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Plasticity criterion for porous medium with cylindrical void

Yves-Patrick Pellegrini

Département de physique théorique et appliquée, CEA, BP 12, 91680 Bruyères-le-Châtel, France

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Note presented by Jean-Baptiste Leblond.

Abstract A simple Gurson-based yield criterion for porous materials with cylindrical voids in plane stress is proposed. With no adjustable parameters, it compares quite satisfactorily with recent numerical data by Francescato et al. for different porosities. It is non-analytic with respect to the porosity, and displays an angulous point. ***To cite this article: Y.-P. Pellegrini, C. R. Mecanique 330 (2002) 763–768.***

porous media / ductile behavior / perfect plasticity / Gurson model / cylindrical voids / homogenization

Un critère de plasticité pour un matériau poreux à vide cylindrique

Résumé On développe à partir du critère de Gurson un nouveau critère de plasticité pour un matériau poreux à vides cylindriques, en contraintes planes. D'une forme simple et sans paramètres ajustables, ce critère est non-analytique par rapport à la porosité et présente un point anguleux. Il reproduit de façon satisfaisante des résultats numériques récents de Francescato et al. pour différentes porosités. ***Pour citer cet article : Y.-P. Pellegrini, C. R. Mecanique 330 (2002) 763–768.***

milieux pareux / comportement ductile / plasticité parfaite / modèle de Gurson / cavités cylindriques / homogénéisation

Version française abrégée

Les modifications empiriques subies par le critère de plasticité de Gurson [1] depuis son introduction n'ont pas permis de surmonter des déficiences qu'illustrent bien les calculs numériques récents de Francescato et al. [8] de la surface de plasticité pour le problème du vide cylindrique en contrainte plane microscopique pure (CPP) :

(i) le critère de Gurson original est exact pour un chargement axisymétrique en déformation plane généralisée, et semble le rester en CPP. Des modifications revenant à redéfinir la fraction volumique f de vides [2,3] lui font perdre cette propriété ;

(ii) le convexe de plasticité de Gurson intersecte l'axe des contraintes moyennes avec une pente infinie. Cette propriété a été mise en défaut dans les calculs CPP de la Réf. [8], et par une récente théorie d'homogénéisation non-linéaire [6] en déformation plane, qui prédisent une pente finie (point anguleux) ;

(iii) le seuil déviatoire du critère réel CPP est apparemment non-analytique en f dans la limite $f \to 0$ [8].

E-mail address: yves-patrick.pellegrini@cea.fr (Y.-P. Pellegrini).

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Il semble [6] que cette propriété [3,14] dépasse le cadre CPP, sans qu'on puisse actuellement en préciser l'étendue.

Cette Note introduit un critère phénoménologique corrigeant les trois défauts précités, en bon accord avec les résultats numériques de la Réf. [8]. Il est construit comme suit : on transforme le critère de Gurson (1) en l'expression équivalente (3), qui absorbe la dépendance en f dans les seuils moyen et déviatoire effectifs (4). Nous limitant au cas du chargement plan, nous cherchons un critère réductible à l'expression donnée par Gurson en situation axisymétrique, sous la forme (5), où g est une fonction inconnue de la norme (de nature déviatoire) $\Sigma_\mathrm{d} = \sqrt{3}/2|\Sigma_{xx} - \Sigma_{yy}|$. On montre que le point anguleux ne peut être reproduit que si g est linéaire en Σ_d pour Σ_d petit (par opposition, le critère de Gurson est quadratique en Σ_d). Une comparaison aux données numériques nous fait choisir, par analogie avec (3), la fonction $g(x) = \sinh[x/(\sqrt{3}k)]/\sinh[Y_\mathrm{ps}/(\sqrt{3}k)]$, où $\sqrt{3}k \equiv Y_0$ est le seuil plastique de Mises de la matrice, et où Y_ps est un nouveau seuil déviatoire en contrainte plane à déterminer. Le critère prend la forme (9).

