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Asymptotics for the eigenelements of vibrating membranes with very heavy thin inclusions

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Abstract We consider the vibrations of a membrane that contains a very thin and heavy inclusion around a curve γ. We assume that the membrane occupies a domain Ω of $\mathbb{R}^2$. The inclusion occupies a layer-like domain $\omega_\varepsilon \subset \Omega$ of width 2ε and it has a density of order $\mathrm{O}(\varepsilon^{-3})$. The density is of order $\mathrm{O}(1)$ outside this inclusion, the *concentrated mass around the curve* γ. ε is a positive parameter, $\varepsilon \in (0,1)$. By means of asymptotic expansions, we describe the behaviour, as $\varepsilon \to 0$, of the eigenelements $(\lambda^\varepsilon, u^\varepsilon)$ of the associated spectral problem. We provide complete asymptotic series for the low frequencies $\lambda^\varepsilon = \mathrm{O}(\varepsilon^2)$, the medium frequencies $\lambda^\varepsilon = \mathrm{O}(\varepsilon)$ and the corresponding eigenfunctions u^ε. ***To cite this article: Y. Golovaty et al., C. R. Mecanique 330 (2002) 777–782.***

vibrations / concentrated masses / spectral analysis / low frequencies / medium frequencies

Comportement asymptotique des éléments propres pour des membranes vibrantes avec des couches très minces et très lourdes

Résumé On considère les vibrations d'une membrane qui contient une très mince et très lourde inclusion placée autour d'une courbe γ. On suppose que la membrane occupe un domaine $\Omega \subset \mathbb{R}^2$, tandis que l'inclusion occupe une couche $\omega_\varepsilon \subset \Omega$ de largeur 2ε, la densité étant d'ordre $\mathrm{O}(\varepsilon^{-3})$. La densité est d'ordre $\mathrm{O}(1)$ en dehors de la petite inclusion : *la masse est concentrée autour de* γ. ε est un petit paramètre, $\varepsilon \in (0,1)$. À l'aide des développements asymptotiques, nous décrivons le comportement, pour $\varepsilon \to 0$, des éléments propres $(\lambda^\varepsilon, u^\varepsilon)$ du problème spectral associé. En fait, nous obtenons les séries asymptotiques complètes pour les basses fréquences $\lambda^\varepsilon = \mathrm{O}(\varepsilon^2)$ et les moyennes fréquences $\lambda^\varepsilon = \mathrm{O}(\varepsilon)$, ainsi que les fonctions propres correspondantes u^ε. ***Pour citer cet article : Y. Golovaty et al., C. R. Mecanique 330 (2002) 777–782.***

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Soit Ω un domaine borné de $\mathbb{R}^2$ avec une frontière régulière $\partial\Omega$. On suppose que Ω est divisé en deux parties Ω_+ et Ω_- par l'interface $\gamma : \Omega = \Omega_+ \cup \Omega_- \cup \gamma$. γ est une courbe fermée telle que $\gamma \subset \Omega$, et Ω_+ et Ω_- sont domaines ouverts, bornés et connexes avec des frontières régulières : $\partial\Omega_+ = \gamma$ et $\partial\Omega_- = \gamma \cup \partial\Omega$. On dénote par ω_ε le ε-voisinage de γ et par Ω_ε le domaine $\Omega_\varepsilon = \Omega \setminus \omega_\varepsilon$, où ε est un petit paramètre telle que $\overline{\omega_\varepsilon} \subset \Omega$.

Soit (n, τ) les coordonnées curvilignes naturelles dans un voisinage de γ, τ est la longueur de l'arc et n la distance tout au long de la normale intérieur $\bar{n}$ dirigée vers Ω_+. Soit $\varkappa(\tau)$ la courbure de γ au point τ. Pour les fonctions régulières f définies dans $\overline{\Omega}_+$ ($\overline{\Omega}_-$, respectivement), $f(0+, \tau)$ et $f|_{\gamma^+}$ ($f(0-, \tau)$ et $f|_{\gamma^-}$, respectivement) dénotent la trace de f sur γ.

