C. R. Mecanique 330 (2002) 819–824

Confined buckling of inextensible rods by convex difference algorithms

Pierre Alart, Stéphane Pagano

LMGC, CNRS-Université de Montpellier II, place E. Bataillon CC048, 34005 Montpellier cedex, France

Received 20 June 2002; accepted 29 October 2002

Note presented by Pierre Suquet.

Abstract In this Note we present an approach to determine the local minima of a specific class of minimization problems. Attention is focused on the inextensibility condition of flexible rods expressed as a nonconvex constraint. Two algorithms are derived from a special splitting of the Lagrangian into the difference of two convex functions (DC). They are compared to the augmented Lagrangian methods used in this context. These DC formulations are easily extended to contact problems and applied to the determination of confined buckling shapes. ***To cite this article: P. Alart, S. Pagano, C. R. Mecanique 330 (2002) 819–824.***

computational solid mechanics / nonlinear mechanics / local minimization / convex difference / augmented Lagrangian / confined buckling

Flambement confiné de tiges inextensibles par des algorithmes de différence convexe

Résumé Dans cette Note nous présentons une approche permettant de déterminer les minima locaux d'une classe spécifique de problèmes de minimisation. On s'intéresse à la condition d'inextensibilité d'une poutre flexible exprimée comme une contrainte non convexe. Deux algorithmes découlent alors de la décomposition du Lagrangien en la différence de deux fonctions convexes (DC). Ils sont comparés à la méthode du Lagrangien augmenté utilisée dans ce contexte. La gestion de conditions de contact est aisément intégrable dans le schéma DC et application est faite au flambement confiné. ***Pour citer cet article : P. Alart, S. Pagano, C. R. Mecanique 330 (2002) 819–824.***

mécanique des solides numérique / mécanique non linéaire / minimisation locale / différence convexe / Lagrangien augmenté / flambement confiné

Version française abrégée

Les grandes déformations introduisent une première forme de non linéarité, dite géométrique, impliquant la perte de convexité de la fonctionnelle à minimiser dans le cas simple d'un comportement élastique. Les états d'équilibre, éventuellement multiples, peuvent être définis comme des minima locaux de ce potentiel. L'existence et l'approximation de solutions minimisantes locales restent un problème ouvert hors du propos de cette Note. Les fonctions s'écrivant comme différence de deux fonctions convexes (DC) constituent un premier ensemble de fonctions non convexes pour lesquelles la convexité « séparée » peut être encore exploitée théoriquement et numériquement : les points critiques des fonctions DC peuvent être caractérisés via des sous-différentiels [1] ; on dispose d'outils algorithmiques puissants associés à la convexité [2,3].

E-mail addresses: alart@lmgc.univ-montp2.fr (P. Alart); pagano@lmgc.univ-montp2.fr (S. Pagano).

S1631-0721(02)01547-4/FLA

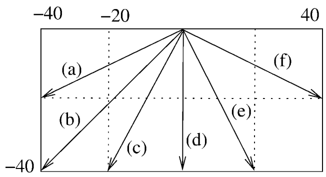

Figure 1. Different loading cases for the force **F**.

Figure 1. *Différent cas de charge* **F**.

Table 1. Behaviour of the algorithms (6 finite elements, 3 × 6 Gauss points).

Tableau 1. *Comportement des algorithmes* (6 *éléments,* 3 × 6 *points de Gauss*).

Chargement	(a)	(b)	(c)	(d)	(e)	(f)
Itération DCalg1	Div	35(7)	30(5)	22(3)	21(2)	29(1)
Itération DCalg2	37	33	28	22	21	29
ALalg ($r = 200$, $\rho = 20$)	58	71	45	36	46	55
ALalg ($r = \rho = 200$)	∞	∞	24	22	18	16
Pts de Gauss $\lambda = \lambda^-$	11	8	8	5	2	0

La détermination de l'équilibre d'une tige inextensible soumis à un chargement et à des conditions aux limites classiques constitue un premier exemple d'étude pour lequel la non convexité est introduite via une contrainte d'égalité : la condition d'inextensibilité (cf. Éqs. (1) et (4)). Par contre l'énergie potentielle (2) reste convexe. Un champ de multiplicateurs scalaires fournit l'expression d'un Lagrangien sur lequel est défini un problème de points-selles locaux (6) équivalent au problème initial. Une étape essentielle de notre approche réside dans la décomposition de ce Lagrangien en la différence de deux fonctions convexes par rapport à la variable déplacement (8), via les parties positive et négative du champ des multiplicateurs (9). Suivant la démarche d'Auchmuty [2] on se ramène à un problème inf–inf–sup (11) sur un Lagrangien de type II (10). On combine la méthode d'Uzawa d'actualisation du multiplicateur avec l'algorithme DC pour définir l'algorithme DCalg1 (cf. Tableau 1). Une deuxième version, DCalg2, consiste à n'effectuer qu'une itération de l'algorithme DC.