Plusieurs auteurs ont envisagé la possibilité d'une dépendance non-analytique du seuil déviatoire quand $f \to 0$ [3,14,6], la proposition de Richmond et Smelser [3] impliquant par exemple $Y \sim Y_0(1 - f^{2/3})$. Nous adoptons pour Y_ps la forme d'essai (7) à n paramètres $a_i < 1$, $i = 1, \ldots, n$, déterminés par ajustement non-linéaire simultané du modèle (9) sur les données numériques de Francescato et al., pour des porosités $f = 0{,}01$, 0,05, 0,10 et 0,16. Nous n'avons retenu, pour chaque porosité, que les deux couples de points les plus proches des deux axes, soit 16 points au total (voir Fig. 1). Des ajustements successifs sont conduits pour n variant de 1 à 4. Les coefficients a_i sont à chaque fois trouvés presque égaux, et le coefficient du terme $f^{2/3}$ dans le développement de Y_ps en puissances de f semble tendre vers 0,5. Cela appelle l'hypothèse selon laquelle $Y_\mathrm{ps} = Y_0(1 - f)\exp(-af^{2/3})$, avec a de l'ordre de 1/2. Un ajustement ultime de ce dernier modèle conduit à $a = 0{,}501$, suggérant que l'expression $Y_\mathrm{ps} = Y_0(1 - f)\exp(-f^{2/3}/2)$ est peut-être exacte. Le modèle (9) qui en résulte est comparé aux données numériques sur la Fig. 1, où Σ_d est représenté en fonction de $\Sigma_\mathrm{m} = (\Sigma_{xx} + \Sigma_{yy})/2$, lorsque $\Sigma_{xy} = 0$. L'accord est parfait au voisinage des axes, et raisonnable pour des chargements intermédiaires. Par ailleurs, il s'améliore partout pour la porosité la plus faible $f = 0{,}01$. Le critère est ensuite étendu, sur la base des invariants du système, aux chargements plans généralisés avec $\Sigma_{xy} \neq 0$. Cette dernière extension n'a pu être testée faute de données numériques.

La méthode semi-empirique utilisée dans cette article pourrait être mise en œuvre afin d'améliorer le critère de Gurson pour un vide sphérique, dès que les données numériques correspondantes seront disponibles.

1. Introduction

Since its introduction in 1977, Gurson's well-known plasticity criterion for porous materials with spherical voids [1] has undergone modifications [2–4] in order to improve its adequacy with experimental or numerical results [5]. Motivated by comparisons to numerical results, Tvergaard replaced the porosity f by $1.5f$ in an attempt to account for localization effects. Drucker's crude cross-sectional arguments for a lower bound give [14] a non-analytical estimation for the deviatoric yield stress of a voided material as $Y_\mathrm{eq} = Y_0(1 - af^\theta)$ when $f \to 0$, where Y_0 is the Mises yield threshold of the matrix and a some coefficient, and where the exponent θ is 2/3 for *spherical* voids and 1/2 for *cylindrical* voids. Elaborating on Drucker's work, Richmond and Smelser used $f^{2/3}$ instead of f in Gurson's formula for spherical voids [3]. An exponent 2/3 for *cylindrical* voids in plane strain conditions was first obtained analytically by Ponte Castañeda (PC) from a second-order effective-medium theory (EMT) [6], and has been extracted afterwards by Pellegrini (unpublished) from his alternative second-order approach [7]. Both approaches, compared to older methods, insist on a more refined treatment of field heterogeneities which is a reason for their non-trivial predictions.

By means of finite-element calculations based on the theory of limit analysis, Francescato et al. recently computed the yield surface of a hollow cylinder aligned along the unit vector $\mathbf{u} = \mathbf{e}_z$, under homogeneous plane boundary conditions with pure plane local stresses (PPσ) [8]. They used both an interior (static) and an exterior (kinematic) approach wich provide tight bounds for the exact solution. Denoting by $\Sigma = \langle\sigma\rangle$ the macroscopic stress, they displayed their results as curves $\Sigma_\mathrm{d} = h(\Sigma_\mathrm{m})$, where $\Sigma_\mathrm{m} = (\Sigma_{xx} + \Sigma_{yy})/2$ is the mean stress and where $\Sigma_\mathrm{d} = (\sqrt{3}/2)|\Sigma_{yy} - \Sigma_{xx}|$, and compared them to four available analytical plasticity criteria (namely the 2d Gurson criterion, and three other related 3d criteria [2–4], due to lack of 2d couterparts). The curves illustrate several fundamental properties or deficiencies of these criteria: (i) the matrix obeying the Mises criterion $\sigma_\mathrm{eq} \leqslant \sqrt{3}k$, Gurson's criterion for *cylindrical* voids [1],