On considère dans Ω le problème aux valeurs propres (1) où $\rho_\varepsilon(x) = p(x)$ si $x \in \Omega_\varepsilon$, et $\rho_\varepsilon(x) = \varepsilon^{-m} q_\varepsilon(x)$ si $x \in \omega_\varepsilon$, m un paramètre positif et p et q_ε des fonctions régulières et positives dans Ω et ω_ε respectivement, $q_\varepsilon(x) = q(\frac{n}{\varepsilon})$. ρ_ε est la fonction densité dans Ω. Nous étudions le comportement asymptotique des éléments propres $(\lambda^\varepsilon, u^\varepsilon)$ de (1), lorsque $\varepsilon \to 0$.

Pour chaque $\varepsilon > 0$ fixé, (1) est un standard problème aux valeurs propres. Soient $\{\lambda_i^\varepsilon\}_{i=1}^\infty$ la suite des valeurs propres positives, avec la convention classique des valeurs propres répétées, et $\{u_i^\varepsilon\}_{i=1}^\infty$ les correspondantes fonctions propres normalisées dans $H_0^1(\Omega)$.

On considère un cas particulier $m = 3$. Le principe du minimaux nous donne les estimations $C\varepsilon^2 < \lambda_i^\varepsilon < C_i\varepsilon^2$ (voir Proposition 1), et permet de postuler le développement asymptotique (3) pour les *basses fréquences* $\lambda^\varepsilon = \mathrm{O}(\varepsilon^2)$. Par analogie avec les systèmes avec des masses concentrées dans points (voir, par exemple, [1–3]), les valeurs propres $\lambda^\varepsilon = \lambda_{i(\varepsilon)}^\varepsilon$ d'ordre $\mathrm{O}(1)$ sont appelées les *hautes fréquences*, et les valeurs propres d'ordre $\mathrm{O}(\varepsilon)$ sont les *moyennes fréquences*.

Nous obtenons les développements asymptotiques pour les moyennes fréquences (voir (4)). Ces développements sont aussi valides pour les basses fréquences (3) lorsque le premier terme dans le développement (4) est zéro (voir Remarque 1). Également, nous obtenons les développements asymptotiques pour les fonctions propres correspondantes (voir (5), (6) et (17)). Dans la Section 2, on obtient les premiers termes de ces développements asymptotiques. On montre que si les fonctions propres associées u^ε ne s'annulent pas asymptotiquement, les fréquences moyennes λ^ε sont approchées par les valeurs propres d'un problème local notamment, le problème de Neumann dans l'intervalle $(-1, 1)$ (voir (13)). En plus, nous montrons que les valeurs propres d'un certain problème de Steklov dans Ω (voir (14)–(16)) sont des termes correcteurs pour ces fréquences ; d'ailleurs, les fonctions propres u^ε sont approchées par convenables raccordements (voir (17)) des fonctions propres de tous les deux problèmes. Dans la Section 3 nous donnons les termes d'ordre supérieur des développements asymptotiques (4)–(6).

Il faut signaler que, puisque les fréquences $\lambda^\varepsilon = \mathrm{O}(\varepsilon^\alpha)$, avec $0 \leqslant \alpha < 2$, s'accumulent sur tout l'axe réel positif (voir [6]), les développements asymptotiques pour les moyennes fréquences sont données par $\lambda^\varepsilon = \lambda_{i(\varepsilon)}^\varepsilon$, avec $i(\varepsilon) \to \infty$ pour $\varepsilon \to 0$ (voir (4) avec $\lambda_0 \neq 0$). Par contre, pour les basses fréquences, pour chaque i fixé, nous trouvons les développements asymptotiques (3) de $\lambda^\varepsilon = \lambda_i^\varepsilon$, où $m = 3$ et μ^0 est la i-ème valeur propre d'un problème de Steklov (voir (14)–(16) et Remarque 1).

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1. Introduction and statement of the problem

In this paper we address the asymptotic behaviour of the vibrations of a system composed of a membrane that contains a thin region which has a density much higher than the rest of the membrane, *a concentrated mass around a curve*. The limit problems and the results that we obtain are very different from those in the literature on vibrating systems with concentrated masses: see [1–3] for concentrated masses at points and [4] and [5] for a concentrated mass around a curve and a hyperplane respectively.

Let Ω be an open bounded domain of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$. We assume that Ω is divided in two parts Ω_+ and Ω_- by the interface γ: $\Omega = \Omega_+ \cup \Omega_- \cup \gamma$. We consider γ a smooth closed curve,

$\gamma \subset \Omega$, such that Ω_+ and Ω_- are open connected bounded domains with smooth boundaries: $\partial\Omega_+ = \gamma$ and $\partial\Omega_- = \gamma \cup \partial\Omega$. We denote by ω_ε the ε-neighborhood of γ and by Ω_ε the domain $\Omega_\varepsilon = \Omega \setminus \omega_\varepsilon$; ε is a small positive parameter such that $\overline{\omega_\varepsilon} \subset \Omega$.

