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On the fundamental sloshing frequency in the ice-fishing problem

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Abstract The sloshing problem is considered in a half-space covered by a rigid dock with apertures. The dependence of the fundamental sloshing frequency on the shape of the free surface region is studied. It is proved that the inequality holds between the fundamental eigenvalues corresponding to two different regions if some conditions are fulfilled. These conditions are verified for particular classes of regions of a fixed area in order to demonstrate that the disk yields the maximum of the fundamental eigenvalue for each of these classes. On the other hand, examples of regions are constructed for which the fundamental eigenfrequency is larger than that for the circular aperture of the same area and even as large as one wishes. ***To cite this article: V. Kozlov, N. Kuznetsov, C. R. Mecanique 330 (2002) 723–728.***

fluid mechanics / sloshing problem / fundamental eigenvalue / integral operator / variational principle

Sur la fréquence de ballottement principale dans le problème de la glace avec trous pour pêcher

Résumé On considère le problème du ballottement dans un demi-espace couvert part un couvercle rigide avec des ouvertures. On étudie la dépendance de la valeur propre fondamentale par rapport à la forme de la région de surface libre, sous l'hypothèse que l'aire de cette région est fixée. ***Pour citer cet article : V. Kozlov, N. Kuznetsov, C. R. Mecanique 330 (2002) 723–728.***

mécanique des fluides / oscillations libres d'une fluide / valeur propre fondamenale / opérateur intégral / principe variationnelle

1. Introduction: statement of the problem

The aim of the present Note is to study the dependence of the fundamental sloshing eigenvalue on the shape of the free surface region in the so-called ice-fishing problem (IFP for the sake of brevity in what follows). Our starting point is the following observation. In the linear theory of surface waves, boundary value problems for the two-dimensional Helmholtz equation arise when it is possible to separate the vertical coordinate; that is, when fluid has a constant depth and is bounded by vertical walls (see, for example, [1],

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Section 2.4). Therefore, many classical results for the Helmholtz equation can be reformulated in terms of surface waves. In particular, the Szegö–Weinberger isoperimetric inequality for free membranes (see, for example, [2], Sections 3.3 and 5) provides the following assertion "*For all vertical-walled containers having a given constant depth and a given area of the free surface the circular cylinder yields the maximum of the fundamental sloshing eigenfrequency*". In the recent note [3], a similar result was obtained for the sloshing problem in a half-plane covered by a rigid dock with two equal gaps. Namely, it was demonstrated that "*the zero spacing between gaps yields the maximum of the fundamental sloshing eigenfrequency*". Here we show that the situation is much more complicated for IFP.

Let an inviscid, incompressible, heavy fluid occupy the half-space $\mathbb{R}^3_- = \{x = (x_1, x_2) \in \mathbb{R}^2,\ y < 0\}$ and be covered by a rigid dock with an aperture F which is an arbitrary bounded open set in the plane $y = 0$. Neglecting the surface tension, we consider free, small-amplitude, time-harmonic oscillations of the fluid and its motion is assumed to be irrotational. Mathematically one has to find the eigenvalues ν for the boundary value problem:

$$\nabla^2 u = 0 \quad \text{in } \mathbb{R}^3_-, \qquad u_y = \nu u \quad \text{on } F, \qquad u_y = 0 \quad \text{on } \partial\mathbb{R}^3_- \setminus \overline{F}$$

where $u \in H^1(\mathbb{R}^3_-)$ and satisfies the orthogonality condition $\int_F u(x,0)\,\mathrm{d}x = 0$. This problem is usually referred to as IFP and its eigenvalues produce the sloshing frequencies $\sqrt{\nu/g}$, where g is the acceleration due to gravity. In this Note we are interested only in the fundamental eigenvalue, that is, the smallest positive one which we denote by ν^F.

