity, the one-dimensional case is considered where $D = [0, 1]$ and where Φ is regular except at the origin $x = 0$. In this case, a singular perturbation problem occurs. Near $x = 0$, local variables and local expansions are introduced:

$$\mathrm{E}_{\nu}^{(n)}\Phi = \sum_{i=1}^{n} \delta_{\nu}^{(i)}(\varepsilon)\Phi_{\nu}^{(i)}(x_{\nu}) \quad \text{with } \Phi - \mathrm{E}_{\nu}^{(n)}\Phi = \mathrm{o}\left(\delta_{\nu}^{(n)}\right) \text{ and } x_{\nu} = \frac{x}{\delta_{\nu}(\varepsilon)} \tag{3}$$

Here, $\delta_{\nu}(\varepsilon) = \mathrm{o}(1)$ except for $\nu = 0$ in which case, $\delta_0(\varepsilon) = 1$ and $x_0 = x$. The asymptotic set δ_{ν} is such that $\nu_1 < \nu_2$ implies $\delta_{\nu_2} \prec \delta_{\nu_1}$. If $\mathrm{E}_{\nu}^{(n)}$ is an expansion operator to the prescribed order δ, it is not necessary to specify the number of terms n. It is sufficient to write $\mathrm{E}_{\nu}\Phi$ to the considered order δ:

$$\Phi - \mathrm{E}_{\nu}\Phi = \mathrm{o}(\delta) \tag{4}$$

When there is only one boundary layer in D where the significant variable is for example $\varepsilon x_1 = \varepsilon X = x$, the x-expansion $\mathrm{E}_0\Phi$ is commonly called the outer expansion and the X-expansion $\mathrm{E}_1\Phi$ is called the inner expansion. The x_{ν}-expansions $\mathrm{E}_{\nu}\Phi$ such that $\varepsilon \prec \delta_{\nu} \prec 1$ are called intermediate expansions.

3. An asymptotic matching principle

The matching process establishes a link between the different significant expansions. The first known process has been proposed by Kaplun and Lagerstrom [2]. The idea is to assume that the extended domains of validity of the outer and inner expansions have a common domain of validity; this is the so-called overlap hypothesis:

$$\mathrm{E}_{\nu}\mathrm{E}_0^{(n)}\Phi = \mathrm{E}_{\nu}\mathrm{E}_1^{(m)}\Phi \tag{5}$$

In the above expression, the number of terms in each expansion must be specified because the approximations are not necessarily defined to the same order. When order functions are used in place of gauge functions, the equality must be replaced by an asymptotic equality such as $\approx$. As it will be seen later, it may happen that no overlap exists and then an intermediate matching is not possible [3,4].

A second matching process is the Van Dyke principle (VDP) [5]:

$$\mathrm{E}_1^{(m)}\mathrm{E}_0^{(n)}\Phi = \mathrm{E}_0^{(n)}\mathrm{E}_1^{(m)}\Phi \tag{6}$$

Computations are easier than with intermediate variables, and so this principle has been very popular in applications. In fact, Eckhaus demonstrated that this principle can be deduced from the overlap hypothesis but this hypothesis is not necessary for the validity of the VDP.

Counter-examples to the above principles are known. For instance, with the function:

$$\Phi(x,\varepsilon) = \frac{1}{\ln x} + \frac{\mathrm{e}^{-x/\varepsilon}}{\ln \varepsilon} + \frac{1}{\ln x - \ln \varepsilon + 1} \tag{7}$$

it is easy to see that there is no overlap. Moreover, for $(m = 1, n = 2)$ or $(n = 1, m = 2)$, the VDP does not work. To remove this failure, a modified Van Dyke principle (MVDP) is proposed. *If* $\mathrm{E}_0\Phi$ *and* $\mathrm{E}_1\Phi$ *are the outer and inner expansions to a given order* δ*, the formal rule of matching is*:

$$\mathrm{E}_1\mathrm{E}_0\Phi = \mathrm{E}_0\mathrm{E}_1\Phi \tag{8}$$

In the case of a single boundary layer, a uniformly valid approximation in D is obtained from the composite expansion:

$$\Phi_{\mathrm{app}} = \mathrm{E}_0\Phi + \mathrm{E}_1\Phi - \mathrm{E}_0\mathrm{E}_1\Phi \tag{9}$$

where each term is defined *to the same order* δ. This is an essential feature and a striking difference with the method proposed by Van Dyke in which the different terms of the composite expansion are not necessarily defined to the same order.

