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On the confinement of a viscous fluid by means of a feedback external field

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Abstract In this work we consider a planar stationary flow of an incompressible viscous fluid in a semi-infinite strip governed by the standard Stokes system. We show how this fluid can be stopped at a finite distance from the entrance of the semi-infinite strip by means of a feedback source depending in a sublinear way on the velocity field. This localization effect is proved by reducing the problem to a non-linear biharmonic type one for which the localization of solutions is obtained through the application of an energy method, in the spirit of the monograph by S.N. Antontsev, J.I. Díaz and S.I. Shmarev (Energy Methods for Free Boundary Problems: Applications to Non-Linear PDEs and Fluid Mechanics, Birkäuser, Boston, 2002). Since the presence of the non-linear terms defined by the source is not standard in fluid mechanics literature, we give also some results about the existence and uniqueness of weak solutions for this problem. ***To cite this article: S.N. Antontsev et al., C. R. Mecanique 330 (2002) 797–802.***

computational fluid mechanics / Stokes system / feedback dissipative field / non-linear higher order equation / energy method / localization effect

Sur le confinement d’un fluide visqueux par le biais d’un champ extérieur avec mémoire

Résumé Dans ce travail nous considérons un écoulement plan stationaire incompressible d’un fluide visqueux situé dans un milieu semi-infini régi par le système de Stokes standard. Nous montrons comment ce fluide peut être arrêté à une distance finie de l’entrée du milieu, utilisant une source avec mémoire dépendant d’une manière sous-linéaire du champ de vitesses. Cet effet de localisation est atteint en réduisant le problème à un problème non linéaire du type biharmonique, où la localisation des solutions est obtenue par l’application d’une méthode d’énergie dans l’esprit de la monographie de S.N. Antontsev, J.I. Díaz and S.I. Shmarev (Energy Methods for Free Boundary Problems: Applications to Non-Linear PDEs and Fluid Mechanics, Birkäuser, Boston, 2002). En outre, du fait que la présence de terms non linéaires definis par la source est non fréquente dans la littérature de la mécanique des fluides, nous donnons aussi des résultats sur l’existence et l’unicité des solutions faibles de ce problème. ***Pour citer cet article : S.N. Antontsev et al., C. R. Mecanique 330 (2002) 797–802.***

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mécanique des fluides numérique / systéme de Stokes / champ avec mémoire dissipatif / equations non-linèaires d'ordres superieures / méthode d'énergie / effet de localisation

1. Introduction

The main problem we consider in this paper deals with the study of the planar stationary flow of an incompressible viscous fluid in a semi-infinite strip $\Omega = (0, +\infty) \times (0, L)$, $L > 0$, of velocity $\mathbf{u}(\mathbf{x}) = (u(\mathbf{x}), v(\mathbf{x}))$, $\mathbf{x} = (x, y) \in \Omega$, satisfying the following Stokes system

$$-\nu \Delta \mathbf{u} = \mathbf{f} - \nabla p \quad \text{in } \Omega \tag{1}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \tag{2}$$

$$\mathbf{u}(0, y) = \mathbf{u}_*(y), \quad y \in (0, L) \tag{3}$$

$$\mathbf{u}(x, 0) = \mathbf{u}(x, L) = \mathbf{0}, \quad x \in (0, +\infty) \tag{4}$$

$$\left|\mathbf{u}(x, y)\right| \to 0, \quad \text{as } x \to +\infty \text{ and } y \in (0, L) \tag{5}$$

where $p = p(x, y)$ stands here for the hydrostatic pressure divided by the constant density of the fluid and ν is the kinematics viscosity coefficient. We assume that the non-zero velocity at the strip entrance satisfies

