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Cosserat modeling of cellular solids

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Abstract Cellular solids inherit their macroscopic mechanical properties directly from the cellular microstructure. However, the characteristic material length scale is often not small compared to macroscopic dimensions, which limits the applicability of classical continuum-type constitutive models. Cosserat theory, however, offers a continuum framework that naturally features a length scale related to rotation gradients. In this paper a homogenization procedure is proposed that enables the derivation of macroscopic Cosserat constitutive equations based on the underlying microstructural morphology and material behavior. ***To cite this article: P.R. Onck, C. R. Mecanique 330 (2002) 717–722.***
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continuum mechanics / homogenization / micromechanics / Cosserat theory / cellular solids / foams / bone / generalized continua

Modélisation des solides cellulaires selon Cosserat

Résumé Les solides cellulaires doivent leurs propriétés mécaniques directement à leur structure microcellulaire. Néanmoins, la longueur caractéristique du matériau est souvent non-négligeable comparée aux dimensions macroscopiques, ce qui limite le domaine de validité des modèles classiques, basés sur une description continue. En revanche, la théorie de Cosserat offre un cadre continu incorporant naturellement une échelle de longueur liée aux gradients de rotation. Dans cette Note nous proposons un procédé d'homogénéisation permettant de dériver, au niveau macroscopique, les équations constitutives de Cosserat, tenant compte de la morphologie de la microstructure concernée ainsi que le comportement du matériau considéré. ***Pour citer cet article : P.R. Onck, C. R. Mecanique 330 (2002) 717–722.***
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milieux continus / homogénéisation / micromécanique / théorie de Cosserat / solides cellulaires / mousses / tissus osseux / milieux continus généralisés

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1. Introduction

Continuum theory is valid when each material point of the macroscopic structure (specimen or component) represents a volume of material (material sample) that has a size D that is much smaller than the specimen size L. Furthermore, the constitutive behavior of this material sample must be representative for all material in the structure. When one of these conditions is violated, continuum theory should not be used and one should resort to methods that take the discrete nature of the materials' microstructure into account. Classical continuum theory is based on the assumption that the transfer of load between two neighboring material points occurs only through a force vector, leading to the definition of (symmetric)

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stresses and strains. This implies that classical continuum theory is well-suited for situations where the variations in stresses and strains (with wavelength λ) are smooth enough so that they can be approximated as being uniform on the scale of the material points ($L > \lambda \gg D$). However, in many situations this is not necessarily the case, e.g., near notch and crack tips, in the case of the localization of deformation and in the case of the formation of boundary layers (e.g., [1–3]). In these situations ($L > \lambda > D$) one often resorts to enhanced continuum theories (also called enriched or generalized) that take into account the nonuniformity of stresses and strains at the scale of the material point.

Three classes of enhanced continuum theories can be identified: (i) higher-grade, (ii) higher-order and (iii) non-local continuum theories. One of the simplest higher-order theories is Cosserat (or micro-polar) theory, in which the interaction between neighboring material points is governed by a moment vector in addition to the force vector from classical continuum theory. As a result, next to displacements, rotations are introduced as kinematic quantities. From a mechanics point of view, many cellular materials (e.g., foams, truss structures, trabecular bone) can be seen as a structure of interconnected beams. In these materials bending is often a prominent deformation mechanism, so that at the microscale both displacements and rotations are present. In a recent theoretical study [2] it was shown that the enhanced or reduced constraint of rotations at the specimen edge is vital in predicting size effects. This, in addition to experimental evidence reported by, e.g., Lakes [4], makes Cosserat theory a suitable candidate for continuum modeling of cellular materials, because it contains rotations as degrees of freedom.

Homogenization procedures for dense solids have been proposed in the literature that assume a Cosserat material at the macroscopic scale and either a classical continuum material [5–7] or a Cosserat material [8,9] at the microscale. Homogenization of (discrete) cellular solids has mainly been performed for periodic beam structures leading to macroscopic couple stress theory [10,11] and Cosserat theory [12–14]. In [8] a heterogeneous beam network is subjected to Cosserat homogenization, assuming a priori that the macroscopic response is elastic and isotropic. In the current paper a more general Cosserat homogenization framework is proposed that yields a macroscopic constitutive response that is an outcome of the procedure and depends on the specific (elastic, visco-plastic) material behavior and topology of the underlying cellular microstructure.