$$\left[\Sigma_\mathrm{eq}/\left(\sqrt{3}k\right)\right]^2 + 2f\cosh(\Sigma_\mathrm{m}/k) - 1 - f^2 = 0 \tag{1}$$

is known to be exact for axisymmetric (AS) generalized plane strain (GPε), a state where $\dot{\varepsilon}_{zz}$ is uniform, owing to the trial velocity field [1,9] used in its derivation. Hereafter, we use the Mises equivalent norm

$$\Sigma^2_\mathrm{eq} = (\Sigma_\mathrm{m} - \Sigma_{zz})^2 + \Sigma^2_\mathrm{d} + 3\Sigma^2_{xy} + 3\left(\Sigma^2_{xz} + \Sigma^2_{yz}\right) = \Sigma^2_\mathrm{AS} + \Sigma^2_\mathrm{r} \tag{2}$$

where ${\Sigma_\mathrm{AS}}^2 = (\Sigma_\mathrm{m} - \Sigma_{zz})^2$ is the only surviving part in AS loading, whereas Σ_r stands for the remainder which leads to inexact values of the criterion for more general loadings. *Focusing from now on on the plane macroscopic stress case*, we thus see that when $\Sigma_\mathrm{d} = 0$, Gruson's criterion provides an exact yield stress $\Sigma_\mathrm{m} = Y^*$, where Y^* is the root of the above-defined function $h(x)$. Hence, any empirical modification consisting in redefining the porosity modifies the threshold value Y^* into an incorrect one. The local normality and equilibrium equations allow us to compute the local stress field associated to the velocity field used in the criterion: it is such that $\sigma_{zz} \neq 0$ even when $\Sigma_{zz} = \langle\sigma_{zz}\rangle = 0$. Hence the criterion cannot rigorously be transposed to a PPσ situation. However, surprisingly enough, the numerical PPσ results of Ref. [8] show that beyond being exact in AS-GPε, the 2d Gurson criterion is apparently also exact for the AS-PPσ case, or at least close to the exact result. This criterion with $\Sigma_\mathrm{d} = 0$ is thus a good starting point to reproduce numerical PPσ data, providing we forbid modifications of f; (ii) the comparison criteria considered in [8] feature an infinite slope at the location $(\Sigma_\mathrm{m}, \Sigma_\mathrm{d})$=$(Y^*, 0)$, in contradiction with the PPσ data which display a finite slope instead (a 'corner'). It is worth noting that both the yield surface of the 3d Rousselier criterion [10] and the theoretical one deduced from the EMT of Ref. [6] in plane strain give such a corner; (iii) the porosity dependence of Gurson's criterion in plane biaxial stress is inconsistent with the numerical results. Francescato et al. found empirically that the replacement of f by $f^{2/3}$ in the 2d Gurson formula gave overall better results, without obtaining perfect agreement partly because of point (i). This suggests however that a power-law non-analyticity in f, already surmised in plane strain, is also present in plane stress.

These remarks motivate us to propose in this Note an empirical plasticity criterion for a cylindrical void which does not suffer these shortcomings, and which is in closest agreement to AS-PPσ numerical results than other available models. The analysis below confirms a power-law dependence of the criterion as $f^{2/3}$ via a particular yield threshold, Y_ps, for which a possibly exact expression is found. From symmetry considerations the criterion is then extended to loadings with nonzero Σ_{xy}.