$$\mathbf{u}_* \in \mathbf{H}^{1/2}(0, L) \tag{6}$$

$\mathbf{u}_* = (u_*, v_*)$, and the compatibility conditions

$$\int_0^L u_*(s)\, \mathrm{d}s = 0 \tag{7}$$

and

$$\mathbf{u}_*(0) = \mathbf{u}_*(L) = \mathbf{0} \tag{8}$$

in the sense of pseudo-traces. In what follows we shall denote problem (1)–(8) as $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$. We also assume that the external body force is given in a feedback form, $\mathbf{f} : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{f}(\mathbf{x}, \mathbf{u}) = (f_1(\mathbf{x}, \mathbf{u}), f_2(\mathbf{x}, \mathbf{u}))$, such that, for every $\mathbf{u} \in \mathbb{R}^2$, $\mathbf{u} = (u, v)$, and for almost all $\mathbf{x} \in \Omega$,

$$-\mathbf{f}(\mathbf{x}, \mathbf{u}) \cdot \mathbf{u} \geqslant \delta \chi_{\mathbf{f}}(\mathbf{x})\, |u|^{1+\sigma} - g(\mathbf{x}) \tag{9}$$

for some $\delta > 0$, $0 < \sigma < 1$ and

$$g \in L^1\left(\Omega^{x_g}\right), \quad g \geqslant 0, \quad g(\mathbf{x}) = 0 \text{ a.e. in } \Omega_{x_g} \tag{10}$$

for some $x_{\mathbf{f}}$, x_g with $0 \leqslant x_g < x_{\mathbf{f}} \leqslant +\infty$ and $x_{\mathbf{f}}$ large enough. Here $\Omega^{x_g} = (0, x_g) \times (0, L)$, $\Omega_{x_g} = (x_g, +\infty) \times (0, L)$ and $\chi_{\mathbf{f}}$ denotes the characteristic function of the interval $(0, x_{\mathbf{f}})$.

We address the resemblance about our formulation and the important question of the confinement of a plasma typical of the magnetohydrodynamics (MHD) to the detailed version of this Note [2].

2. Existence and uniqueness result

Since the presence of the non-linear terms defined by $\mathbf{f}(\mathbf{x}, \mathbf{u})$ is not standard in fluid mechanics literature, we collect in this section some results about the existence and uniqueness of weak solutions for problem $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$.

We shall search solutions such that $\int_\Omega |\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x} < +\infty$. Moreover, due to the fact that the Poincaré inequality holds, our searched solution will be an element of the Sobolev space $\mathbf{H}^1(\Omega)$ simplifying, in this way, the functional framework needed for other unbounded domains.

In order to define the notion of weak solution, we introduce the functional spaces

$$\widetilde{\mathbf{H}}(\Omega) = \Big\{\mathbf{u} \in \mathbf{H}^1(\Omega) : \operatorname{div}\mathbf{u} = 0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_*(\cdot),\ \mathbf{u}(x,0) = \mathbf{u}(x,L) = \mathbf{0},\ x \in (0,+\infty),\ \lim_{x\to+\infty} |\mathbf{u}| = 0\Big\}$$

and

$$\widetilde{\mathbf{H}}_0(\Omega) = \Big\{\mathbf{u} \in \mathbf{H}^1(\Omega) : \operatorname{div}\mathbf{u} = 0,\ \mathbf{u}(0,\cdot) = \mathbf{0},\ \mathbf{u}(x,0) = \mathbf{u}(x,L) = \mathbf{0},\ x \in (0,+\infty),\ \lim_{x\to+\infty} |\mathbf{u}| = 0\Big\}$$

In this section we shall assume that $\mathbf{f} : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$, with $\mathbf{f}(\mathbf{x},\mathbf{u}) = (f_1(\mathbf{x},\mathbf{u}), f_2(\mathbf{x},\mathbf{u}))$, $\mathbf{u} = (u, v)$,

$$\mathbf{f}(\mathbf{x},\mathbf{u}) = -\delta\chi_{\mathbf{f}}(\mathbf{x})\big(|u|^{\sigma-1}u(\mathbf{x}), 0\big) - \mathbf{h}(\mathbf{x},\mathbf{u}) \tag{11}$$

for some $\delta > 0$, $0 \leqslant x_{\mathbf{f}} \leqslant +\infty$ and $0 < \sigma < 1$. Here $\mathbf{h}(\mathbf{x},\mathbf{u})$ is a Carathéodory function such that