Let (n, τ) be the natural orthogonal curvilinear coordinates in a neighborhood of γ; τ is the arc length and n the distance along the inner normal $\bar{n}$ towards Ω_+. Let $\varkappa(\tau)$ denote the curvature of the curve γ at the point τ. For smooth functions f defined in $\overline{\Omega}_+$ ($\overline{\Omega}_-$, respectively), both $f(0+, \tau)$ and $f|_{\gamma^+}$ ($f(0-, \tau)$ and $f|_{\gamma^-}$, respectively) denote the trace of f on γ.

We consider in Ω the eigenvalue problem:

$$\begin{cases} \Delta u^\varepsilon + \lambda^\varepsilon \rho_\varepsilon u^\varepsilon = 0 & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\rho_\varepsilon(x) = p(x)$ if $x \in \Omega_\varepsilon$, and $\rho_\varepsilon(x) = \varepsilon^{-m} q_\varepsilon(x)$ if $x \in \omega_\varepsilon$, for m a positive parameter and p and q_ε some smooth positive functions in Ω and ω_ε respectively, $q_\varepsilon(x) = q(\frac{n}{\varepsilon})$. ρ_ε is the density function in Ω. We study the asymptotic behaviour of the eigenelements $(\lambda^\varepsilon, u^\varepsilon)$ of (1), as $\varepsilon \to 0$.

For each fixed $\varepsilon > 0$, (1) is a standard eigenvalue problem. Let us consider $\{\lambda_i^\varepsilon\}_{i=1}^\infty$ the sequence of positive eigenvalues, with the classical convention of repeated eigenvalues. Let $\{u_i^\varepsilon\}_{i=1}^\infty$ be the corresponding eigenfunctions, which are a basis in $H_0^1(\Omega)$. An estimate for the eigenvalues can be obtained by using the minimax principle (cf., for instance, [1] for the technique):

PROPOSITION 1. – *For each $i = 1, 2, 3, \ldots$, we have*:

$$C < \lambda_i^\varepsilon < C_i \quad \text{when } 0 \leqslant m \leqslant 1, \quad \text{and} \quad C\varepsilon^{m-1} < \lambda_i^\varepsilon < C_i \varepsilon^{m-1} \quad \text{when } m \geqslant 1 \tag{2}$$

where C, C_i are constants independent of ε and $C_i \to \infty$ when $i \to \infty$.

Proposition 1 allows us to postulate an expansion for $\lambda^\varepsilon = \lambda_i^\varepsilon$, for a given $i \in \mathbb{N}$:

$$\lambda^\varepsilon = \varepsilon^{m-1}\mu^0 + \varepsilon^m \mu^1 + \cdots \quad \text{when } m \geqslant 1 \tag{3}$$

and, consequently, to obtain different limit problems for μ^0 depending on the value of the parameter m. These eigenvalues $\lambda^\varepsilon = \mathrm{O}(\varepsilon^{m-1})$ are the so-called *low frequencies*. By analogy with systems with concentrated masses at points (cf., for instance, [1,2] and [3]), the eigenvalues $\lambda^\varepsilon = \lambda_{i(\varepsilon)}^\varepsilon$ of order $\mathrm{O}(1)$ are referred to as the *high frequencies*. In this way, the eigenvalues λ^ε of order $\mathrm{O}(\varepsilon^{m-2})$ are in the range of the *medium frequencies*.

In what follows we consider the case $m = 3$ (cf. [6] for other values of m), as a very distinctive case: the low frequencies, high frequencies and medium frequencies have to be considered in order to describe the asymptotic behaviour of the vibrations. From formal asymptotics, as $\varepsilon \to 0$, the behaviour of the low frequencies is described by the spectrum of a Steklov problem in Ω with the "term mass" appearing only on γ. The high frequency vibrations deal with a classical Dirichlet problem in either Ω_+ or Ω_-. Thus, both kinds of frequencies, low and high give rise to global vibrations of the structure.

