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Yield criteria for porous media in plane strain: second-order estimates versus numerical results

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Abstract This Note presents a comparison of some recently developed "second-order" homogenization estimates for two-dimensional, ideally plastic porous media subjected to plane strain conditions with corresponding yield analysis results using a new linearization technique and systematically optimized finite elements meshes. Good qualitative agreement is found between the second-order theory and the yield analysis results for the shape of the yield surfaces, which exhibit a *corner* on the hydrostatic axis, as well as for the dependence of the effective flow stress in shear on the porosity, which is found to be *non-analytic* in the dilute limit. Both of these features are inconsistent with the predictions of the standard Gurson model. ***To cite this article: J. Pastor, P. Ponte Castañeda, C. R. Mecanique 330 (2002) 741–747.***

porous media / homogenization / limit analysis / optimization

Critère de plasticité des matériaux poreux en déformation plane : Estimations du second ordre et résultats numériques

Résumé Cette Note présente une comparaison entre d'une part, les estimations issues d'une récente théorie d'homogénéisation, dite de « deuxième ordre », pour les matériaux parfaitement plastiques poreux en déformation plane, et d'autre part, les résultats homologues obtenus par analyse limite grâce une nouvelle technique de linéarisation du problème et une optimisation systématique des maillages éléments finis utilisés. Qualitativement parlant on observe un bon accord entre les deux approches sur la forme de la surface limite, avec mise en évidence d'un point anguleux sur l'axe hydrostatique, et sur la dépendance de la contrainte équivalente en cisaillement avec la porosité, contrainte dont la limite pour les faibles porosités apparaît non analytique. Ces deux caractéristiques ne sont pas prévues par le modèle de Gurson standard. ***Pour citer cet article : J. Pastor, P. Ponte Castañeda, C. R. Mecanique 330 (2002) 741–747.***

matériaux poreux / homogénéisation / analyse limite / optimisation

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1. Introduction

Quite serendipitously, two recent works (Francescato et al. [1] and Ponte Castañeda [2,3]) have given what appear to be qualitatively similar predictions for the macroscopic yield criteria of porous media. Both works are concerned with in-plane deformation of two-dimensional porous media with ideally plastic matrices and yield anomalous predictions. By anomalous we mean different from the corresponding predictions of the well-known Gurson model [4]. Thus, both works suggest that the yield surfaces of such materials exhibit a corner on the hydrostatic axis. In addition, for shear loading and low porosities, they exhibit a *non-analytic* dependence of the effective flow strength on porosity. By contrast, the Gurson model is smooth on the hydrostatic axis and exhibits a more standard linear dependence of the strength on the porosity.

The first of the two studies [1] consists of finite element discretizations of the limit analysis problems for a *hollow cylinder*, which is a commonly used model for porous media (e.g., the Gurson model). For a given type of boundary condition (uniform strain or stress), this approach provides upper and lower bounds for the associated limit loads, depending on whether the kinematical or static approach are used. By making use of well-known arguments (e.g., [5]), it is also possible to reinterpret these results as upper and lower bounds for the effective flow strength of the *composite cylinder* model.

The second study [3] arises from an application to porous media of a recently developed "second-order" homogenization theory due to Ponte Castañeda [2]. Like the earlier second-order theory [6], this new theory makes use of a linear "thermoelastic" comparison composite, whose constituent phases are identified with appropriate linearizations of the given nonlinear phases about certain reference strains, which, in turn, are identified self-consistently with the phase averages of the strain in the linear comparison composite. Also, as in this author's earlier "variational" theory [7], the new theory incorporates direct dependence on the fluctuations of the strain field in the phases of the linear comparison composite. However, the choice for the relevant comparison moduli in the new theory is more general, being somewhat intermediate between the "secant" and "tangent" moduli used in the earlier theories. This new theory yields estimates that are exact to second order in the contrast and are therefore quite accurate for weakly inhomogeneous composites. However, it has also been shown to give accurate predictions for high-contrast systems [3].