$$\mathbf{h}(\mathbf{x},\mathbf{u}) \cdot \mathbf{u} \geqslant -g(\mathbf{x}) \quad \text{for every } \mathbf{u} \in \mathbb{R}^2 \text{ and a.e. } \mathbf{x} \in \Omega \tag{12}$$

for some

$$g \in L^1\big(\Omega^{x_g}\big), \quad g \geqslant 0, \quad g(\mathbf{x}) = 0 \text{ a.e. in } \Omega_{x_g} \tag{13}$$

with $0 \leqslant x_g < x_{\mathbf{f}}$, and

$$H_M \in L^1\big(\Omega^{x_{\mathbf{f}}}\big) \quad \text{for all } M > 0 \tag{14}$$

where $H_M(\mathbf{x}) = \sup_{|\mathbf{u}|\leqslant M} |\mathbf{h}(\mathbf{x},\mathbf{u})|$.

Notice that no upper restriction on the growth of $|\mathbf{f}(\mathbf{x},\mathbf{u})|$ with respect to $\mathbf{u}$ is imposed. Due to that, sometimes this type of non-linear terms are called *strongly non linear*.

DEFINITION 2.1. – We say that a vector function $\mathbf{u}$ is a weak solution of problem $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$ if:

(i) $\mathbf{u} \in \widetilde{\mathbf{H}}(\Omega)$;
(ii) $\mathbf{f}(\mathbf{x},\mathbf{u}) \in \mathbf{L}^1_{\mathrm{loc}}(\Omega)$;
(iii) $\nu\int_\Omega \nabla\mathbf{u} : \nabla\varphi\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\varphi\,\mathrm{d}\mathbf{x}$, for all $\varphi \in \widetilde{\mathbf{H}}_0(\Omega) \cap \mathbf{L}^\infty(\Omega)$ with compact support.

THEOREM 2.2. – *Under the above assumptions* (11)–(14) *on* $\mathbf{f}(\mathbf{x},\mathbf{u})$, *there exists, at least, one weak solution of problem* $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$. *Moreover,* $\mathbf{f}(\mathbf{x},\mathbf{u}) \cdot \mathbf{u}$ *lies in* $\mathrm{L}^1(\Omega)$ *and* $\mathbf{u}$ *satisfies to the energy estimate*

$$\int_\Omega \big(|\nabla\mathbf{u}|^2 + \chi_{\mathbf{f}}|u|^{1+\sigma} + \big|\mathbf{h}(\mathbf{x},\mathbf{u})\cdot\mathbf{u}\big|\big)\,\mathrm{d}\mathbf{x} \leqslant C(L,\delta,\nu,\sigma)\big(\|\mathbf{u}_*\|^2_{\mathbf{H}^{1/2}(0,L)} + \|g\|_{L^1(\Omega^{x_g})} + 1\big)$$

Problem $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$ *has only one solution, if, in addition, the inequality*

$$\big(\mathbf{f}(\mathbf{x},\mathbf{u}_1) - \mathbf{f}(\mathbf{x},\mathbf{u}_2)\big)\cdot(\mathbf{u}_1 - \mathbf{u}_2) \leqslant 0$$

holds for every $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^2$ *and almost every* $\mathbf{x} \in \Omega$.

The existence of a weak solution is proved by invoking some known results for the Stokes problem in domains with an unbounded boundary (see, e.g., Galdi [3], Chapter VI) and using the Shauder fixed point theorem in the spirit of Bernis [4], Theorem 6.1, with a truncation on the vector field $\mathbf{u}$ conform Vrabie [5], Definition 3.4.3, and using also some arguments from measure theory which can be found in Brezis and Browder [6].

To prove the uniqueness of solutions we use a vector version of a result due to Brezis and Browder [7] assuring that if $T \in L^1(\Omega) \cap H^{-1}(\Omega)$ and $u \in H^1(\Omega)$ are such that $T(\mathbf{x})u(\mathbf{x}) \geqslant 0$ a.e. in Ω, then $Tu \in L^1(\Omega)$ and

$$\langle T, u\rangle_{H^{-1}(\Omega)\times H^1(\Omega)} = \int_\Omega T(\mathbf{x})u(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

See [2] for a complete proof.

