C. R. Mecanique 330 (2002) 735–740

Asymptotic analysis of integral equations for a great interval and its application to stellar radiative transfer

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Received 14 June 2002; accepted after revision 17 September 2002

Note presented by Évariste Sanchez-Palencia.

Abstract We consider the integral form of the radiative transfer equation over a large interval. This equation describes the radiative transfer of energy in a star. The asymptotic expansion of the solution is constructed and justified. The method of asymptotic partial decomposition of domain is applied. Numerical results are discussed. ***To cite this article: G. Panasenko et al., C. R. Mecanique 330 (2002) 735–740.***

waves / integral radiative transfer equation / great integration interval / asymptotic partial decomposition of domain

Analyse asymptotique d'une équation intégrale sur un grand intervalle et son application au transfert radiatif stellaire

Résumé Nous considérons la forme intégrale de l'équation de transfert sur un grand intervalle. Cette équation décrit le transfert radiatif de l'énergie dans une étoile. Nous construisons et justifions le développement asymptotique de la solution lorsque la longueur de l'intervalle d'intégration tend vers l'infini. La méthode de décomposition asymptotique partielle du domaine est appliquée. Les résultats numériques sont discutés. ***Pour citer cet article : G. Panasenko et al., C. R. Mecanique 330 (2002) 735–740.***

ondes / équation intégrale du transfert stellaire / grand intervalle d'intégration / méthode de décomposition asymptotique partielle du domaine

1. Introduction

Models of stellar atmospheres are described by complex non-linear systems of equations. The main part of these systems are represented by a great number of linear equations (see [1]), so the problem of fast solvers for these equations is very important. Multiple scattering of radiation is described by an integral equation

$$S_{\tau_0}(\tau) = \frac{1}{2}\varpi_{\tau_0}(\tau)\int_0^{\tau_0} \mathcal{E}\big(|\tau-\sigma|\big) S_{\tau_0}(\sigma)\,\mathrm{d}\sigma + F_{\tau_0}(\tau), \quad \tau \in [0,\tau_0] \tag{1}$$

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S1631-0721(02)01525-5/FLA

posed on a great interval $[0, \tau_0]$, $\tau_0 \gg 1$. The length τ_0 of the integration interval corresponds to the optical thickness of the atmosphere and usually it is a great parameter. S_{τ_0} is the unknown function of the optical depth variable τ, depending on τ_0. $\mathcal{E}$ is an exponentially decaying continuous function on $]0, +\infty[$ which admits a weak singularity at zero:

$$\forall r \in]0, +\infty[, \quad 0 < \mathcal{E}(r) < M \exp(-\alpha r), \quad M > 0,\ \alpha > 0 \tag{2}$$

$$\int_0^{+\infty} \mathcal{E}(r)\,\mathrm{d}r = 1 \tag{3}$$

If one applies directly some numerical methods (see [1–3]) by discretization of the interval $[0, \tau_0]$ with a step h, then the integral equation is reduced to a linear algebraic system of equations. Its dimension is $\tau_0/h \gg 1$. In astrophysics, τ_0 can be as great as 10^9. In these situations the dimension of the discretized algebraic system is tremendous. On the other hand, we assume that ϖ_{τ_0} (the albedo) only depends on τ/τ_0, i.e., $\varpi_{\tau_0} = \varpi(\tau/\tau_0)$. It is the case for example when the albedo is given on a grid of a uniformly distributed nodes on $[0, \tau_0]$, so that ϖ_{τ_0} is some interpolation of these known values. Free term F_{τ_0} will be written in the following form

$$F_{\tau_0}(\tau) := F_\ell(\tau) + \left(1 - \varpi\left(\frac{\tau}{\tau_0}\right)\right) F_0\left(\frac{\tau}{\tau_0}\right) + F_r(\tau - \tau_0) \tag{4}$$

where F_l and F_r describe the contribution of the external sources (via the boundary conditions) and F_0 the thermal emission of the internal sources. In our model, functions F_l and F_r only depend on the distances τ and $(\tau_0 - \tau)$ to the nearest boundary respectively. They are thus independent of τ_0. Moreover, we suppose that F_l and F_r are exponentially decaying measurable functions in the following sense:

$$\exists c_1, c_2 > 0,\ \forall \tau \in [0, +\infty[, \quad \left|F_\ell(\tau)\right| \leqslant c_1 \exp(-c_2\tau), \quad \forall \tau \in]-\infty, 0], \quad \left|F_r(\tau)\right| \leqslant c_1 \exp(c_2\tau) \tag{5}$$