The purpose of this Note is to compare the predictions [3] of the new "second-order" theory for 2D, ideally plastic, porous materials in plane strain with the corresponding results from limit analysis. In addition to the intrinsic value of this special case, it is known that ideally plastic porous systems are extremely nonlinear systems, so that the proposed comparisons will also be of interest within the general framework of nonlinear homogenization. While the results given by Francescato et al. [1] are for the related case of plane stress, earlier, less accurate results have been given for plane strain [8]. In this Note, more accurate plane strain results will be presented making use of a new, very efficient linearization technique, together with systematic optimization of the relevant meshes, in order to attain the required level of precision.

It is important to remark that evidence for the type of anomalous behaviors discussed in this paper has been found earlier by other authors in related, but not identical problems. Thus, for example, Michel et al. [9] found corners along the hydrostatic axis of the homogenized yield surfaces for perforated sheets (plane stress conditions) with a square distribution of holes. There is also the well-known criterion of Rousselier [10], which was postulated to have a corner on the hydrostatic axis. Experimental evidence for non-analytic dependence of the flow stress of low-hardening porous metals have been reported by Spitzig et al. [11]. On the basis of these experimental observations, these authors proposed a modified Gurson model with non-analytic dependence of the effective flow stress in shear on porosity. Pellegrini [12] has recently proposed an empirically derived yield criterion for porous media in plane stress that agrees remarkably well with the corresponding limit analysis results of Francescato et al. [1]. On the other hand, it is also important to emphasize that most theoretical results to date, including the Hashin–Shtrikman bounds of Ponte Castañeda

[7] and Suquet [13], do not exhibit any of these unusual features, predicting smooth yield surfaces for porous media with linear dependence on the porosity for the effective flow stress in shear.

2. Summary of the methodologies

2.1. Limit analysis estimates

The reader is referred to [14] for details on the standard static and kinematic numerical methods of limit analysis (LA); here, the focus is on a new linearization scheme used to improve the convergence of these two methods. For a von Mises (or Tresca) material subjected to plane strain conditions, the criterion is written as:

$$\Phi(\boldsymbol{\sigma}) = \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\tau_{12}^2} - 2k \leqslant 0 \tag{1}$$

where k is the flow stress in shear of the material. Various authors have proposed replacing (1) with a polyhedral approximation:

$$(\sigma_{11} - \sigma_{22})\cos\theta_i + 2\tau_{12}\sin\theta_i - 2k \leqslant 0, \quad \text{with } \theta_i = \frac{2\pi i}{m},\ m \geqslant 3,\ i = 1, m \tag{2}$$

Using (2) as the new plasticity criterion, the static and kinematic methods of LA lead directly to linear optimization problems, to be solved by LP codes such as IBM's OSL (simplex method), or XA (Interior Point Method). When very precise results are needed as is the case here, the main drawback of this technique lies in the high number of inequations generated from (2). A very recent approximation technique proposed in [15], and reformulated by Glineur [16], replaces the linearization (2) with an ingenious, higher-order polyhedral approximation. Thus, writing the cone in (1) as $\sqrt{Y^2 + Z^2} \leqslant r$, $r \geqslant 0$, it follows that:

$$\begin{aligned} &\alpha_0 = Y, \qquad \beta_0 = Z, \qquad \theta_i = \frac{\pi}{2^i}, \qquad r = \alpha_q \cos\left(\frac{\pi}{2^q}\right) + \beta_q \sin\left(\frac{\pi}{2^q}\right) \\ &\alpha_{i+1} = \alpha_i \cos\theta_i + \beta_i \sin\theta_i, \qquad \beta_{i+1} \geqslant |\beta_i \cos\theta_i - \alpha_i \sin\theta_i|, \quad 0 \leqslant i < q \end{aligned} \tag{3}$$

For the case of the von Mises criterion, $r = 2k$. Relations (3) are equivalent to (2) with $m = 2^q$, involving $3q + 1$ rows and $2q$ columns α_i, β_i. This formulation results in an efficient compromise on the generation of lines and columns, allowing a degree of approximation that is practically unattainable with (2). For instance, for the Gurson problem of interest here, we always used $q = 10$ with no loss of convergence, i.e., a value of $m = 1024$ that is prohibitive when using (2). In addition, the time necessary to read and preprocess the data was drastically lowered and the final accuracy remains excellent.