In this paper, we provide asymptotics for the medium frequencies (cf. (4)). These asymptotics include the asymptotics for the low frequencies (3), as a particular case, when the first term in the expansion is zero (see Remark 1). Also, we provide asymptotics for the corresponding eigenfunctions (cf. (17)). In Section 2, we obtain the first terms of these asymptotic expansions. We prove that, provided that the associated eigenfunctions u^ε do not vanish asymptotically, the medium frequencies λ^ε are approached by the eigenvalues of a local problem, the Neumann problem in the interval $(-1, 1)$. In addition, we show that the eigenvalues of a Steklov type problem on Ω appear as correcting terms for these frequencies; the eigenfunctions u^ε are approached by "matched eigenfunctions" of both problems (cf. (17)). In Section 3 we give the high order terms of the asymptotic expansions.

Note that, since the large frequencies $\lambda^\varepsilon = \mathrm{O}(\varepsilon^\alpha)$, with $0 \leqslant \alpha < m - 1$, accumulate on the whole positive real axis (cf. [6]), the asymptotic expansions for the medium frequencies are given for $\lambda^\varepsilon = \lambda_{i(\varepsilon)}^\varepsilon$, with $i(\varepsilon) \to \infty$ as $\varepsilon \to 0$ (see (4) when $\lambda_0 \neq 0$). Instead, for the low frequencies, for each fixed i, we find the asymptotic expansions (3) of $\lambda^\varepsilon = \lambda_i^\varepsilon$, where $m = 3$ and μ^0 is the i-th eigenvalue of a Steklov problem (see (14)–(16) and Remark 1).

2. Asymptotics for the medium frequencies

We use the technique of the matched asymptotic expansions (cf. Section VI.6 in [3], for instance) to derive the asymptotic behaviour of the eigenelements of (1). Since a boundary layer phenomenon occurs in a neighborhood of γ, we introduce local coordinates. That is, we perform a dilatation of a neighborhood of γ, in the n direction, by introducing the local variable $\xi = n/\varepsilon$ which transforms $(n, \tau) \in (-\varepsilon, \varepsilon) \times [0, \ell)$ into $(\xi, \tau) \in (-1, 1) \times [0, \ell)$, ℓ being the length of γ.

We postulate an asymptotic expansion for the eigenvalues λ^ε, an outer expansion for the corresponding eigenfunctions u^ε and a local expansion in a neighborhood of the boundary layer γ:

$$\lambda^\varepsilon \sim \varepsilon(\lambda_0 + \varepsilon\lambda_1 + \cdots) \tag{4}$$

$$u^\varepsilon(x) \sim v_0(x) + \varepsilon v_1(x) + \cdots \quad \text{if } x \in \Omega \setminus \gamma \tag{5}$$

$$u^\varepsilon(x) \sim w_0(\xi, \tau) + \varepsilon w_1(\xi, \tau) + \cdots \quad \text{if } (\xi, \tau) \in (-A, A) \times [0, \ell),\ \forall A > 0 \tag{6}$$

respectively, where w_k are *ℓ-periodic* functions with respect to τ. Besides, we assume that the first term λ_0 in (4) can be 0 while either v_0 in (5) or w_0 in (6) are different from zero (see Remark 1).

We write the Laplace operator in curvilinear coordinates (n, τ), where we introduce the change of variable $n = \varepsilon\xi$, and we consider, for sufficiently small ε, the power series $(1 - \varepsilon\xi\varkappa)^{-1} = \sum_{s=0}^{\infty}(\varepsilon\xi\varkappa)^s$. Then, gathering the different powers of ε we can write

$$\Delta_{\xi,\tau} \sim \varepsilon^{-2} \sum_{i=0}^{\infty} \varepsilon^i L_i(\partial_\xi, \partial_\tau)$$

where operators L_i are given by: $L_0 = \partial_\xi^2$, $L_1 = -\varkappa(\tau)\partial_\xi$, $L_2(\partial_\xi, \partial_\tau) = -\varkappa^2\xi\partial_\xi + \partial_\tau^2$, and

$$L_i(\partial_\xi, \partial_\tau) = -\varkappa^i\xi^{i-1}\partial_\xi + (i-1)\varkappa^{i-2}\xi^{i-2}\partial_\tau^2 + \tfrac{1}{2}(i-1)(i-2)\varkappa^{i-3}\varkappa'\xi^{i-2}\partial_\tau, \quad \text{when } i \geqslant 3 \tag{7}$$