2. Comparison theorem for fundamental eigenvalues

In [4], it is demonstrated that IFP is equivalent to the following integral equation in $L_2(F)$:

$$u_0 = \nu(I - M)\widehat{K}(I - M)u_0 \tag{1}$$

where $u_0(x) = u(x,0)$, $x \in F$, and $u(x,y)$ is a solution of IFP, I is the identity operator in $L_2(F)$,

$$\big(\widehat{K}u_0\big)(x) = \frac{1}{2\pi}\int_F \frac{u_0(\xi)\,\mathrm{d}\xi}{|x-\xi|}, \qquad Mu_0 = \frac{1}{|F|}\int_F u_0(x)\,\mathrm{d}x, \quad \text{and} \quad |F| = \mathrm{mes}_2\, F$$

The operator in (1) is compact, self-adjoint, and positive semidefinite in $L_2(F)$. If u_0 is an eigenfunction of (1) corresponding to ν, then

$$u(x,y) = \frac{1}{2\pi}\int_F \frac{u_0(\xi)\,\mathrm{d}\xi}{[|x-\xi|^2 + y^2]^{1/2}} - M\widehat{K}u_0 \tag{2}$$

satisfies IFP.

Let F_1 and F_2 be bounded open regions in the plane $y = 0$. The following theorem is proved by using the variational principle for Eq. (1) and provides rather general conditions allowing us to compare ν^{F_1} and ν^{F_2}.

PROPOSITION 1. – *Let $u_0^{F_1} \in L_2(F_1)$ be an eigenfunction of* (1) *with $F = F_1$ corresponding to ν^{F_1} and let the following two assumptions hold*:

$$\int_{F_2} u_0^{F_1}\,\mathrm{d}x = 0, \qquad \int_{F_2}\big(u_0^{F_1}\big)^2\,\mathrm{d}x - \int_{F_1}\big(u_0^{F_1}\big)^2\,\mathrm{d}x + \frac{\nu^{F_1}}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{V(x)V(\xi)\,\mathrm{d}x\,\mathrm{d}\xi}{|x-\xi|} \geqslant 0$$

where $V(x) = [\chi_2(x) - \chi_1(x)]u_0^{F_1}(x)$, χ_1 and χ_2 are the indicator functions of F_1 and F_2, respectively, and the same notation $u_0^{F_1}$ is used for the trace of extension (2) *on the plane $y = 0$. Then $\nu^{F_2} \leqslant \nu^{F_1}$.*

As in the water-wave application of the Szegö–Weinberger isoperimetric property, we use the unit disk $D = \{|x| < 1,\ y = 0\}$ as the gauge free surface domain and compare fundamental eigenvalues of IFP for various regions with $\nu^D = 2.754\ldots$ (see [4] and [5] for this and other numerical results concerning sloshing frequencies for D).

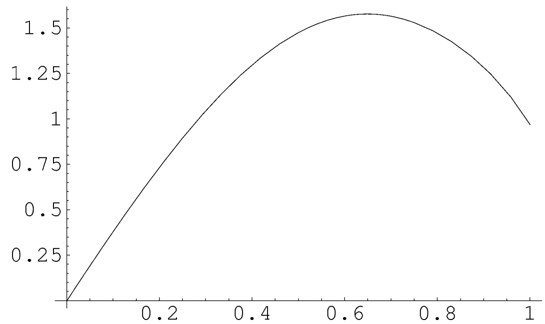

Figure 1. Normalised fundamental eigenfunction $v(r)$ from Eq. (3).

LEMMA 1. – *For $F = D$ the eigenspace of* (1) *corresponding to ν^D is spanned by $v(r)x_i/r$, $r = |x|$, $i = 1, 2$, where $v(r) > 0$ is the fundamental eigensolution of*

$$v(r) = \nu^D \int_0^1 v(s)s\,\mathrm{d}s \int_0^\infty J_1(kr)J_1(ks)\,\mathrm{d}k, \quad r \in (0, 1) \tag{3}$$

The kernel of this equation has a logarithmic singularity as $|r - s| \to 0$.

Using properties of the Bessel function J_1, one obtains that: (i) $v(0) = 0$; (ii) if v is normalised so that $v(1) = 1$, then the following asymptotic formula holds:

$$v(r) - 1 = \nu^D \pi^{-1}(r - 1)\log|r - 1| + \mathrm{O}\big(|r - 1|\big) \quad \text{as } r \to 1 \tag{4}$$

In Fig. 1, the normalised fundamental eigenfunction $v(r)$ of (3) is plotted for $0 \leqslant r \leqslant 1$. Formula (3) with $r > 1$ gives the radial factor in the trace on the plane $y = 0$ of the IFP fundamental eigenfunction and $v(r)$ is convex and monotonically decreasing for $r > 1$, and tends to zero as $r \to \infty$.