For the kinematic case, the dissipated power involves the integral of $2k\sqrt{\dot{\varepsilon}_{11}^2 + \dot{\varepsilon}_{12}^2}$ for plane strain, which can be easily written in the form (3), with r a positive real, by adding constraints such as $\sqrt{\dot{\varepsilon}_{11}^2 + \dot{\varepsilon}_{22}^2} \leqslant r$, making the objective function linear in all the r variables.

Both programs, where the respective fields are allowed to be discontinuous accross any finite element side (as in [14]), lead to linear programming problems (58.181 rows $\times$ 39.748 columns in the static case; 21.893 rows $\times$ 18.917 columns in the kinematic case for the 672-triangle mesh of [1]), which are solved using XA. For the 672-triangle mesh, the CPU times are about 30 seconds on a Macintosh PowerPC G4-867MHz. The admissibility of the fields was systematically controlled *a posteriori*, in order to eliminate results produced by ill-conditioned meshes.

2.2. Second-order homogenization estimates

The yield criterion, predicted by the second-order theory [2], for the porous, two-dimensional, ideally plastic material, using the Hashin–Shtrikman (HS) estimate for the relevant linear comparison composite, may be written in the form [3]:

$$\left[\frac{\bar{\sigma}_e}{(1-f)\sigma_0}\right]^2 + \frac{m(3-m)}{1+m} = 1 \tag{4}$$

Here $\bar{\sigma}_e$ is the average von Mises equivalent stress in plane strain (i.e., $\bar{\sigma}_e = \sqrt{3}[\bar{\sigma}_{12}^2 + (\bar{\sigma}_{11} - \bar{\sigma}_{22})^2/4]^{1/2}$, where 1 and 2 correspond to the plane of deformation), $\sigma_0 = \sqrt{3}k$ is the effective flow stress in uniaxial tension, f is the porosity, and m is a function of f, $\bar{\sigma}_e$ and the average hydrostatic stress $\bar{\sigma}_m = (\bar{\sigma}_{11} + \bar{\sigma}_{22})/2$, which is given by the (quartic) equation:

$$\frac{m^{3/4}}{1-m} = \left(\frac{f}{2}\right)^{1/2}\left[1 + 3\left(\frac{\bar{\sigma}_m}{\bar{\sigma}_e}\right)^2\right]^{1/2} \tag{5}$$

It is interesting to observe that

$$m \sim \left[\frac{f}{2}\left(1 + 3\left(\frac{\bar{\sigma}_m}{\bar{\sigma}_e}\right)^2\right)\right]^{2/3} \quad \text{as } f \to 0$$

for fixed (finite) triaxiality. In particular this means that the effective flow stress for pure shear conditions ($\bar{\sigma}_m = 0$) is given by $\tilde{\sigma}_0/\sigma_0 \sim 1 - (3/2)(f/2)^{2/3}$ in the *dilute* limit.

On the other hand, in the purely hydrostatic limit (i.e., $\bar{\sigma}_e = 0$), the estimate (4), together with expression (5) for m, leads to the prediction $\bar{\sigma}_m = (1-f)/\sqrt{3f}$ for the effective flow stress. It is noted that this result is in precise agreement with the corresponding predictions arising from the variational bounding methods [13]. However, unlike the variational bounds, which predict a smooth yield surface at this point, the second-order estimate (4) can be shown to lead to a corner on the hydrostatic axis. To see this, note that an asymptotic perturbation expansion of the estimate (4) about $\bar{\sigma}_e = 0$ leads to the expression:

$$|\bar{\sigma}_m| + \frac{C}{\sqrt{f}}\,\bar{\sigma}_e = \frac{1-f}{\sqrt{3f}} \tag{6}$$

where C is some positive number. The existence of a corner can then be simply deduced from this result.