2. Cosserat theory: equilibrium and kinematics

In a Cosserat model the independent kinematic degrees of freedom are the displacements U_i and the microrotations Φ_i, $i = 1, 2, 3$. In addition to the classical Cauchy stress Σ_{ij} a material point can support couple stresses M_{ij} (moment per unit area). Force and moment equilibrium dictates

$$\Sigma_{ji,j} = 0 \quad \text{and} \quad M_{ji,j} + e_{ijk}\Sigma_{jk} = 0 \tag{1}$$

where e_{ijk} is the usual permutation tensor. Note that moment equilibrium (1(b)) implies that the Cauchy stress is not symmetric, due to the presence of the couple stresses. Equilibrium on the surface of the body gives

$$T_i = \Sigma_{ji}n_j \quad \text{and} \quad Q_i = M_{ji}n_j \tag{2}$$

where T_i is the surface traction, Q_i the surface couple and n_j the unit vector normal to the surface. The principle of virtual work reads

$$\int_V (\Sigma_{ji}\delta\Gamma_{ji} + M_{ji}\delta K_{ji})\,\mathrm{d}V = \int_S (T_i\delta U_i + Q_i\delta\Phi_i)\,\mathrm{d}S \tag{3}$$

where body forces and couples are omitted for simplicity. The (small) strains Γ_{ji} and curvatures K_{ji} are defined as

$$\Gamma_{ji} = U_{i,j} - e_{kji}\Phi_k \quad \text{and} \quad K_{ji} = \Phi_{i,j} \tag{4}$$

We can decompose the non-symmetric stresses and strains in a symmetric and anti-symmetric part

$$\Sigma_{ji} = S_{ji} + T_{ji} \quad \text{and} \quad \Gamma_{ji} = E_{ji} + A_{ji} \tag{5}$$

where

$$E_{ji} \equiv \frac{1}{2}(\Gamma_{ji} + \Gamma_{ij}) = \frac{1}{2}(U_{i,j} + U_{j,i}) \quad \text{and} \quad A_{ji} \equiv \frac{1}{2}(\Gamma_{ji} - \Gamma_{ij}) = e_{jik}(\Omega_k - \Phi_k). \tag{6}$$

Note that the antisymmetric strain A_{ji} is a measure for the relative rotation between the microrotation Φ_k and the macrorotation $\Omega_k = \frac{1}{2}e_{kij}U_{j,i}$. With this, the principal of virtual work can be written as

$$\int_V (S_{ji}\delta E_{ji} + T_{ji}\delta A_{ji} + M_{ji}\delta K_{ji})\,\mathrm{d}V = \int_S (T_i\delta U_i + Q_i\delta\Phi_i)\,\mathrm{d}S \tag{7}$$

For the special case that the micro- and macrorotations are constrained to be equal, $\Phi_k = \Omega_k$, the relative strain A_{ji} vanishes and the anti-symmetric stress T_{ji} does not contribute to the internal work. In that case the Cosserat (micropolar) theory reduces to couple stress theory [15].

3. Cosserat homogenization framework

3.1. Kinematic boundary conditions

The Cosserat framework of the previous section must be complemented by constitutive equations, relating the dual measures S_{ji} to E_{ji}, T_{ji} to A_{ji} and M_{ji} to K_{ji}. One way to do this is to use a homogenization procedure as outlined in Fig. 1. At the macroscopic scale it is assumed that the material

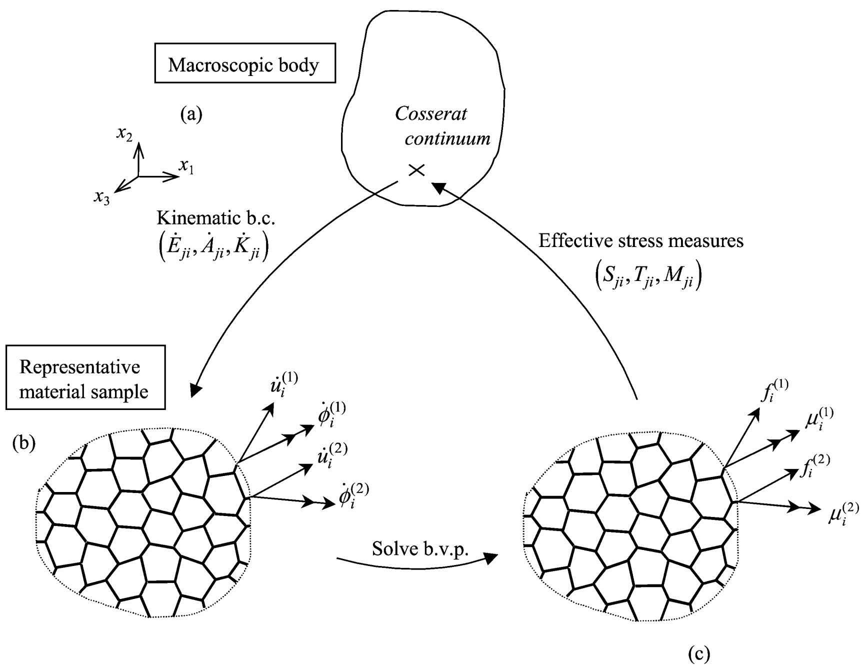

Figure 1. Cosserat homogenization procedure using kinematic boundary conditions.