It is very interesting to note the thought process of Van Dyke: *"Fortunately, since the two expansions have a common region of validity, it is easy to construct from them a single uniformly valid expansion"*. Considering the examples where there is no overlap region, the opposite is believed to be true. *The structure of a uniformly valid expansion must firstly be assumed and then a method of constructing this expansion is deduced.* This can be the method of multiple scales, the WKB method [6] or any other method.

4. The successive complementary expansions method (SCEM)

According to the SCEM, a UVA of the function Φ to the order $\bar{\delta}_n$ is assumed to be:

$$\Phi_{\mathrm{a}}(x, X, \varepsilon) = \sum_{i=1}^{n} \bar{\delta}_i(\varepsilon)\left[\bar{\varphi}_i(x, \varepsilon) + \bar{\Psi}_i(X, \varepsilon)\right] \tag{10}$$

This method has some resemblance with the method of multiple scales since the expansion contains the two variables x and X but the heaviness is removed because the x- and X-variables are not contained in the same functions. Moreover, as can be seen in applications, the appearance of ε in the functions is essential. Nevertheless, the function Φ_{a} can be expanded as:

$$\Phi_{\mathrm{a}} = \Phi_{\mathrm{ar}} + \mathrm{o}(\delta_m) \quad \text{with } \Phi_{\mathrm{ar}}(x, X, \varepsilon) = \sum_{i=1}^{m} \delta_i(\varepsilon)\left[\varphi_i(x) + \psi_i(X)\right] \tag{11}$$

The set of order functions $\bar{\delta}_i$ can be the same as the set δ_i but this is not necessary since new terms can appear and since the functions δ_i are now gauge functions such as $\bar{\delta}_n = \mathrm{O}(\delta_m)$. In this regular case, similar attempts are described by Nayfeh [6] but their interest was not fully appreciated and the methods were not further exploited.

The essential property of expansion (11) is that the UVA is the sum of *two regular expansions*:

$$\Phi_0(x, \varepsilon) = \sum_{i=1}^{m} \delta_i(\varepsilon)\varphi_i(x) \quad \text{and} \quad \Phi_1(X, \varepsilon) = \sum_{i=1}^{m} \delta_i(\varepsilon)\psi_i(X) \tag{12}$$

Then, to the order δ_m, the following identities hold:

$$\Phi_0 = \mathrm{E}_0\Phi_0 \quad \text{and} \quad \Phi_1 = \mathrm{E}_1\Phi_1 \tag{13}$$

Thus, it can be written:

$$\mathrm{E}_0\Phi_{\mathrm{ar}} = \mathrm{E}_0\Phi_0 + \mathrm{E}_0\Phi_1 \quad \text{and} \quad \mathrm{E}_1\Phi_{\mathrm{ar}} = \mathrm{E}_1\Phi_0 + \mathrm{E}_1\Phi_1 \tag{14}$$

leading to,

$$\Phi_{\mathrm{ar}} = \mathrm{E}_0\Phi_{\mathrm{ar}} + \mathrm{E}_1\Phi_{\mathrm{ar}} - (\mathrm{E}_0\Phi_1 + \mathrm{E}_1\Phi_0) \tag{15}$$

Using once again the same process, it is obtained:

$$\mathrm{E}_0\mathrm{E}_1\Phi_{\mathrm{ar}} = \mathrm{E}_0\Phi_1 + \mathrm{E}_0\mathrm{E}_1\Phi_0 \quad \text{and} \quad \mathrm{E}_1\mathrm{E}_0\Phi_{\mathrm{ar}} = \mathrm{E}_1\Phi_0 + \mathrm{E}_1\mathrm{E}_0\Phi_1 \tag{16}$$

which gives:

$$\mathrm{E}_0\Phi_1 + \mathrm{E}_1\Phi_0 = \mathrm{E}_0\mathrm{E}_1\Phi_{\mathrm{ar}} + \mathrm{E}_1\mathrm{E}_0\Phi_0 - \mathrm{E}_0\mathrm{E}_1\Phi_0 \tag{17}$$

and:

$$\mathrm{E}_0\Phi_1 + \mathrm{E}_1\Phi_0 = \mathrm{E}_1\mathrm{E}_0\Phi_{\mathrm{ar}} + \mathrm{E}_0\mathrm{E}_1\Phi_1 - \mathrm{E}_1\mathrm{E}_0\Phi_1 \tag{18}$$