Combining Proposition 1 with the above description of the fundamental eigenfunctions in the case when the free surface is the unit disk D, we obtain the following:

COROLLARY 1. – *Let F be a bounded open region such that the following two conditions hold*:

$$\int_F v(r)\,\frac{x_i}{r}\,\mathrm{d}x = 0, \quad i = 1, 2 \tag{5}$$

there exists a nonzero vector (α_1, α_2) such that

$$\left(\int_F - \int_D\right) v^2(r)\frac{(\alpha_1 x_1 + \alpha_2 x_2)^2}{r^2}\,\mathrm{d}x + \frac{\nu^D}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{V(x)V(\xi)\,\mathrm{d}x\,\mathrm{d}\xi}{|x - \xi|} \geqslant 0 \tag{6}$$

where

$$V(x) = \big[\chi_F(x) - \chi_D(x)\big]v(r)\frac{\alpha_1 x_1 + \alpha_2 x_2}{r} \tag{7}$$

and χ_F and χ_D are the indicator functions of F and D, respectively. Then $\nu^F \leqslant \nu^D$ and the equality is attained only if $F = D$.

The next assertion is a straightforward modification of a lemma proven in [6] and demonstrates that (5) holds when the origin is chosen properly. *Let F be a bounded region in the plane $y = 0$, then there exists such a position of the origin within the convex hull of F that* (5) *is valid for $i = 1, 2$.*

Let us give several applications of Corollary 1 to concrete types of the free surface regions of the same area as the unit disk D. Examples 2.1 and 2.2 are concerned with simple families of regions depending on one real parameter, whereas Examples 2.3 and 2.4 deal with small perturbations of D having the form $r = 1 + \varepsilon\psi_0(\theta) + \cdots$. For all of the examples, the unit disk yields a local maximum of the fundamental eigenvalue.

2.1. Annular domains

Let the free surface be an annulus $F_R = \{R < r < \sqrt{R^2+1},\ y = 0\}$, where $R > 0$. It is clear that $|F_R| = |D|$. By R_* we denote the smallest positive root of

$$\int_R^{\sqrt{R^2+1}} v^2(r) r \,\mathrm{d}r = \int_0^1 v^2(r) r \,\mathrm{d}r$$

In order to ensure the existence of R_* we note two facts about the left-hand side in the latter equation: (i) it is larger than the right-hand side for small $R > 0$ because $v(0) = 0$; (ii) it tends to zero as $R \to \infty$ because $v(r) \to 0$ as $r \to \infty$ (in fact, the behaviour of v shown in Fig. 1 guarantees that there is only one root of the last equation). Therefore,

$$\int_R^{\sqrt{R^2+1}} v^2(r) r \,\mathrm{d}r \geqslant \int_0^1 v^2(r) r \,\mathrm{d}r \quad \text{for } R \leqslant R_* \tag{8}$$

It is obvious that (5), $i = 1, 2$, holds for F_R. Moreover, (8) implies that (6) is valid either for $\alpha_1 = 1$ and $\alpha_2 = 0$ or for $\alpha_1 = 0$ and $\alpha_2 = 1$. Then Corollary 1 yields that $\nu^{F_R} < \nu^D$ for $0 < R \leqslant R_*$.

2.2. Thin ice bridge over a circular fluid domain

Let $F_\varepsilon = D_{1+\delta} \setminus \{-\infty < x_1 < +\infty,\ |x_2| < \varepsilon,\ y = 0\}$. Here $\varepsilon > 0$ and $D_{1+\delta}$ is the disk centred at the origin and having radius $1 + \delta$, where δ is chosen so that $|F_\varepsilon| = \pi$. It is clear that (5) is fulfilled for $i = 2$ and (6) is true for $\alpha_1 = 0$ and $\alpha_2 = 1$ when ε is small because the function $v(|x|, 0)\,(x_2/|x|)$ vanishes along the x_1 axis. By Corollary 1 we get that $\nu^{F_\varepsilon} < \nu^D$ for small $\varepsilon > 0$.