3. Comparisons and discussion

In Fig. 1, the predictions of the second-order HS estimate (4) are compared with the Gurson and limit analysis results for a value of the porosity equal to $f = 0.16$. Concerning the limit analysis predictions, the dark triangles correspond to an imposed uniform strain boundary condition, while the clear triangles correspond to an imposed uniform traction. For each of these two types of boundary conditions there are two sets of results – up and down triangles – corresponding respectively to the kinematic and static approaches. They provide bounds on the exact solution for a given type of boundary condition. On the other hand, it is known (see, for example, [5]) that the uniform strain and uniform traction boundary conditions provide upper and lower bounds, respectively, for the effective behavior of porous media with a composite-cylinder type of microstructure. Therefore, the outer (dark down triangles) and inner (clear up triangles) are upper and lower bounds for the effective behavior of porous media with this type of microstructure. Recalling that for two-dimensional, incompressible linear-elastic composites, the Hashin–Shtrikman bounds are reproduced closely by the composite cylinder microstructure, it follows that the second-order HS estimates, which arise from the corresponding linear HS estimates, can also be loosely identified with the composite cylinder microstructure, in which case the above-mentioned limit-analysis bounds should be satisfied. Of course, the Gurson model corresponds exactly to the composite-cylinder microstructure, since it is based on an approximate solution of the hollow cylinder problem with uniform strain boundary conditions.

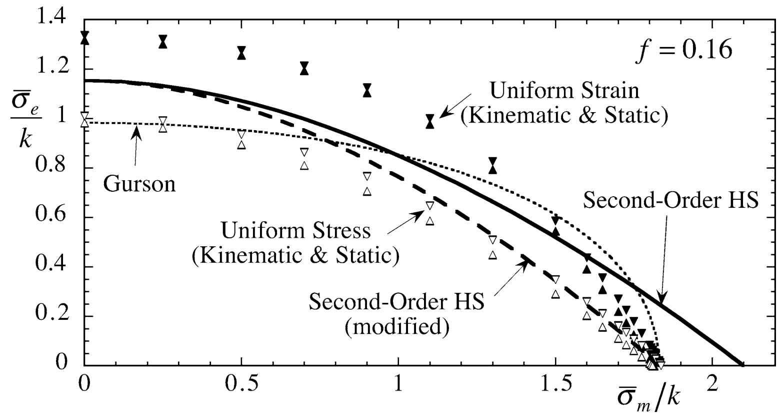


Figure 1. Homogenized yield surfaces for ideally plastic, porous materials subjected to plane strain loading (the cylindrical pores are transverse to the plane of deformation): comparison of the limit analysis predictions with Gurson's model and the second-order HS estimates (original and modified).

Figure 1. *Critères de plasticité macroscopique pour les matériaux poreux parfaitement plastiques sous chargement en déformation plane* (*la cavité cylindrique est orthogonale au plan de la déformation*) : *comparaison des prédictions de l'analyse limite avec ceux du modèle de Gurson et les estimations HS du second ordre* (*originales et modifiées*).

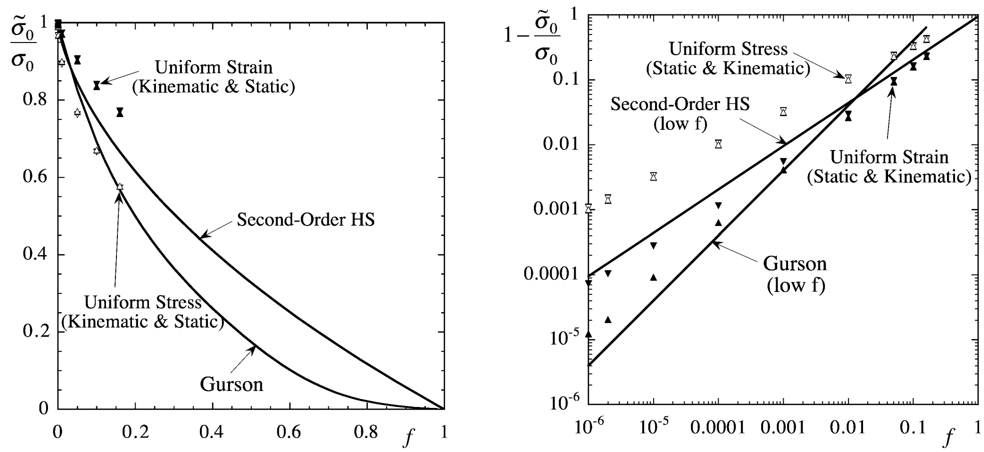
However, as seen in Fig. 1, the second-order HS estimate (4) actually violates these bounds for sufficiently high triaxiality. This is a well-known problem (see, for example, Ponte Castañeda and Suquet [17]) with the linear comparison composite approaches, which can be partially remedied by use of more general comparison composites [5]. A less rigorous, but simpler approach is to modify in an ad hoc fashion the hydrostatic term in the expression (5) for m, such that

$$\frac{m^{3/4}}{1-m} = \left(\frac{f}{2}\right)^{1/2} \left\{1 + \frac{3}{f}\left[\frac{1-f}{\log f}\frac{\bar{\sigma}_m}{\bar{\sigma}_e}\right]^2\right\}^{1/2} \tag{7}$$