2. Empirical derivation of the plasticity criterion for a cylindrical void

Since our new criterion heavily relies on the cylindrical Gurson criterion, we begin by an analysis of the latter. With the identity $\cosh(x) = 2\sinh^2(x/2) + 1$, we transform (1) into the equivalent form

$$(\Sigma_\mathrm{eq}/Y_\mathrm{eq})^2 + \left\{\sinh\left[\Sigma_\mathrm{m}/(2k)\right]/\sinh\left[Y_\mathrm{m}/(2k)\right]\right\}^2 = 1 \tag{3}$$

where the dependence in f now solely enters the effective mean and deviatoric thresholds

$$Y_\mathrm{m} = -k\log(f), \qquad Y_\mathrm{eq} = \sqrt{3}k(1-f) \tag{4}$$

Our first step is to modify the criterion so that it provides a good approximation to the numerical results of Francescato et al. [8] for arbitrary macroscopic biaxial plane stress loading. The modification therefore concerns the dependence of the criterion in Σ_d. In order not to jeopardize the assumed exactness in the AS case, we look for the new criterion in the form

$$g(\Sigma_{\rm d}) + F(\Sigma_{\rm m}) = 1, \quad \text{where } F(\Sigma_{\rm m}) = (\Sigma_{\rm m}/Y_{\rm eq})^2 + \left\{\sinh\left[\Sigma_{\rm m}/(2k)\right]/\sinh\left[Y_{\rm m}/(2k)\right]\right\}^2 \tag{5}$$

is the exact part borrowed from (3) when $\Sigma_{\rm d} = 0$, and where $g(x)$ is a function to be determined, such that $g(0) = 0$. This form is motivated by an analogy to Eq. (3), where the yield surface is descibed by the sum of two functions, each one depending on one particular invariant. Indeed, it is clear that plasticity criteria can be built, to a first approximation, by summing up suitable functions of stress invariants, where each of these functions corresponds to a particular type of loading. We note in passing that in our notations, $F(Y^*) = 1$ where Y^* is defined above.

The behavior of g is then precised by assuming a power-law asymptotic behavior $g(x) \sim x^\alpha$ for $x \to 0$, where α must be determined. The data being represented as $\Sigma_{\rm d} = h(\Sigma_{\rm m})$, we have here $h(x) = g^{-1}(1 - F(x))$. A first-order expansion of $h(\Sigma_{\rm m})$ around $\Sigma_{\rm m} = Y^*$ shows that a finite slope at the location $(\Sigma_{\rm m}, \Sigma_{\rm d}) = (Y^*, 0)$ is possible only for $\alpha = 1$; otherwise the slope is either 0 ($\alpha < 1$), or infinite ($\alpha > 1$) as in Gurson's criterion where $\alpha = 2$. We found the simplest choice $g(\Sigma_{\rm d}) = \Sigma_{\rm d}/Y_{\rm ps}$, where $Y_{\rm ps}$ is an f-dependent yield stress to be determined, to yield poor agreement with the numerical data. Instead we retained, again by analogy with (5), the form (which still complies with the $\alpha = 1$ requirement):

$$g(\Sigma_{\rm d}) = \sinh(\Sigma_{\rm d}/k)/\sinh(Y_{\rm ps}/k) \tag{6}$$

Next, Drucker's argument put aside PC's analytical results and the results of [8] suggest us that our $Y_{\rm ps}$ may behave non-analytically as $Y_{\rm ps} - \sqrt{3}k \sim f^{2/3}$ when $f \to 0$. Now, in this problem – as well as in the spherical void one – all the effective moduli relative to deviatoric loading obtained analytically so far (be them linear or non-linear) consist in a factor $1 - f$ multiplying some function of f: the exact shear elastic modulus [11], the deviatoric yield stress of the Gurson formula, or that of the non-linear Hashin–Shtrikman upper bound [12] are significant examples of this fact. Both arguments motivate us to look for $Y_{\rm ps}$ in the n-parameter product representation

$$Y_{\rm ps} = Y_{\rm ps}^{(n)} \equiv \sqrt{3}k(1 - f)\prod_{i=1}^{n}\left(1 - a_i f^{2/3}\right) \tag{7}$$

where the free coefficients a_i are less than 1, so that $Y_{\rm ps}$ has no root other than 1 in the range $f \in [0, 1]$. The extra factors $1 - a_i x$ provide a convenient representation for a wide class of functions of $x = f^{2/3}$ which are analytic at $x = 0$ (and non-analytic at $f = 0$).