behaves as a Cosserat solid (see Fig. 1(a)) with the deformation rates given instantaneously by $\dot{E}_{ji}$, $\dot{A}_{ji}$ and $\dot{K}_{ji}$. Then a representative material sample (representative volume element, RVE) with volume V (see Fig. 1(b)) is identified. The RVE can have any disordered cell structure; the cell structure of Fig. 1(b) is an arbitrary choice shown for illustrative purposes. The RVE is subjected to the boundary conditions

$$\dot{u}_i = \dot{E}_{ji} x_j, \qquad \dot{\phi}_i = \frac{1}{2} e_{lji} \dot{A}_{jl} + \dot{K}_{ji}(x_j - X_j) \tag{8}$$

with X_j the coordinates of an arbitrary reference point of the RVE, added to ensure objectivity of the resulting work rate (9). The material sample responds in terms of forces f_i and moments μ_i, at each point on the boundary of the sample (Fig. 1(c)). The average work rate can be written as

$$\overline{\dot{W}} = \frac{1}{V} \left\{ \sum_k \left(f_i^{(k)} \dot{u}_i^{(k)} + \mu_i^{(k)} \dot{\phi}_i^{(k)} \right) \right\} \tag{9}$$

The homogenization is based on the equivalence between $\overline{\dot{W}}$ and the mechanical work rate in the macroscopic material point $\dot{W}$,

$$\overline{\dot{W}} \equiv \dot{W} = S_{ji} \dot{E}_{ji} + T_{ji} \dot{A}_{ji} + M_{ji} \dot{K}_{ji} \tag{10}$$

Substitution of (8) in (9) and (10) results in the following definitions for the effective macroscopic stress and couple stress measures:

$$S_{ji} = \frac{1}{V} \sum_k \frac{1}{2} \left(f_i^{(k)} x_j^{(k)} + f_j^{(k)} x_i^{(k)} \right) \tag{11}$$

$$T_{ji} = \frac{1}{2V} e_{ijl} \sum_k \mu_l^{(k)} \tag{12}$$

$$M_{ji} = \frac{1}{V} \sum_k \mu_i^{(k)} \left(x_j^{(k)} - X_j \right) \tag{13}$$

Note that the expression for S_{ji} is identical to the average stress tensor used in the homogenization of granular media (e.g., [16]).

3.2. Static boundary conditions

A dual problem can be formulated by applying uniform surface tractions (see Fig. 2)

$$t_i = (S_{ji} + T_{ji}) n_j, \qquad m_i = m_i^0 + M_{ji} n_j \tag{14}$$

with

$$m_i^0 = -\frac{\overline{V}_{jk}}{S} e_{ijl} T_{kl}, \quad \text{where } \overline{V}_{jk} = \sum_k x_j^{(k)} n_k^{(k)} \, \mathrm{d}s^{(k)} \text{ and } S = \sum_k \mathrm{d}s^{(k)} \tag{15}$$

The surface tractions and couples are transmitted to the discrete material sample through local force and moment equilibrium (see Fig. 2(b)):

$$f_i^{(k)} = t_i^{(k)} \, \mathrm{d}s^{(k)} \quad \text{and} \quad \mu_i^{(k)} = m_i^{(k)} \, \mathrm{d}s^{(k)} \tag{16}$$

where $f_i^{(k)}$ and $\mu_i^{(k)}$ are the force and moment acting on the individual cell wall (k), which is associated with the surface area $\mathrm{d}s^{(k)}$. Note that these boundary conditions satisfy force equilibrium, $\sum_k f_i^{(k)} = 0$, and