Remark 1. – The above existence theorem admits many different variations: $\sigma \geqslant 1$, case of $\sigma = 0$; more general unbounded sets Ω; etc. Our presentation is strictly motivated by the results on the localization effect in the simplest case of Ω given by the constant semi-infinite strip.

3. Localization effect

In the previous section, the existence of a weak solution having a finite global energy

$$E := \int_\Omega \left(|\nabla \mathbf{u}|^2 + \chi_{\mathbf{f}}|u|^{1+\sigma}\right)\mathrm{d}\mathbf{x}$$

has been established.

THEOREM 3.1. – *Assume* $\mathbf{f}$ *satisfies* (9) *and* (10)*. Then*:

(i) *if* $x_{\mathbf{f}} = +\infty$ *(*$x_{\mathbf{f}}$ *is given in* (10)*) and* $\mathbf{u}$ *is any weak solution of* $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$ *with finite energy* E*, then* $\mathbf{u}(x, y) = \mathbf{0}$ *for* $x > a'$*, where*

$$a' = \frac{7+\sigma}{1-\sigma}\left(C\frac{\nu^2}{\min^2(\nu,\delta)}\right)^{2/(3+\sigma)}\left(\frac{L}{\pi}\right)^{(1+\sigma)/(3+\sigma)} E^{(1-\sigma)/(2(3+\sigma))} \tag{15}$$

with C *a positive constant depending on* σ*;*

(ii) *if* $x_{\mathbf{f}} < +\infty$ *and* $x_{\mathbf{f}} > a'$ *(*a' *given by* (15)*), then there exists at least one weak solution* $\mathbf{u}$ *of* $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$ *with a finite energy* E*, such that* $\mathbf{u}(x, y) = \mathbf{0}$ *for* $x > a'$*;*

(iii) *if, in addition to the conditions of* (ii)*, we assume* $\mathbf{f}$ *non-increasing, then the conclusion of* (ii) *holds for the unique solution of* $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$*.*

In order to prove this theorem, we introduce the associated stream function, say ψ, with the weak solution $\mathbf{u}$, i.e.,

$$u = \psi_y \quad \text{and} \quad v = -\psi_x \quad \text{in } \Omega. \tag{16}$$

By classical methods, we can reduce the study of problem $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$, to the consideration of the higher order problem

$$\mathcal{P}_\psi \qquad \begin{cases} \nu\Delta^2\psi + \dfrac{\partial f_1}{\partial y}(\mathbf{x}, \psi_y, -\psi_x) - \dfrac{\partial f_2}{\partial x}(\mathbf{x}, \psi_y, -\psi_x) = 0 \quad \text{in } \Omega \\ \psi(x,0) = \psi(x,L) = 0 \quad \text{for } x \in (0,+\infty) \\ \dfrac{\partial\psi}{\partial n}(x,0) = \dfrac{\partial\psi}{\partial n}(x,L) = 0 \quad \text{for } x \in (0,+\infty) \\ \psi(0,y) = \int_0^y u_*(s)\,\mathrm{d}s, \quad \dfrac{\partial\psi}{\partial n}(0,y) = v_*(y) \quad \text{for } y \in (0,L) \\ \psi(x,y),\ \left|\nabla\psi(x,y)\right| \to 0 \quad \text{as } x \to +\infty \text{ and for } y \in (0,L) \end{cases}$$

where the pressure term does not appear anymore.