A non-linear fit of the resulting model built from gathering Eqs. (5)–(7) is performed *simultaneously* over four data sets corresponding to $f = 0.01$, $f = 0.05$, $f = 0.10$ and $f = 0.16$, taken from the plots of [8]. Here, we only aim at reproducing correctly the threshold $Y_{\rm ps}$, since the sinh form of $g(\Sigma_{\rm d})$ can at best be only approximate. Hence, the fit is carried out over the set built from the two leftmost and righmost points for each porosity, i.e., 16 data points (see Fig. 1). The data we used were estimated from an eye-average of the static and kinematic curves of [8], close to one another. The relative error between the data of Francescato et al. and ours is estimated, from the errors on the abscissa, to be of

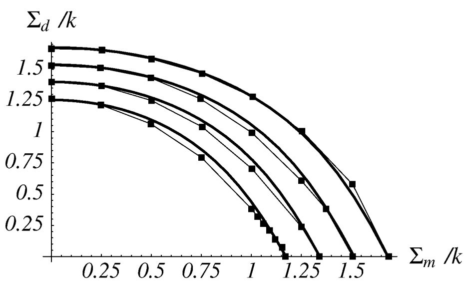

Figure 1. Yield surfaces in plane stress for a cylindrical void: $\Sigma_{\rm d}/k$ vs. $\Sigma_{\rm m}/k$ for different porosities f. From top to bottom: $f = 0.01$, 0.05, 0.10, 0.16. Symbols: data taken from [8]; solid line: model (9).

Figure 1. *Surfaces de plasticité en contrainte plane pour un vide cylindrique* : $\Sigma_{\rm d}/k$ *en fonction de* $\Sigma_{\rm m}/k$ *pour differentes porosités* f. *De haut en bas* : $f = 0{,}01$, $0{,}05$, $0{,}10$, $0{,}16$. *Symboles* : *données numériques de* [8] ; *ligne continue* : *modèle* (9).

order 5×10^{-3}. For $n = 1, 2, 3, 4$, we get the numerical values $\{a_1\} = 0.472$, $\{a_1, a_2\} = \{0.243, 0.243\}$, $\{a_1, a_2, a_3\} = \{0.166, 0.163, 0.162\}$, $\{a_1, a_2, a_3, a_4\} = \{0.124, 0.123, 0.123, 0.123\}$, respectively. In turn, these sets imply expansions

$$Y_{\text{ps}}^{(n)} = \sqrt{3}k(1-f)\left[1 + \sum_{p=1}^{n}(-1)^p b_p \left(f^{2/3}\right)^p\right] \tag{8}$$

with $\{b_1\} = \{0.472\}$, $\{b_1, b_2\} = \{0.486, 0.059\}$, $\{b_1, b_2, b_3\} = \{0.491, 0.080, 0.004\}$, $\{b_1, b_2, b_3, b_4\} = \{0.493, 0.091, 0.007, 0.0002\}$, respectively. The coefficient b_1 in these expansions (that of $f^{2/3}$) seems to converge smoothly towards $1/2$. In addition, the monotonic behaviour of all the coefficients with n provides some evidence for the relevance of the trial form used. Moreover, the obtained sets $\{a_i\}_{i=1,n}$ are consistent with an expression $Y_{\text{ps}}^{(n)} = \sqrt{3}k(1-f)[1 - f^{2/3}/(2n)]^n \to \sqrt{3}k(1-f)\mathrm{e}^{-f^{2/3}/2}$. As a further test, we assume a form $Y_{\text{ps}} = \sqrt{3}k(1-f)\exp(-af^{2/3})$ with a undetermined, and we carry out a ultimate non-linear fit of the model. The result, $a = 0.501$, supports our previous findings. We therefore arrive at the criterion:

$$\frac{\sinh(\Sigma_{\text{d}}/k)}{\sinh(Y_{\text{ps}}/k)} + \frac{\Sigma_{\text{m}}^2}{Y_{\text{eq}}^2} + \left\{\frac{\sinh[\Sigma_{\text{m}}/(2k)]}{\sinh[Y_{\text{m}}/(2k)]}\right\}^2 = 1 \tag{9}$$

where $Y_{\text{m}} = -k\log(f)$, $Y_{\text{eq}} = \sqrt{3}k(1-f)$, $Y_{\text{ps}} = \sqrt{3}k(1-f)\mathrm{e}^{-f^{2/3}/2}$, and where the the threshold Y_{ps} is possibly exact.