Functions ϖ and F_0 are also independent of τ_0; they are defined on $[0, 1]$ where they are assumed to be sufficiently regular. Moreover we suppose that

$$\exists \kappa > 0,\ \forall t \in [0, 1] \quad 0 \leqslant \varpi(t) < 1 - \kappa < 1 \tag{6}$$

In term of variable $t := \tau/\tau_0$, Eq. (1) is reformulated in the following manner:

$$S_\delta(t) = \frac{\varpi(t)}{2\delta} \int_0^1 \mathcal{E}\left(\frac{|t-s|}{\delta}\right) S_\delta(s)\,\mathrm{d}s + F_\ell\left(\frac{t}{\delta}\right) + \left(1 - \varpi(t)\right) F_0(t) + F_r\left(\frac{t-1}{\delta}\right) \tag{7}$$

where we defined $\delta := 1/\tau_0 \ll 1$ and $S_\delta(t) := S_{\tau_0}(t\tau_0)$.

Thanks to hypothesis (2)–(6), the existence and uniqueness of the solution in $L^\infty([0, 1])$ is an immediate consequence of the fixed point theorem of contractant operators.

2. Asymptotic analysis

As τ_0 tends to infinity, the previous hypotheses allow us to construct an asymptotic expansion $S^{[m]}$ of solution S_δ with the following structure:

$$S^{[m]}(t) := S^\delta_{\mathrm{reg}}(t) + S^\ell_{\mathrm{BL}}\left(\frac{t}{\delta}\right) + S^r_{\mathrm{BL}}\left(\frac{t-1}{\delta}\right), \quad t \in [0, 1] \tag{8}$$

where S^δ_{reg} is a regular series in powers of δ^2, calculated explicitly. S^ℓ_{BL} et S^r_{BL} are two series in power of δ which represent two boundary layers. Their respective coefficients $S^\ell_{\mathrm{BL}\,j}(t/\delta)$ and $S^r_{\mathrm{BL}\,j}((1-t)/\delta)$ are functions which satisfy the following estimations:

$$\begin{aligned} &\exists c_{1j}, c_{2j} > 0,\ \forall \tau \in [0, +\infty], \quad \left|S^\ell_{\mathrm{BL}\,j}(\tau)\right| \leqslant c_{1j} \exp(-c_{2j}\tau) \\ &\forall \tau \in]-\infty, 0], \quad \left|S^r_{\mathrm{BL}\,j}(\tau)\right| \leqslant c_{1j} \exp(c_{2j}\tau) \end{aligned} \tag{9}$$

In particular, the following theorem could be proved:

THEOREM 2.1. – *Let* ϖ, $F_0 \in C^1([0,1])$. *For all* $t \in [0,1]$, *we define*

$$S^{[0]}(t) := F_0(t) + S^\ell_{\mathrm{BL0}}\left(\frac{t}{\delta}\right) + S^r_{\mathrm{BL0}}\left(\frac{t-1}{\delta}\right) \tag{10}$$

where S^ℓ_{BL0} *and* S^r_{BL0} *are the solutions of the following equations*:

$$S^\ell_{\mathrm{BL0}}(\tau) = \frac{\varpi(0)}{2}\int_0^{+\infty} \mathcal{E}\big(|\tau-\sigma|\big) S^\ell_{\mathrm{BL0}}(\sigma)\,\mathrm{d}\sigma + F_\ell(\tau) - \frac{\varpi(0)F_0(0)}{2}\int_\tau^{+\infty} \mathcal{E}(r)\,\mathrm{d}r, \quad \tau \in \mathbb{R}_+$$

$$S^r_{\mathrm{BL0}}(\tau) = \frac{\varpi(1)}{2}\int_{-\infty}^{0} \mathcal{E}\big(|\tau-\sigma|\big) S^r_{\mathrm{BL0}}(\sigma)\,\mathrm{d}\sigma + F_r(\tau) - \frac{\varpi(1)F_0(1)}{2}\int_{|\tau|}^{+\infty} \mathcal{E}(r)\,\mathrm{d}r, \quad \tau \in \mathbb{R}_-$$

which can be exactly solved (*see* [4]). *Then*

$$\big\|S^{[0]} - S_\delta\big\|_{L^\infty([0,1])} = \mathcal{O}(\delta) \tag{11}$$

This result can be generalized in case when function ϖ is smooth everywhere with exception of one point t_0 in the interval $]0,1[$. In that case, an additional exponentially decaying boundary layer appears in the neighborhood of this point.