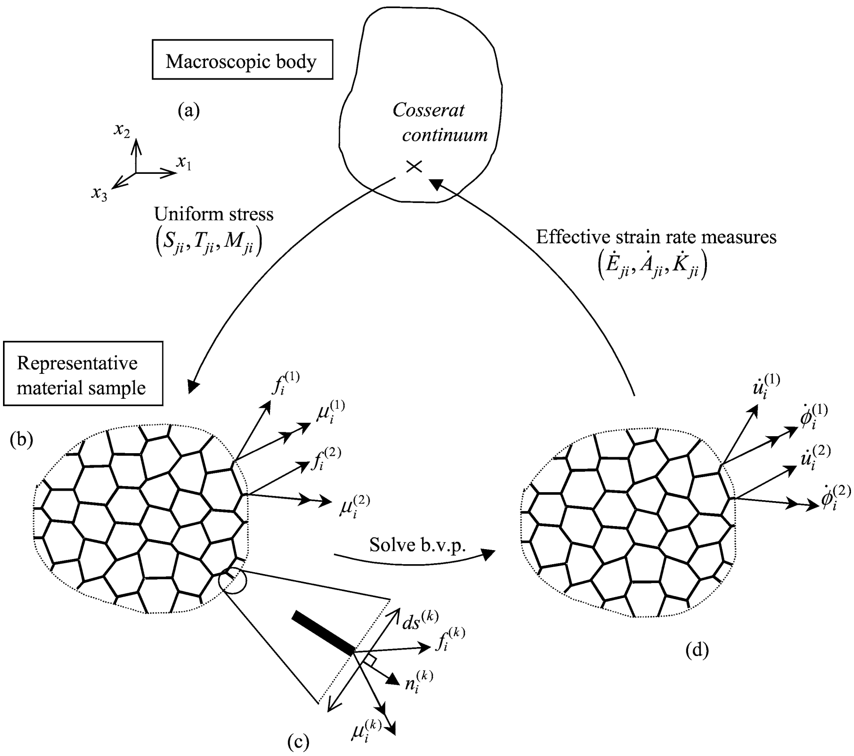

Figure 2. Cosserat homogenization procedure with static boundary conditions.

moment equilibrium, $\sum_k (e_{ijl} x_j^{(k)} f_l^{(k)} + \mu_i^{(k)}) = 0$, of the representative material sample. Using similar homogenization arguments as in Section 3.1 the effective macroscopic strain rate and curvature rate measures can be identified as:

$$\dot{E}_{ji} = \frac{1}{V} \sum_k \frac{1}{2} \left(n_j^{(k)} \dot{u}_i^{(k)} + n_i^{(k)} \dot{u}_j^{(k)} \right) \mathrm{d}s^{(k)} \tag{17}$$

$$\dot{A}_{ji} = \frac{1}{V} \sum_k \left(\frac{1}{2} \left(n_j^{(k)} \dot{u}_i^{(k)} - n_i^{(k)} \dot{u}_j^{(k)} \right) - e_{qpi} \frac{\bar{V}_{pj}}{S} \dot{\phi}_q^{(k)} \right) \mathrm{d}s^{(k)} \tag{18}$$

$$\dot{K}_{ji} = \frac{1}{V} \sum_k n_j^{(k)} \dot{\phi}_i^{(k)} \, \mathrm{d}s^{(k)} \tag{19}$$

Note that the expressions for $\dot{E}_{ji}$ and $\dot{K}_{ji}$ are the discrete equivalents of the average continuum strain rates $\frac{1}{V} \int \frac{1}{2} (n_j \dot{u}_i + n_i \dot{u}_j) \, \mathrm{d}S$ and curvature rates $\frac{1}{V} \int n_j \dot{\phi}_i \, \mathrm{d}S$, respectively. By using similar static boundary conditions and associated effective strain and curvature rate measures it has been demonstrated in [17] that the macroscopic response for a rectangular elasto-viscoplastic lattice can be obtained analytically.

4. Discussion

The key difference between Cosserat and couple stress homogenization lies in the application of the skew-symmetric part of the strain tensor A_{ji}, see Eq. (8), and its conjugate stress measure T_{ji}, Eqs. (14) and (15). Applying A_{ji} in terms of displacements results in a rigid body rotation of the representative material sample which obviously will not induce a mechanical response. By applying A_{ji} in terms of boundary rotations as proposed here (see Eq. (8)), we apply a relative rotation, being the difference between the micro- and macrorotation (see Eq. (6)). Similar considerations are used in [6]. By applying static boundary conditions, Cosserat theory is essentially different from classical continuum and couple stress theory in the sense that a body under uniform tractions $t_i = \Sigma_{ji} n_j$ and couple stresses $m_i = M_{ji} n_j$ does not satisfy moment equilibrium, because the stress tensor is not symmetric. To restore equilibrium we propose here to apply an additional uniform moment m_i^0 (see Eq. (14)) that results in boundary conditions that are mathematically conjugate to the kinematic boundary conditions (8).

The current paper provides a general framework to identify the effective macroscopic Cosserat properties. However, there are still some open issues that need to be addressed.

- Applying a uniform moment m_i^0 (Eq. (14)) is not a unique way of restoring equilibrium in case of static boundary conditions (see [17] for an alternative set of static boundary conditions). Different boundary conditions will lead to a different material response.
- Since Cosserat theory features a length scale associated with rotation gradients, there will inherently be a dependence of the effective 'bending' properties on the size of the representative volume element.

Work is in progress to investigate the effects of boundary conditions and RVE-size on the overall properties. The quality of the obtained overall properties for a certain cellular material (e.g., foams, bone) needs to be assessed by comparing continuum Cosserat with discrete calculations for a specific application.

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