The analysis of the behaviour of Φ_0 for $x \to 0$ and Φ_1 for $X \to \infty$ gives:

$$\mathrm{E}_0\Phi_1 = \mathrm{E}_0\mathrm{E}_1\Phi_1 \equiv \mathrm{E}_1\mathrm{E}_0\Phi_1 \quad \text{and} \quad \mathrm{E}_0\mathrm{E}_1\Phi_0 \equiv \mathrm{E}_1\mathrm{E}_0\Phi_0 = \mathrm{E}_1\Phi_0 \tag{19}$$

which is nothing else but the VDP principle for regular functions. It is concluded that:

$$\mathrm{E}_0\Phi_1 + \mathrm{E}_1\Phi_0 = \mathrm{E}_0\mathrm{E}_1\Phi_{\mathrm{ar}} = \mathrm{E}_1\mathrm{E}_0\Phi_{\mathrm{ar}} \tag{20}$$

showing the structure of the standard matching:

$$\Phi_{\mathrm{ar}} = \mathrm{E}_0\Phi_{\mathrm{ar}} + \mathrm{E}_1\Phi_{\mathrm{ar}} - \mathrm{E}_0\mathrm{E}_1\Phi_{\mathrm{ar}} = \mathrm{E}_0\Phi_{\mathrm{ar}} + \mathrm{E}_1\Phi_{\mathrm{ar}} - \mathrm{E}_1\mathrm{E}_0\Phi_{\mathrm{ar}} \tag{21}$$

Thus, the asymptotic matching principle is contained into the SCEM.

As an example, consider the function Φ defined by Eq. (7). To the order $-1/\ln\varepsilon$, the first outer approximation is:

$$\varphi_1(x,\varepsilon) = \frac{1}{\ln x} - \frac{1}{\ln\varepsilon} \tag{22}$$

Since this approximation is not uniformly valid, a UVA to the order $-1/\ln\varepsilon$ is looked for as:

$$\Phi_{a1}(x,X,\varepsilon) = \varphi_1(x,\varepsilon) + \psi_1(X,\varepsilon) \quad \text{with } X = \frac{x}{\varepsilon} \tag{23}$$

From $E_1(\Phi - \varphi_1)$, it is seen that:

$$\Psi_1(X,\varepsilon) = \frac{1}{\ln X + 1} + \frac{1+e^{-X}}{\ln\varepsilon} \quad \text{and} \quad \Phi_{a1}(x,X,\varepsilon) = \frac{1}{\ln x} + \frac{1}{\ln X + 1} + \frac{e^{-X}}{\ln\varepsilon} \tag{24}$$

where Φ_{a1} is a UVA to the considered order. In fact, Φ_{a1} is the exact solution.

5. Application to the Oseen–Stokes model

The equation to solve is:

$$\frac{d^2\Phi}{dx^2} + \frac{1}{x}\frac{d\Phi}{dx} + \Phi\frac{d\Phi}{dx} = 0 \tag{25}$$

where the function Φ is defined in the domain $x \geqslant \varepsilon$ with the boundary conditions:

$$x = \varepsilon : \Phi = 0, \quad x \to \infty : \Phi = 1 \tag{26}$$

A regular outer approximation is looked for as $1 + \delta_1(\varepsilon)\bar{\varphi}_1(x)$ where δ_1 is an order function such as $\delta_1 \to 0$ when $\varepsilon \to 0$. The $\bar{\varphi}_1$-equation is:

$$\frac{d^2\bar{\varphi}_1}{dx^2} + \frac{1}{x}\frac{d\bar{\varphi}_1}{dx} + \frac{d\bar{\varphi}_1}{dx} = 0 \tag{27}$$

The solution of Eq. (27) satisfying the boundary conditions exactly is such that:

$$\bar{\Phi}_{a1} = 1 - \frac{E_1(x)}{E_1(\varepsilon)} \quad \text{with } E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \text{ and } \delta_1(\varepsilon) = \frac{1}{E_1(\varepsilon)} \tag{28}$$

With the SCEM, a UVA is obtained by adding to $\bar{\Phi}_{a1}$ a function $\delta_1(\varepsilon)\bar{\psi}_1(X,\varepsilon)$ with $X = x/\varepsilon$. It is shown that $\bar{\psi}_1 = 0$ so that $\bar{\Phi}_{a1}$ is a first UVA. Chen et al. [7] obtained the same result using the renormalization group.