Then, by replacing expansions (4)–(6) in (1), after considering the classical transmission conditions on the boundary of ω_ε ($[u^\varepsilon]_{\partial\omega_\varepsilon} = [\partial u^\varepsilon/\partial n]_{\partial\omega_\varepsilon} = 0$) and the classical Taylor expansions in a neighborhood of $n = 0$ (for fixed τ), we easily obtain, at the first step, that the leading terms in (4)–(6) satisfy equations:

$$\frac{\partial^2 w_0}{\partial\xi^2} + \lambda_0 q(\xi) w_0(\xi, \tau) = 0, \quad \xi \in (-1, 1),\ \tau \in \gamma \tag{8}$$

$$\frac{\partial w_0}{\partial\xi}(-1, \tau) = 0, \qquad \frac{\partial w_0}{\partial\xi}(1, \tau) = 0 \tag{9}$$

$$\Delta v_0 = 0 \quad \text{in } \Omega \setminus \gamma, \qquad v_0 = 0 \quad \text{on } \partial\Omega \tag{10}$$

$$v_0(0-, \tau) = w_0(-1, \tau), \qquad v_0(0+, \tau) = w_0(1, \tau) \tag{11}$$

We observe that τ is a parameter in (8), (9) and v_0, the solution of (10), (11), is a well determined function once that w_0 in (8), (9) is also determined. Nevertheless, it is known (cf. Section IV.3 of [3]) that the spectrum of problem (8), (9) is formed by a countable infinity of eigenvalues, $\{\lambda_{0,i}\}_{i=0}^{\infty}$, $\lambda_{0,i} \to \infty$ as $i \to \infty$, each $\lambda_{0,i}$ with infinite multiplicity; the associated eigenfunctions are:

$$w_0(\xi, \tau) = W_{0,i}(\xi)\alpha(\tau) \tag{12}$$

where $\alpha(\tau)$ is an arbitrary function of τ, $\tau \in \gamma$, $\alpha(\tau)$ ℓ-periodic, and $W_{0,i}(\xi)$, satisfying $\int_{-1}^{1} q W_{0,i}^2\,\mathrm{d}\xi = 1$, is the eigenfunction associated with the single eigenvalue $\lambda_{0,i}$ of the problem:

$$\begin{cases} \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + \lambda q(\xi) u = 0, & \xi \in (-1, 1) \\ u'(-1) = 0, \quad u'(1) = 0 \end{cases} \tag{13}$$

In what follows, for each fixed $i = 1, 2, \ldots$, we denote by σ_+ and σ_- the non null constants: $\sigma_+ = W_{0,i}(+1)$ and $\sigma_- = W_{0,i}(-1)$. σ_+ and σ_- are related by $\int_{-1}^{1} W_{0,i} \frac{dW_{0,i}}{d\xi}\, d\xi = \frac{1}{2}(\sigma_+^2 - \sigma_-^2)$. Then, from (11) and (12), we deduce that $\alpha(\tau)$, or equivalently w_0, and v_0 determine each other reciprocally by the relation: $\alpha(\tau) = \frac{1}{\sigma_-} v_0(0-, \tau) = \frac{1}{\sigma_+} v_0(0+, \tau)$. In order to determine both functions α and v_0, we consider the second order terms in (4)–(6) that satisfy (18), (19) for $k = 1$. That is, a non homogeneous problem associated with (8), (9). The compatibility condition for (18), (19), in the case where $k = 1$, along with (10) and (11), allows us to write an eigenvalue problem whose eigenelements determine λ_1 and v_0:

$$\Delta v_0 = 0 \quad \text{in } \Omega \setminus \gamma, \qquad v_0 = 0 \quad \text{on } \partial\Omega \tag{14}$$

$$\sigma_+ v_0|_{\gamma^-} = \sigma_- v_0|_{\gamma^+} \tag{15}$$

$$\sigma_+^2 \frac{\partial v_0}{\partial n}\bigg|_{\gamma^+} - \sigma_-\sigma_+ \frac{\partial v_0}{\partial n}\bigg|_{\gamma^-} - \left(\frac{1}{2}\big(\sigma_+^2 - \sigma_-^2\big)\varkappa(\tau) - \lambda_1\right) v_0\bigg|_{\gamma^+} = 0 \tag{16}$$

Now, by setting $v_0(x) = \sigma_+\Phi(x)$ if $x \in \Omega_+$ and $v_0(x) = \sigma_-\Phi(x)$ if $x \in \Omega_-$, we write (14)–(16) as a classical eigenvalue Steklov problem in $H_0^1(\Omega)$ for the eigenfunction Φ with the spectral parameter λ_1 appearing only in the equation on γ. It is known that this problem has a real and discrete spectrum converging to ∞. Therefore, (λ_1, v_0) is an eigenlement of (14)–(16), where we choose v_0 of norm 1 in $L^2(\gamma^+)$. Also, from (11), w_0 is a well determined eigenfunction associated with the eigenvalue λ_0 of (8), (9).