3. Asymptotic decomposition of domain

Applying the described above asymptotic information we construct a version of the method of partial asymptotic decomposition of domain (MPADD) (see [5]) in order to reduce problem (1) to the resolution of two integral equations posed on intervals much smaller than $[0, \tau_0]$. We suppose that ϖ, $F_0 \in C^{m+1}([0,1])$, where $m \in \mathbb{N}$. We define, for all $t \in [0,1]$,

$$S^m_{\mathrm{reg}}(t) := \sum_{k=0}^{[m/2]} \delta^{2k} S_{2k}(t) \tag{12}$$

where

$$S_0(t) := F_0(t) \quad \text{and} \quad S_{2k}(t) := \frac{\varpi(t)}{1-\varpi(t)} \sum_{j=1}^{k} \frac{I_{2j}}{(2j)!} \frac{\mathrm{d}^{2j} S_{2(k-j)}}{\mathrm{d}t^{2j}}(t), \quad k > 0$$

with $I_{2j} := \int_0^{+\infty} \mathcal{E}(r) r^{2j}\,\mathrm{d}r$. Define, for all $\widehat{m} \in]0,+\infty[$, $d := \widehat{m}|\ln\delta|$. Let V^m_ℓ be the solution of

$$V^m_\ell(\tau) = \frac{\varpi(\tau/\tau_0)}{2}\int_0^{d} \mathcal{E}\big(|\tau-\sigma|\big) V^m_\ell(\sigma)\,\mathrm{d}\sigma + \mathcal{F}^m_\ell(\tau), \quad \tau \in [0,d] \tag{13}$$

where

$$\mathcal{F}^m_\ell(\tau) := \frac{\varpi(\tau/\tau_0)}{2}\int_0^{\tau_0} \mathcal{E}\big(|\tau-\sigma|\big) S^m_{\mathrm{reg}}(\sigma/\tau_0)\,\mathrm{d}\sigma - S^m_{\mathrm{reg}}(\tau/\tau_0) + \big(1-\varpi(\tau/\tau_0)\big) F_0(\tau/\tau_0) + F_\ell(\tau) \tag{14}$$

and V^m_r the solution of

$$V^m_r(\tau) = \frac{\varpi(\tau/\tau_0)}{2}\int_{\tau_0-d}^{\tau_0} \mathcal{E}\big(|\tau-\sigma|\big) V^m_r(\sigma)\,\mathrm{d}\sigma + \mathcal{F}^m_r(\tau), \quad \tau \in [\tau_0-d, \tau_0] \tag{15}$$

where

$$\mathcal{F}^m_r(\tau) := \frac{\varpi(\tau/\tau_0)}{2}\int_0^{\tau_0} \mathcal{E}\big(|\tau-\sigma|\big) S^m_{\mathrm{reg}}(\sigma/\tau_0)\,\mathrm{d}\sigma - S^m_{\mathrm{reg}}(\tau/\tau_0) + \big(1-\varpi(\tau/\tau_0)\big) F_0(\tau/\tau_0) + F_r(\tau-\tau_0) \tag{16}$$

THEOREM 3.1. – *Let S_d^m be defined for all $\tau \in [0, \tau_0]$ by*

$$S_d^m(\tau) := \begin{cases} V_\ell^m(\tau) + S_{\text{reg}}^m(\tau/\tau_0) & \text{if } \tau \in [0, d] \\ S_{\text{reg}}^m(\tau/\tau_0) & \text{if } \tau \in]d, \tau_0 - d[\\ V_r^m(\tau) + S_{\text{reg}}^m(\tau/\tau_0) & \text{if } \tau \in [\tau_0 - d, \tau_0] \end{cases} \tag{17}$$

Then for all $m \in \mathbb{N}$ there exists $\widehat{m} \in]0, +\infty[$ such that

$$\left\| S_d^m - S_{\tau_0} \right\|_{L^\infty([0,\tau_0])} = \mathcal{O}\left(\tau_0^{-m-1}\right) \tag{18}$$

Problem (1) reduces to the solution of (13) and (15), that needs less time, and less memory capacity. This method allows us to solve Eq. (1) when τ_0 is very large and to accelerate its resolution. These points are fundamental in astrophysical applications (see [1,2]).