In Fig. 1 are displayed comparisons with the numerical data for the four porosities. We see that our empirical formula succeeds in reproducing the data with reasonable accuracy. Similar plots in [8], where comparison criteria [1–4] are also displayed clearly demonstrate the new criterion (9) to do a better job in fitting the data. The best result is obtained for $f = 0.01$, whereas discrepancies show up in the intermediate range $0 \ll \Sigma_{\text{m}} \ll Y^*$ for increasing porosities. This indicates that the functional form of the first term in the left-hand side of (9) is only approximate.

The last step of the analysis consists in extending the criterion to more general plane stress loadings with nonzero Σ_{xy}. Owing to the symmetry of the problem, any scalar quantity (including the criterion itself) depends on Σ through the five scalar invariants $I_1 = \operatorname{tr}\Sigma$, $I_2 = \operatorname{tr}\Sigma^2$, $I_3 = \det\Sigma$, $I_4 = \mathbf{u}\cdot\Sigma\cdot\mathbf{u}$ and $I_5 = \mathbf{u}\cdot\Sigma^2\cdot\mathbf{u}$. From these invariants, we conclude that Σ_{d} (not an invariant) must at least appear in the invariant combination of plane stresses $\Sigma_{\text{ps}}^2 \equiv \Sigma_{\text{d}}^2 + 3\Sigma_{xy}^2$ whenever it is present. Accordingly, for the particular orientation $\mathbf{u} = \mathbf{e}_z$, the minimal modifications required to preserve invariance lead to the generalized criterion built from (9) by replacing Σ_{d} by Σ_{ps}. An arbitrary orientation of the symmetry axis $\mathbf{u}$ can then be used, provided we express the relevant quantities in terms of the invariants as $\Sigma_{\text{m}} = (I_1 - I_4)/2$, $\Sigma_{\text{ps}}^2 = 3I_2/2 - 3I_1^2/4 + 3I_1I_4/2 + 3I_4^2/4 - 3I_5$.

3. Conclusion

To summarize, we have derived, by using physical and mathematical arguments in a heuristic approach, a plasticity criterion for a voided axisymmetric cell. The criterion reproduces well the numerical results of Francescato et al. obtained by limit analysis techniques in pure plane stress, with biaxial moacroscopic loading. It features a corner near the average stress axis, and possesses a non-analytic power-law dependence in the porosity f with exponent $2/3$, as first suggested by Francescato et al. The new analysis of their data which we present here allowed us to precise the result, and to propose a closed analytical form for the threshold Y_{ps}, of deviatoric nature. The exponent $2/3$ had previously been extracted analytically from recent 2d EMTs in plane strain. Our work confirms that it shows up in plane stress as well, a situation not completely devoid of practical interest (perforated sheets). Due to the lack of numerical results, we cannot for the time being overcome the restriction of macroscopic plane stress and extend our empirical criterion to more general loadings (e.g., by complementing it with some function of additional suitable invariants) without making undue assumptions. We believe however that the ideas used in the above heuristic derivation could be used as well to improve Gurson's criterion for spherical voids as soon as three-dimensional high quality numerical data are available.

As to the corner, we feel important to mention that Francescato et al. also performed limit analysis calculations in an alternative situation of generalized plane strain/macroscopic plane stress where $\dot{\varepsilon}_{zz}$ is uniform and $\sigma_{zz} \neq 0$, but where $\Sigma_{zz} = \langle \sigma_{zz} \rangle = 0$: they found [13] that the criterion does not display a corner, but an infinite slope instead, much like Gurson's. The presence of the corner is therefore very sensitive to the situation at hand, further work being required to fully understand the reasons for its presence or disappearance.

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