DEFINITION 3.2. – A function ψ is called a weak solution of problem $\mathcal{P}_\psi$ if:
(i) $\psi \in \mathrm{H}^2(\Omega)$, $\mathbf{f}(\mathbf{x}, \psi_y, -\psi_x) \in \mathbf{L}^1_{\mathrm{loc}}(\Omega)$;
(ii) $\psi(0, y) = \int_0^y u_*(s)\,\mathrm{d}s$, $\frac{\partial\psi}{\partial n}(0, y) = v_*(y)$ for $y \in (0, L)$, $\psi(x, 0) = \psi(x, L) = \frac{\partial\psi}{\partial n}(x, 0) = \frac{\partial\psi}{\partial n}(x, L) = 0$ for $x \in (0, +\infty)$ and $\psi(0, 0) = \psi(0, L) = 0$;
(iii) $\psi,\ |\nabla\psi| \to 0$, when $x \to +\infty$;
(iv) $\nu\int_\Omega \Delta\psi\,\Delta\phi\,\mathrm{d}\mathbf{x} - \int_\Omega (f_1(\mathbf{x}, \psi_y, -\psi_x)\phi_y - f_2(\mathbf{x}, \psi_y, -\psi_x)\phi_x)\,\mathrm{d}\mathbf{x} = 0$, for all $\phi \in \mathrm{H}^2_0(\Omega)W^{1,\infty}(\Omega)$ with compact support.

If $\mathbf{u}$ is a weak solution of problem $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$ in the sense of Definition 2.1, then ψ, given by (16), is a weak solution of problem $\mathcal{P}_\psi$ in the sense of Definition 3.2.

To establish the localization effect, we will apply the so-called energy methods for free boundary problems (see Antontsev et al. [1]) introduced by Antontsev [8], improved by Díaz and Véron [9,10] and extended by several authors amongst whom Bernis [11].

We shall use here the technique of integrating over a family of variable half-planes, which requires zero boundary conditions. We observe that the only non-zero boundary condition in problem $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$, or problem $\mathcal{P}_\psi$, is on the boundary $x = 0$. Thus, following Bernis [11], we are lead to introduce a weighted function which will cancel the terms on this boundary. For $m \geqslant 2$, let $\psi(\mathbf{x})(x - a)_+^m = \psi(\mathbf{x})(x - a)^m$ if $x > a$ and $\psi(\mathbf{x})(x - a)_+^m = 0$ otherwise, where $a \geqslant 0$ is a variable parameter and ψ is a weak solution of $\mathcal{P}_\psi$. We remark that this function is not, in general, an admissible test function, because Ω is unbounded. But, we can overcome this difficulty following Bernis [12], Appendix II, and we take that function as a test function in Definition 3.2. Then, proceeding as in [2], Lemmas 3.2 and 3.6, we obtain

$$E_4(a) \leqslant C\frac{\nu^2}{\min^2(\nu, \delta)}\left(\frac{L}{\pi}\right)^{(1+\sigma)/2}\big(E_2(a)\big)^{(5-\sigma)/4} \tag{17}$$

and

$$E_2(a) \leqslant \frac{1}{2}E_0(a) + C\frac{\nu^2}{\min^2(\nu, \delta)}\left[2 + \left(\frac{L}{\pi}\right)^2\right]\left(\frac{L}{\pi}\right)^{2(1+\sigma)/(3+\sigma)}\big(E_0(a)\big)^{4/(3+\sigma)} \tag{18}$$

where C means two different positive constants depending on σ and x_g is given in (10). Notice that in the inequalities (17) and (18) it arise the energy type terms which depend on a

$$E_m(a) = \int_\Omega \big(\big|\mathrm{D}^2\psi\big|^2 + |\psi_y|^{1+\sigma}\big)(x - a)_+^m\,\mathrm{d}\mathbf{x}$$

Finally, since $\sigma \in (0, 1)$, using a result of Bernis [12], Appendix III, we get the desired result.

Assume now that $x_\mathbf{f} < +\infty$. Then we construct a weak solution in the following way $\mathbf{u}(\mathbf{x}) = \widetilde{\mathbf{u}}(\mathbf{x})$ if $x \leqslant a'$ and $\mathbf{u}(\mathbf{x}) = \mathbf{0}$ otherwise, with $\widetilde{\mathbf{u}}(\mathbf{x})$ weak solution of $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$ with $x_\mathbf{f} = +\infty$. By the proof of the above case and the assumption $a' < x_\mathbf{f}$, we get that $\mathbf{u}(\mathbf{x})$ is a weak solution of the original problem $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$.