2.3. Small perturbation of the unit disk, nondegenerate case

Let $F_\varepsilon = \{r < 1 + \varepsilon\psi(\theta, \varepsilon)\}$, where $\psi(\theta, \varepsilon) = \psi_0(\theta) + \psi_1(\theta, \varepsilon)$. Here ψ_0 and ψ_1 are 2π-periodic, continuous functions and $\psi_1 \to 0$ as $\varepsilon \to 0$ uniformly in $\theta \in [0, 2\pi]$. From the assumption that $|F_\varepsilon| = \pi$, it follows that

$$\int_0^{2\pi} \psi_0(\theta) \,\mathrm{d}\theta = 0 \tag{9}$$

Moreover, the following two equalities

$$\int_0^{2\pi} \psi_0(\theta) \cos\theta \,\mathrm{d}\theta = 0 \quad \text{and} \quad \int_0^{2\pi} \psi_0(\theta) \sin\theta \,\mathrm{d}\theta = 0 \tag{10}$$

must hold when (5) is fulfilled for $i = 1, 2$. Let us assume that either

$$\int_0^{2\pi} \psi_0(\theta) \cos 2\theta \,\mathrm{d}\theta \neq 0 \quad \text{or} \quad \int_0^{2\pi} \psi_0(\theta) \sin 2\theta \,\mathrm{d}\theta \neq 0 \tag{11}$$

Since (9) and (10) are invariant with respect to rotation about the origin, we can suppose without loss of generality that the first of these integrals is nonzero and positive. These relations allow us to verify that (6) holds for $\alpha_1 = 1$ and $\alpha_2 = 0$ in the case when $F = F_\varepsilon$ and $\varepsilon > 0$ is sufficiently small. Now Corollary 1 yields that $\nu^{F_\varepsilon} < \nu^D$ for such values of ε. In particular, *for all ellipses, having area equal to π and small value of eccentricity, the unit disk yields the maximum fundamental sloshing frequency.*

2.4. Small perturbation of the unit disk, degenerate case

Let F_ε be the same as in 2.3, but now we additionally require ψ_0 to be Lipschitz continuous and suppose that it not only satisfies the orthogonality conditions (9) and (10), but both integrals in (11) do vanish (therefore, the proof applicable for 2.3 fails to guarantee the result now). Moreover, let the following asymptotic representation $\psi_1(\theta,\varepsilon)=\varepsilon\log\varepsilon\Psi_1(\theta)+\mathrm{O}(\varepsilon)$ hold uniformly in $\theta\in[0,2\pi]$ as $\varepsilon\to 0$. Then from the area preserving condition $|F_\varepsilon|=\pi$, we get that $\int_0^{2\pi}\Psi_1(\theta)\,\mathrm{d}\theta=0$ must be valid along with (9). Furthermore, (5) and Lemma 1 imply that

$$\int_0^{2\pi}\Psi_1(\theta)\cos\theta\,\mathrm{d}\theta=-\frac{\nu^D}{2\pi}\int_0^{2\pi}\psi_0^2(\theta)\cos\theta\,\mathrm{d}\theta$$

and the similar equality with sine instead of cosine must be valid along with (10). Finally, let us assume that either

$$\int_0^{2\pi}\Psi_1(\theta)\cos 2\theta\,\mathrm{d}\theta\neq 0\quad\text{or}\quad\int_0^{2\pi}\Psi_1(\theta)\sin 2\theta\,\mathrm{d}\theta\neq 0$$

Again the invariance with respect to rotation about the origin, allows us without loss of generality to suppose that the first of these integrals is nonzero and negative. These assumptions allow us to verify that (6) is valid for $\alpha_1=1$ and $\alpha_2=0$ if $\varepsilon>0$ is sufficiently small. Then Corollary 1 yields that $\nu^{F_\varepsilon}<\nu^D$ for such values of ε.

In Examples 2.1–2.3, it was sufficient to show that the difference of the first two terms on the left-hand side of (6) is positive, thus guaranteeing the inequality between eigenvalues to be valid. In the present example, all terms on the left-hand side of (6) are involved in the demonstration that (6) holds.

3. Regions with the fundamental eigenvalue larger than ν^D

The aim of this section is to provide conditions on a plane region F, guaranteeing that the fundamental eigenvalue of the corresponding IFP is larger than ν^D.