With this modified form for m, the expression (4) for the second-order HS estimate reduces to $\bar{\sigma}_m = -\log f/\sqrt{3}$ in the purely hydrostatic limit, which is in agreement with the exact result. Thus, with this modification, the second-order HS estimate is found to satisfy the limit analysis bounds.

Aside from this problem, the second-order HS estimate is seen to agree *qualitatively* with the limit analysis results which also clearly exhibit a *corner* on the hydrostatic axis of the yield surface. (Note, however, that the predicted value of the slope at the corner is not necessarily accurate.) This is in contrast with the Gurson model, which although yielding the correct limit for purely hydrostatic loading, does not exhibit a corner on the yield surface at this point. As a consequence of this fact, the Gurson model is seen to violate the limit analysis bounds (in particular, the upper bound) for high triaxialities. Not only that, but in addition, the Gurson model is also found to violate the lower bound for low triaxialities at this relatively high level of porosity. By contrast, the second-order HS estimate is found to be square in the middle of the upper and lower bounds of limit analysis at low triaxialities.

In Fig. 2, the dependence of the effective flow stress in shear ($\bar{\sigma}_m = 0$) is investigated in more detail. Thus, in Fig. 2(a) a comparison is made of the second-order HS and Gurson estimates with the limit analysis results. It is observed that while the second-order HS estimates appear to be consistent with the bounds, the Gurson model gives results that actually violate the bounds for both sufficiently low and high porosities. To be able to compare the predictions of the two theories with the limit analysis results for low porosities, they are plotted using a logarithmic scale in Fig. 2(b). There it can be seen that while the Gurson theory gives predictions that violate the limit analysis upper bound (note that the upper bounds appear as lower bounds



(a) (b)

Figure 2. (a) Comparison of the Gurson and second-order HS predictions for the effective flow stress in shear as functions of porosity with the corresponding limit analysis results. (b) Same as part (a), but for dilute porosity levels, plotted in logarithmic scales.

Figure 2. (a) *Contrainte équivalente effective en cisaillement en fonction de la porosité* : *comparaison des prédictions du modèle de Gurson et HS/second ordre.* (b) *idem* (a), *mais pour le cas dilué.*

when the results are plotted in this format) for sufficiently low porosities, the HS second-order estimates satisfy the bounds.

In this connection, it is interesting to remark that the limit analysis predictions resulting from the uniform strain and uniform stress types of boundary conditions appear to lead to two different types of dilute estimates. Thus, the uniform stress results lead to a prediction of the type $\tilde{\sigma}_0/\sigma_0 - 1 \sim f^{1/2}$, whereas the uniform strain results appear to be more consistent with a prediction of the type $\tilde{\sigma}_0/\sigma_0 - 1 \sim f^{2/3}$. At first, it is a bit surprising that the two types of boundary conditions should lead to different predictions for dilute levels of porosity – it is expected that the boundary conditions on the hollow cell problem should not play a role in the dilute limit. Hovewer, it should be recalled that traction and velocity boundary conditions are not always equivalent in homogenization of ideally plastic materials [18,19].

In view of all of this, it is remarkable that the second-order HS estimate leads to a dilute prediction that is consistent with the uniform-strain yield analysis, giving a dependence of $f^{2/3}$ on the porosity, as already mentioned in the context of the discussion of expressions (4) and (5). By contrast, the Gurson model gives a dependence of f^1, which is inconsistent with the uniform-strain bound.

Although perhaps of little practical importance, it remains to determine the correct exponent (1/2, 2/3, or something else) in the dilute limit. In particular, it would be useful from a theoretical perspective to determine whether the prediction of 2/3 given by the second-order theory [3] is the correct prediction, at least for this particular type of Hashin–Shtrikman-like estimate which is closely associated with the composite cylinder microstructure. (It has been suggested by Suquet [20] that the actual exponent may depend on microstructural information beyond overall isotropy.) If the theory did give the correct prediction, it would be an unexpected and truly remarkable result, and a positive one for nonlinear homogenization in general, given the intrinsic difficulty of this very strongly nonlinear example.