4. Numerical examples

Here we present some numerical experiments for the method of asymptotic partial decomposition of domain described above. We consider problem (1) with the following data:

$$\varpi_{\tau_0}(\tau) = \varpi_\star \exp(-\tau/\tau_0), \qquad \mathcal{E}(r) = E_1(r) := \int_0^1 \frac{\exp(-r/\mu)}{\mu}\,\mathrm{d}\mu, \qquad F_0(\tau) = 1$$
$$F_\ell(\tau) = 0, \qquad F_r(\tau) = \exp(\tau), \qquad m = 0 \tag{19}$$

These data satisfy conditions (2)–(6). E_1 is the first exponential integral function (see [6]). In this case S_{reg}^0, $\mathcal{F}_\ell^0$ and $\mathcal{F}_r^0$ are given by

$$S_{\text{reg}}^0(\tau) = 1, \qquad \mathcal{F}_\ell^0(\tau) = -\varpi_\star \exp(-\tau/\tau_0)\big(E_2(\tau) + E_2(\tau_0 - \tau)\big) + \exp(-\tau)$$
$$\mathcal{F}_\ell^0(\tau) = -\varpi_\star \exp(-\tau/\tau_0)\big(E_2(\tau) + E_2(\tau_0 - \tau)\big) + \exp(\tau - \tau_0) \tag{20}$$

where E_2 is the second exponential integral function (see [6]). Moreover, $\|S_{\tau_0}\|_{L^\infty([0,\tau_0])} = 1$. In order to compare this method with a direct discretization method, we compute this example with $\tau_0 = 100$. Figs. 1–3

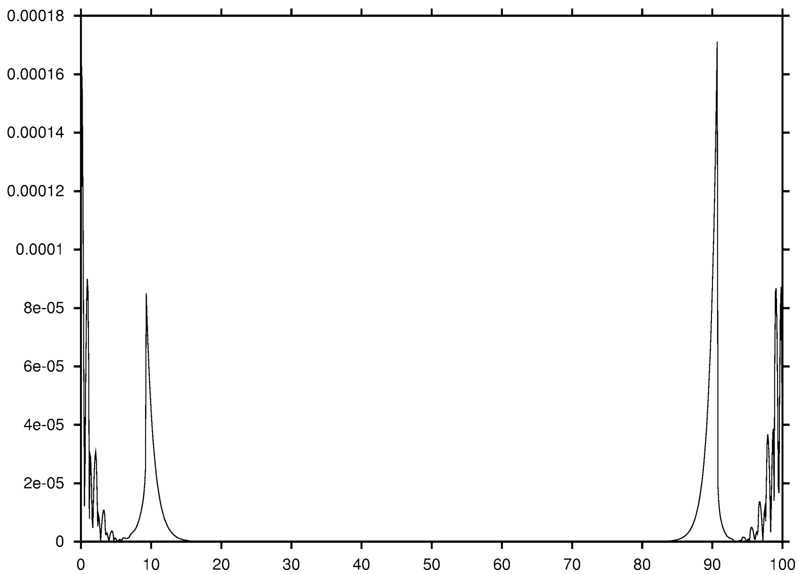

Figure 1. Relative error between the MPADD solution computed with $\varpi_\star = 0.7$, $\widehat{m} = 2$ and the solution obtained by the fine mesh (1000 nodes) direct discretization method.

Figure 1. *Erreur relative entre la solution MPADD calculée avec* $\varpi_\star = 0{,}7$, $\widehat{m} = 2$ *et la solution obtenue avec une méthode de discrétisation directe fine* (1000 *nœuds*).