See [2] for a complete proof.

Remark 2. – Obviously, we obtain an analogous localization effect if we replace the role of variables x and y for the study of unbounded sets of the form $\Omega = (0, L) \times (0, +\infty)$.

Remark 3. – We obtain the same localization effect if we consider the non-constant semi-infinite strip $\Omega = (0, +\infty) \times (L_1(x), L_2(x))$, with $L_1, L_2 \in \mathrm{C}^2(0, +\infty)$, $0 < k_1 \leqslant |L_2(x) - L_1(x)| \leqslant k_2 < +\infty$, $|L_1'(x)|$, $|L_2'(x)| \leqslant k_3 < +\infty$, and $|L_1''(x)|,\ |L_2''(x)| \leqslant k_4 < +\infty$ for all $x \geqslant 0$.

Remark 4. – In the case of $\sigma = 1$, the above arguments lead to the inequality

$$E_m(a) \leqslant C E_{m-2}(a) \quad \text{for } a \geqslant x_g$$

and then an exponential decay is derived (this type of decay is optimal).

Remark 5. – The localization effect can be extended to the limit case $\sigma = 0$ following the approach presented in Díaz [13].

Remark 6. – The main result of this section applies to the family of stationary problems obtained by implicit discretization of the associated parabolic problem, i.e. to the family of problems

$$\begin{cases} \dfrac{\mathbf{u}_n - \mathbf{u}_{n-1}}{\tau} - \nu\Delta\mathbf{u}_n = \mathbf{f}(\mathbf{x}, \mathbf{u}_n) - \nabla p_n & \text{in } \Omega \\ \operatorname{div}\mathbf{u}_n = 0 & \text{in } \Omega \end{cases}$$

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References

[1] S.N. Antontsev, J.I. Díaz, S.I. Shmarev, Energy Methods for Free Boundary Problems, Applications to Non-Linear PDEs and Fluid Mechanics, Birkhäuser, Boston, 2002.
[2] S.N. Antontsev, J.I. Díaz, H.B. de Oliveira, Stopping a viscous fluid by a feedback dissipative external field: I. The stationary Stokes problem, to appear.
[3] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations: Linearised Steady Problems, Springer, New York, 1994.
[4] F. Bernis, Elliptic and parabolic semilinear problems without conditions at infinity, Arch. Rational Mech. Anal. 106 (1989) 217–241.
[5] I.I. Vrabie, Compactness Methods for Non-Linear Evolutions, Pitman, London, 1987.
[6] H. Brezis, F.E. Browder, Strongly non-linear elliptic boundary value problems, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (IV) 5 (1978) 587–603.
[7] H. Brezis, F.E. Browder, Some properties of higher order Sobolev spaces, J. Math. Pures Appl. 61 (1982) 245–259.
[8] S.N. Antontsev, On the localization of solutions of non-linear degenerate elliptic and parabolic equations, Dokl. Akad. Nauk SSSR 260 (6) (1981) 1289–1293 (in Russian);
English translation in Dokl. Math. 24 (2) (1981) 420–424.
[9] J.I. Díaz, L. Véron, Compacité du support des solutions d'équations quasi linéaires elliptiques ou paraboliques, C. R. Acad. Sci. Paris, Série I 297 (3) (1983) 149–152.
[10] J.I. Díaz, L. Véron, Local vanishing properties of solutions of elliptic and parabolic quasilinear equations, Trans. Amer. Math. Soc. 290 (2) (1985) 787–814.
[11] F. Bernis, Compactness of the support for some non-linear elliptic problems of arbitrary order in dimension n, Comm. Partial Differential Equations 9 (3) (1984) 271–312.
[12] F. Bernis, Qualitative properties for some non-linear higher order degenerate parabolic equations, Houston J. Math. 14 (1988) 319–352.
[13] J.I. Díaz, On the formation of the free boundary for the obstacle and Stefan problems via an energy method, in: L. Ferragut, A. Santos (Eds.), Actas XVII CEDYA/VII CMA (CD-Rom), S.P. Universidad Salamanca, 2001.