PROPOSITION 2. – *Let F be an open subset of $D_2=\{r<2,\ y=0\}$ and such that the following assumptions hold*:
(i) $|F|=|D|$, (5) *is valid, and $\varepsilon=|F\setminus D|$ is smaller than a certain absolute constant* ε_0;
(ii) *for all $\alpha_1,\alpha_2\in\mathbb{R}$ the following inequality holds*:

$$\left(\int_F-\int_D\right)v^2(r)\frac{(\alpha_1x_1+\alpha_2x_2)^2}{r^2}\,\mathrm{d}x+\frac{\nu^D}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{V(x)V(\xi)\,\mathrm{d}x\,\mathrm{d}\xi}{|x-\xi|}\leqslant-\kappa\left(\alpha_1^2+\alpha_2^2\right)$$

Here v and V are the functions defined by (3) *and* (7), *respectively, and κ is a constant such that $|\kappa|<\varepsilon_0$. Then*

$$\frac{1}{\nu^F}\leqslant\frac{1}{\nu^D}-C\kappa+C_\beta\varepsilon^{1+\beta}$$

where C is an absolute positive constant, β is an arbitrary number in $(0,1)$, *and C_β is a positive constant depending only on β.*

The following example illustrates this theorem and demonstrates that even a simply connected fishing hole close to the unit disk D in the sense of Hausdorff can have the fundamental eigenvalue larger than that for D. However, the boundary of the corresponding hole is rather complicated (it is not convex and is as long as approximately 6π).

3.1. Ice inclusions in a circular free-surface domain

Let $F_\delta=D_{1+\delta_0}\setminus\{a-\delta\leqslant r\leqslant a\}$, where $0<a<1$, $0<\delta<a$, and $\delta_0=\sqrt{1+2a\delta-\delta^2}-1=a\delta-2^{-1}(1+a^2)\delta^2+\mathrm{O}(\delta^3)$, and so $|F_\delta|=\pi$. For F_δ, (5) holds in view of the axial symmetry. Moreover, if δ is sufficiently small, then $F_\delta\subset D_2$ and $\varepsilon=|D\setminus F_\delta|=\pi\delta(2a-\delta)<\varepsilon_0$.

Under these assumptions, Proposition 2 yields that

$$\frac{1}{\nu^{F_\delta}}\leqslant\frac{1}{\nu^D}-C\delta\left[v^2(a)-v^2(1)\right]+\mathrm{o}(\delta)\tag{12}$$

According to (4), there exists a value of a that is close to one and provides the inequality $v(a) > v(1)$. Then (12) implies that $\nu^D < \nu^{F_\delta}$.

In our example, F_δ is the union of a disk and an annulus and so it is not a domain. Let us modify F_δ in order to obtain a *domain* F'_δ such that $\nu^{F'_\delta}$ is still larger than ν^D. For this purpose, we connect the ice annulus $\{a - \delta \leqslant r \leqslant a\}$ with the main sheet of ice $\{y = 0\} \setminus D_{1+\delta_0}$ by an ice bridge, having its sides parallel, say, to the x_1 axis and its width proportional to δ^2. On the opposite side of the ice annulus, we make a gap in it thus connecting the main circular free surface with the annular strip of the water surface. Let the sides of this gap also be parallel to the x_1 axis, but the width of the gap must be proportional to δ in order to equalize the areas of the gap and the bridge. Thus we obtain F'_δ, for which our considerations remain valid up to changing the term $\mathrm{o}(\delta)$ in (12), but this does not violate the inequality for the fundamental eigenvalues.

4. Free surface regions providing arbitrarily large fundamental eigenvalues

Let F be the union of N disks $D^{(i)}$ $(i = 1, \ldots, N)$, having the same radius $N^{-1/2}$, which implies that $|F| = \pi$. Let these disks be placed on the plane $y = 0$ so that the distance from $D^{(i)}$ to $D^{(j)}$ is larger than L for $i \neq j$.

Starting from the variational principle for compact, self-adjoint operators, we observe that

$$\frac{1}{\nu^F} \leqslant \max \frac{(\widehat{K}\, w, w)_F}{\|w\|_F^2}$$

where the maximum being taken over all nonzero elements of $L_2(F)$. Splitting w into the following sum:

$$w = \sum_{i=1}^{N} w_i, \quad \text{where } w_i = \begin{cases} w & \text{in } D^{(i)} \\ 0 & \text{elsewhere} \end{cases}$$

stretching coordinates, and applying the Schwarz inequality, one arrives at

$$\left(\nu^F\right)^{-1} \leqslant C N^{-1/2} + (2L)^{-1}$$

where C is an absolute constant. This inequality shows that $\nu^F \to \infty$ as $L, N \to \infty$.

This result is natural from the physical point of view. Indeed, sloshing in a hole that is sufficiently remote from others is an independent process and the fundamental frequency of sloshing in a single hole can attain as large value as one wishes by making the radius of hole sufficiently small.

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