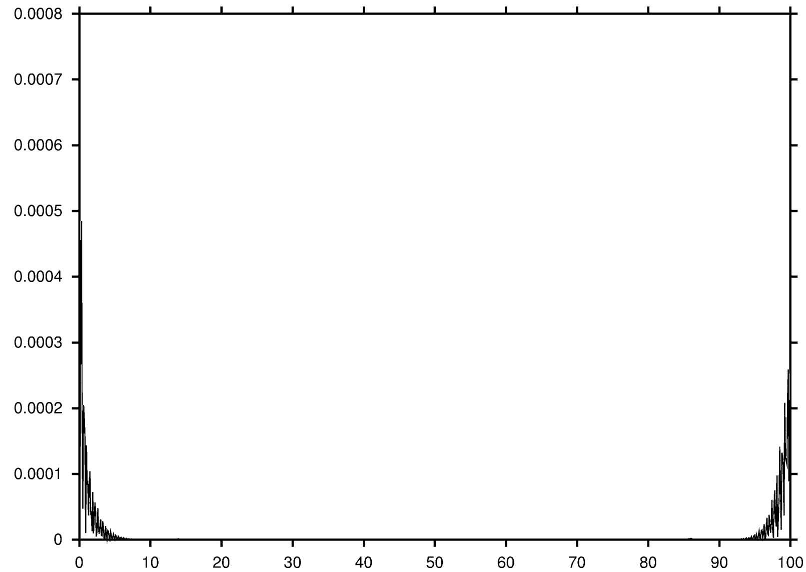

Figure 2. Relative error between the MPADD solution computed with $\varpi_\star = 0.7$, $\widehat{m} = 3$ and the solution obtained by the fine mesh (1000 nodes) direct discretization method.

Figure 2. *Erreur relative entre la solution MPADD calculée avec* $\varpi_\star = 0{,}7$, $\widehat{m} = 3$ *et la solution obtenue avec une méthode de discrétisation directe fine* (1000 *nœuds*).

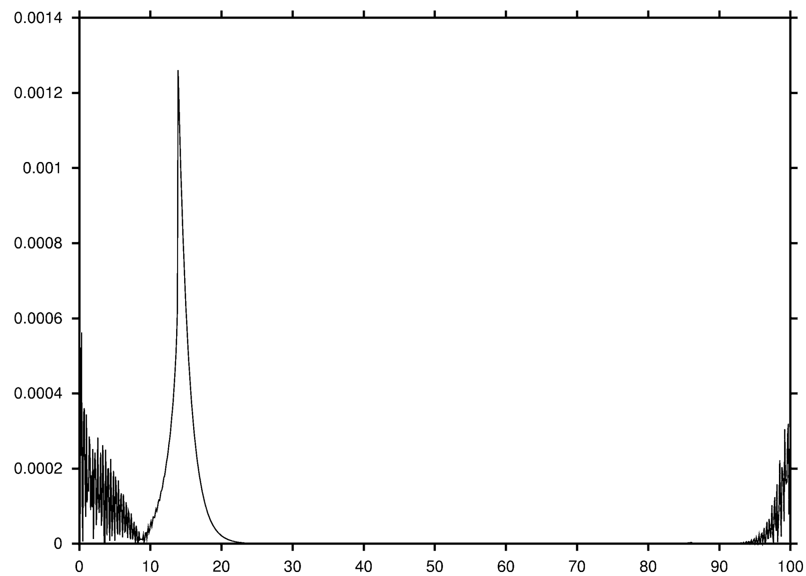

Figure 3. Relative error between the MPADD solution computed with $\varpi_\star = 0.999$, $\widehat{m} = 3$ and the solution obtained by the fine mesh (1000 nodes) direct discretization method.

Figure 3. *Erreur relative entre la solution MPADD calculée avec* $\varpi_\star = 0{,}999$, $\widehat{m} = 3$ *et la solution obtenue avec une méthode de discrétisation directe fine* (1000 *nœuds*).

show the relative error between the MPADD solution computed with different values of $\varpi_\star$ and $\widehat{m}$ and a numerical solution obtained by a fine mesh (1000 nodes) direct discretization method (see [3]). Figs. 1 and 2 show the important decay of the relative error when $\widehat{m}$ is changed from 2 to 3. Fig. 3 shows that the estimates become worse when $\varpi_\star$ is too close to 1.

Peaks on Figs. 1 and 3 correspond to the points of discontinuity of the MPADD solution (see Eq. (17)): they decrease when $\widehat{m}$ increases.

Finally, as the MPADD involves the resolution of two small systems instead of a large one, this method is much faster than the direct method, as can be seen in Table 1.

Table 1. Comparison of computation times.
Tableau 1. *Comparaison des temps de calcul.*

	CPU time
MPADD with $\widehat{m} = 2$	4 s 340 ms
MPADD with $\widehat{m} = 3$	11 s 530 ms
Direct method	3 mn 8 s 630 ms

Acknowledgements. The authors are grateful to A. Amosov, M. Ahues and A. Largillier for fruitful discussions on this work.

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