The behaviour of $E_1(\varepsilon)$ for small values of ε is: $E_1(\varepsilon) \approx -\ln\varepsilon - \gamma + \varepsilon + \cdots$ with $\gamma = 0.577215$ (Euler constant). It is also shown that a second order UVA is: $\bar{\Phi}_{a2} = 1 - E_1(x)/E_1(\varepsilon) + \delta_2(\varepsilon)\bar{\varphi}_2(x,\varepsilon)$. The $\bar{\varphi}_2$-equation is:

$$\frac{d^2\bar{\varphi}_2}{dx^2} + \frac{1}{x}\frac{d\bar{\varphi}_2}{dx} + \frac{d\bar{\varphi}_2}{dx} + \frac{1}{\delta_2 E_1^2(\varepsilon)}E_1(x)\frac{dE_1(x)}{dx} = 0 \tag{29}$$

The solution of Eq. (29) such that $\bar{\Phi}_{a2}$ satisfies the boundary conditions leads to:

$$\bar{\Phi}_{a2} = 1 - \frac{E_1(x)}{E_1(\varepsilon)} + \frac{F_1(x)}{E_1^2(\varepsilon)} - \frac{F_1(\varepsilon)E_1(x)}{E_1^3(\varepsilon)}, \quad F_1(x) = 2E_1(2x) - e^{-x}E_1(x) \tag{30}$$

with the choice $\delta_2 = F_1(\varepsilon)/E_1^3(\varepsilon)$. The behaviour of $F_1(\varepsilon)$ for small values of ε is: $F_1(\varepsilon) \approx -\ln\varepsilon - \gamma - 2\ln 2 - \varepsilon\ln\varepsilon + (3-\gamma)\varepsilon + \cdots$

The results given in Fig. 1 show that $\bar{\Phi}_{a1}$ and $\bar{\Phi}_{a2}$ are excellent approximations of the numerical solution of the exact equation even for large values of ε. It should be noted that the order functions δ_1 and δ_2 are determined by applying the exact boundary conditions. On the contrary, when using the MMAE, there are many choices for the order functions and the range of accuracy of the approximations depends strongly on the choice of the order functions [8,9]. A standard choice for the order functions is $-1/\ln\varepsilon$ and $1/(\ln\varepsilon)^2$; with this choice, it is clear that the approximation can not be valid if $\varepsilon = 1$ and the accuracy of the approximation can be good only for small values of ε. It can be noted that logarithmic order functions appear if E_1 and F_1 are expanded for small values of ε. It is also stressed that the SCEM eliminates the difficulty of logarithms thanks to the use of appropriate order functions and nonregular expansions.

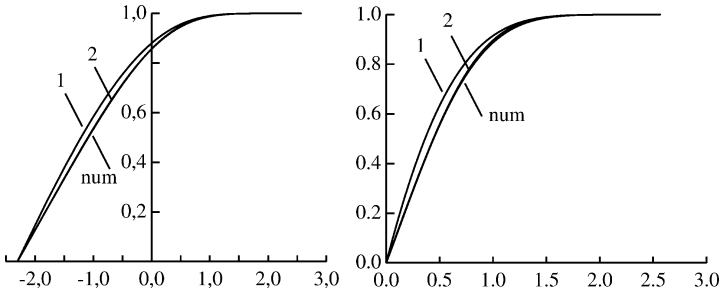

Figure 1. Stokes–Oseen model. Comparison of the exact solution (numerical solution) with SCEM solutions: 1 = first order solution (Eq. (28)), 2 = second order solution (Eq. (30)).

Figure 1. *Modèle de Stokes–Oseen. Comparaison de la solution exacte* (*solution numérique*) *avec les solutions MDSC* : 1 = *solution de premier ordre* (*équation* (28)), 2 = *solution de deuxième ordre* (*équation* (30)).

6. Discussion and conclusion

Other applications of the Successive Complementary Expansions Method (SCEM) than those presented in this paper have been performed. In particular, the SCEM has been applied with success to differential equations. In its regular form, the application of the SCEM is as straightforward as the standard method of matched asymptotic expansions. Indeed, the final results are identical. The application of the SCEM in its nonregular form is more involved and the user has to think a little more. The advantage, however, is that more flexibility is allowed. In particular, it is possible to concentrate more information in the first terms of the expansion. This is certainly an interesting feature because the series used in the asymptotic methods are very often divergent. In addition, these methods are applicable when the small parameter (let us say ε) goes to zero. In practice, however, it is desirable to apply the results when ε is not really small. In this respect, it is valuable to include the first elements of the expansion terms which are negligible when ε goes to zero but which are no longer negligible when ε is not really small. In the same way, it is interesting to respect the boundary conditions exactly and not asymptotically.

The SCEM has also been applied to study different approximations of Navier–Stokes equations at high Reynolds number. This application leads to the notion of interactive boundary layer. If terms negligible when the Reynolds number goes to infinity are included in the boundary layer equations, it appears that the inviscid equations can not be solved independently from the boundary layer equations; the extended boundary layer equations and the inviscid flow equations must be solved simultaneously. It is also shown that the model of the interactive boundary layer contains the standard boundary layer theory and the triple deck theory [10].

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