Hence, we have the composite expansions in Ω for the eigenfunctions u^ε of (1) associated with the eigenvalues $\lambda^\varepsilon \sim \varepsilon\lambda_{0,i} + \varepsilon^2\lambda_{1,j} + O(\varepsilon^3)$:

$$u^\varepsilon(x) \sim \begin{cases} v_{0,j}(x) & \text{for } x \in \Omega \setminus \omega_\varepsilon \\ \frac{1}{W_{0,i}(1)} W_{0,i}\left(\frac{n}{\varepsilon}\right) v_{0,j}(0+, \tau) + v_{0,j}(n, \tau) - v_{0,j}(0+, \tau) & \text{for } (n, \tau) \in \omega_\varepsilon,\ n \geqslant 0 \\ \frac{1}{W_{0,i}(1)} W_{0,i}\left(\frac{n}{\varepsilon}\right) v_{0,j}(0+, \tau) + v_{0,j}(n, \tau) - v_{0,j}(0-, \tau) & \text{for } (n, \tau) \in \omega_\varepsilon,\ n \leqslant 0 \end{cases} \tag{17}$$

where $\{(\lambda_{0,i}, W_{0,i})\}_{i=0}^{\infty}$ are the eigenelements of (13) and $\{(\lambda_{1,j}, v_{0,j})\}_{j=1}^{\infty}$ are the eigenelements of (14)–(16).

It should be noticed that the limit problem (8), (9), (14)–(16) gives us the two-term expansion of λ^ε. Also, we note that for each fixed i we have a splitting of the eigenvalues at second order.

3. The complete asymptotic series

In this section we assume that λ_1 is a single eigenvalue of (14)–(16). The process used to find the leading terms in (4)–(6) can be continued to systematically construct the other terms of the asymptotic expansions (4)–(6). We outline here below the general procedure.

To the order $O(\varepsilon)$, we find λ_2, w_1 and v_1, depending on λ_1, λ_0, v_0 and w_0. In general, to the order $O(\varepsilon^k)$, we compute λ_{k+1}, v_k and w_k in terms of $\{\lambda_i\}_{i=0}^{k}$ and $\{v_i, w_i\}_{i=0}^{k-1}$.

Indeed, let us suppose that values λ_i for $i = 0, 1, \ldots, k$, and functions v_i, w_i for $i = 0, 1, \ldots, k-1$, in the asymptotics (4)–(6), are already determined. Then, considering (7) for $k \geqslant 1$, we obtain that λ_{k+1}, v_k and w_k satisfy equations:

$$\frac{\partial^2 w_k}{\partial \xi^2} + \lambda_0 q w_k = \varkappa \frac{\partial w_{k-1}}{\partial \xi} - \lambda_k q w_0 + h_k(\xi, \tau), \quad \xi \in (-1, 1),\ \tau \in \gamma \tag{18}$$

$$\frac{\partial w_k}{\partial \xi}(\pm 1, \tau) = \frac{\partial v_{k-1}}{\partial n}(0\pm, \tau) + \eta_k^{\pm}(\tau) \tag{19}$$

$$\Delta v_k = g_k(x) \quad \text{in } \Omega \setminus \gamma, \qquad v_k = 0 \quad \text{on } \partial\Omega \tag{20}$$

$$v_k(0\pm, \tau) - w_k(\pm 1, \tau) = \varphi_k^{\pm}(\tau) \tag{21}$$

$$\sigma_+^2 \frac{\partial v_k}{\partial n}\bigg|_{\gamma^+} - \sigma_-\sigma_+ \frac{\partial v_k}{\partial n}\bigg|_{\gamma^-} - \left(\frac{1}{2}\left(\sigma_+^2 - \sigma_-^2\right)\varkappa - \lambda_1\right) v_k\bigg|_{\gamma^+} = -\lambda_{k+1} v_0|_{\gamma^+} + S_k \quad \text{on } \gamma \tag{22}$$