Regardless of the outcome of such an effort, it is conjectured here that the second-order theory would predict a similar (singular) dependence in the dilute limit for *simple shear* (plane strain) loading of isotropic porous media in three dimensions. This would still be in disagreement with the corresponding predictions of the three-dimensional version of the Gurson model (for spherical pores). However, it is expected that the standard linear (non-singular) dependence on porosity in this limiting case will survive for *axisymmetric* types of boundary conditions. This difference in the expected predictions between plane strain and axisymmetric shear for systems with spherical pores could be understood in part by remarking that the plane strain configuration would allow the formation of shear bands even for a fully dense material (no pores), while the axisymmetric configuration would preclude the formation of such bands.

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References

[1] P. Francescato, J. Pastor, H. Thai-The, Étude du critère de plasticité des matériaux poreux, C. R. Acad. Sci. Paris, Série IIb 329 (2001) 753–760.
[2] P. Ponte Castañeda, Second-order homogenization estimates for nonlinear composites incorporating field fluctuations. I – Theory, J. Mech. Phys. Solids 50 (2002) 737–757.
[3] P. Ponte Castañeda, Second-order homogenization estimates for nonlinear composites incorporating field fluctuations. II – Applications, J. Mech. Phys. Solids 50 (2002) 759–782.
[4] A. Gurson, Continuum theory of ductile rupture by void nucleation and growth: Part I. Yield criteria and flow rules for porous ductile media, J. Engrg. Math. Tech. 99 (1977) 1–15.
[5] N. Bilger, F. Auslender, M. Bornert, R. Masson, New bounds and estimates for porous media with rigid perfectly plastic matrix, C. R. Mécanique 330 (2002) 127–132.
[6] P. Ponte Castañeda, Exact second-order estimates for the effective mechanical properties of nonlinear composite materials, J. Mech. Phys. Solids 44 (1996) 827–862.
[7] P. Ponte Castañeda, The effective mechanical properties of nonlinear isotropic composites, J. Mech. Phys. Solids 39 (1991) 45–71.
[8] H. Thai-The, P. Francescato, J. Pastor, Limit analysis of unidirectional porous media, Mech. Res. Comm. 25 (1998) 535–542.
[9] J.C. Michel, H. Moulinec, P. Suquet, Effective properties of composite materials with periodic microstructure: a computational approach, Comput. Methods Appl. Mech. Engrg. 172 (1999) 109–143.
[10] G. Rousselier, Ductile fracture model and their potential in local approach of fracture, Nuclear Engrg. Design 105 (1987) 97–111.
[11] W.A. Spitzig, R.E. Smelser, O. Richmond, The evolution of damage and fracture in iron compacts with various initial porosities, Acta Metall. 36 (1988) 1201–1211.
[12] Y.-P. Pellegrini, Plasticity criterion for porous medium with cylindrical void, C. R. Mécanique 330 (2002), submitted.
[13] P. Suquet, On bounds for the overall potential of power-law materials containing voids with an arbitrary shape, Mech. Res. Comm. 19 (1992) 51–58.
[14] J. Pastor, H. Thai-The, P. Francescato, New bounds of the height limit of a vertical slope, Internat. J. Numer. Anal. Methods Geomech. 24 (2000) 165–182.
[15] A. Ben-Tal, A. Nemirovski, On polyhedral approximations of the second-order cone, Math. Oper. Res. 26 (2001) 193–205.
[16] F. Glineur, Topics in convex optimization, Thèse de la Faculté Polytechnique de Mons, Belgique, 2001.
[17] P. Ponte Castañeda, P. Suquet, Nonlinear composites, Adv. Appl. Mech. 34 (1998) 171–302.
[18] S. Turgeman, J. Pastor, Comparaison des charges limites d'une structure réelle et homogénéisée, J. Méc. Théor. Appl. 6 (1987) 121–143.
[19] G. Bouchitté, P. Suquet, Charges limites, plasticité et homogénéisation : le cas d'un bord chargé, C. R. Acad. Sci. Paris, Série I 305 (1987) 441–444.
[20] P. Suquet, Private communication, 2001.