where functions f_k, g_k, $\eta_k^{\pm}$, $\varphi_k^{\pm}$ and S_k are known, namely

$$h_1 = 0, \quad \eta_1^{\pm} = 0, \quad \text{and} \quad h_k = -\lambda_1 q w_{k-1} + f_k(\xi, \tau) \quad \text{when } k \geqslant 2$$

$$f_2 = -L_2 w_0, \quad f_k = -\sum_{i=2}^{k} L_i w_{k-i} - \sum_{i=2}^{k-1} \lambda_i q w_{k-i}, \quad g_k = -\sum_{i=1}^{k} \lambda_{i-1} p v_{k-i}$$

$$\eta_k^{\pm}(\tau) = \sum_{i=1}^{k-1} \frac{(\pm 1)^i}{i!} \frac{\partial^{i+1} v_{k-i-1}}{\partial n^{i+1}}(0\pm, \tau), \qquad \varphi_k^{\pm}(\tau) = -\sum_{i=1}^{k} \frac{(\pm 1)^i}{i!} \frac{\partial^i v_{k-i}}{\partial n^i}(0\pm, \tau)$$

$$S_k(\tau) = \sigma_+ \psi_k(\tau) + \sigma_+ \varkappa \int_{-1}^{1} W_{0,i}\, \partial_\xi W_k \, \mathrm{d}\xi - \left(\frac{1}{2}\left(\sigma_+^2 - \sigma_-^2\right)\varkappa - \lambda_1\right)\left(W_k(1, \tau) + \varphi_k^+(\tau)\right)$$

Functions ψ_k and W_k appearing in the formula for S_k are defined as follows:

$$\psi_k(\tau) = \int_{-1}^{1} W_{0,i} f_{k+1} \, \mathrm{d}\xi + \sigma_- \eta_{k+1}^-(\tau) - \sigma_+ \eta_{k+1}^+(\tau), \quad w_k(\xi, \tau) = \alpha_k(\tau) W_{0,i}(\xi) + W_k(\xi, \tau)$$

where w_k is a solution of (18), (19), α_k is an arbitrary function of τ, $\tau \in \gamma$, $\alpha_k(\tau)$ ℓ-periodic, and W_k is the unique solution of the problem such that $\int_{-1}^{1} q W_{0,i} W_k \, \mathrm{d}\xi = 0$.

Condition (22) has been obtained, on account of (21), by imposing the compatibility condition for the non homogeneous problem associated with (8), (9): (18), (19), where k is replaced by $k+1$. Besides α_k and v_k determine each other by $\alpha_k(\tau) = \frac{1}{\sigma_\pm}(v_k(0\pm, \tau) - W_k(\pm 1, \tau) - \varphi_k^{\pm}(\tau))$.

(20)–(22) and the Fredholm alternative for a non homogeneous Steklov problem in $H_0^1(\Omega)$ associated with (14)–(16) leads us to obtain λ_{k+1}:

$$\lambda_{k+1} = \sigma_+ \int_{\gamma^-} \left(\sigma_+\left(\varphi_k^- + W_k(-1, \tau)\right) - \sigma_-\left(\varphi_k^+ + W_k(1, \tau)\right)\right) \frac{\partial v_0}{\partial n} \, \mathrm{d}\gamma + \int_{\gamma^+} S_k v_0 \, \mathrm{d}\gamma + \sigma_+^2 \int_\Omega g_k v_0 \, \mathrm{d}x$$

Also, we obtain the solution v_k of (20)–(22), uniquely determined by prescribing the orthogonality condition with v_0: $\int_\gamma u^\varepsilon v_0 \, \mathrm{d}\gamma = 1 \Leftrightarrow \int_\gamma v_k v_0 \, \mathrm{d}\gamma = \delta_{k,0}$.

Therefore, we can continue the process for any k; $k = 1, 2, 3, \ldots$.

Remark 1. – We observe that $\lambda_{0,0} = 0$ is the first eigenvalue of problem (13); the corresponding eigenfunctions $W_{0,0}$ are the constants. Thus, the associated values σ_+ and σ_- coincide and the term $(\sigma_+^2 - \sigma_-^2)\varkappa(\tau)/2$ in problem (14)–(16) does not appear. In that case, the eigenvalues of (14)–(16) are strictly positive numbers. This is in good agreement with the fact that (4), for $\lambda_0 = \lambda_{0,0} = 0$, gives asymptotics for the low frequencies of (1).

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