Pour apprécier les apports d'une approche DC, on effectue une comparaison avec la méthode du Lagrangien augmenté (LA) développée dans [5] sur le même problème. Les algorithmes LA et DC présentent quelques similitudes : trois champs (cf. Éqs. (10) et (12)), trois étapes, une deuxième étape (1b) locale résolue analytiquement. Les différences principales sont : un champ de multiplicateurs vectoriel en LA, scalaire en DC ; deux paramètres numériques r et ρ en LA, un seul en DC. La comparaison numérique s'effectue sur un problème de flexion avec une force terminale dont la direction varie (cf. Fig. 1). Avec des critères d'erreurs similaires, les algorithmes DC convergent plus vite que l'algorithme LA en terme d'itérations (cf. Tableau 1). La boucle interne de DCalg1, dont le nombre d'itérations est donné entre parenthèses, n'améliore pas, tout du moins dans cet exemple, la convergence par rapport à DCalg2. Enfin on apprécie l'influence de la non convexité sur le nombre d'itérations à travers le nombre de points de Gauss où le multiplicateur est négatif, autrement dit où la non convexité est active. Afin d'aborder les problèmes de flambement on introduit un terme évanescent dans la formulation qui modifie légèrement l'algorithme DCalg1 tout en stabilisant son comportement (cf. Éq. (13)). On accède ainsi à des branches de flambement via des initialisations grossières (cf. Fig. 2). Le calcul des configurations de flambement d'une tige confinée dans un conduit nécessite de modifier l'étape (1a) de l'algorithme en y ajoutant les conditions de contact traduites en terme de contraintes convexes. On préserve ainsi la structure et les propriétés de l'algorithme DC.

In nonlinear mechanics, and specially when large deformations occur, we have to deal with a nonconvex potential energy. Consequently, the equilibrium states may be defined as the local minima of this potential. From a practical point of view, we have to determine numerically local minima in finite dimensional spaces by using the most general way possible. The functions which may be written as the difference of two convex functions (DC) constitute a first set of nonconvex functions with interesting properties: the critical points of the DC functions may be characterized via sub-differentials [1]; we profit by the efficient numerical tools associated with the separate convexity [2,3]. A special feature of our problem is that the nonconvexity is involved by an equality constraint.

1. A nonconvex equality constraint

We consider a thin rod of length ℓ and we assume that the deformed configuration takes place in a 1–2 plane. The configuration is described by a vector function $\mathbf{y}$ of the curvilinear abscissa s. We restrict ourselves here to the case of an inextensible rod. The inextensibility condition may be written as an equality constraint on the first derivative, with respect to the curvilinear abscissa, of the configuration function,

$$\left\|\mathbf{y}'(s)\right\|^2 = \left(\frac{\mathrm{d}y_1}{\mathrm{d}s}(s)\right)^2 + \left(\frac{\mathrm{d}y_2}{\mathrm{d}s}(s)\right)^2 = 1 \tag{1}$$

The bulk energy is then associated with the bending term. If the rod is submitted to a volume loading field $\mathbf{f}$ and to a terminal loading force $\mathbf{F}$, the potential energy takes the simplified form,

$$J(\mathbf{y}) = \frac{1}{2}\int_0^\ell E(s)I(s)\left\|\mathbf{y}''(s)\right\|^2 \mathrm{d}s - \int_0^\ell \mathbf{f}(s)\cdot\mathbf{y}(s)\,\mathrm{d}s - \mathbf{F}\cdot\mathbf{y}(\ell) \tag{2}$$

where E is the Young modulus and I is the inertial momentum (i.e., EI is the (positive) bending stiffness) which may depend on s. Consequently an equilibrium state of the rod may be characterized as a local minimum of the potential energy restricted to the kinematically admissible configurations (cf. Eq. (5)). The set of admissible configurations is the intersection of two sets, the first one containing the boundary conditions, the second one accounting for the inextensibility condition,

$$\mathcal{A} = \left\{\mathbf{y} \in H^2(0,\ell) \text{ plus adequate boundary conditions}\right\} \tag{3}$$

$$\mathcal{C} = \left\{\mathbf{y} \in H^2(0,\ell);\, h\big(\mathbf{y}'(x)\big) = \left\|\mathbf{y}'(s)\right\|^2 - 1 = 0 \text{ a.e. } s \in \,]0,\ell[\right\} \tag{4}$$

Since the objective function J is convex the nonconvexity is due to the nonconvexity of the set $\mathcal{C}$. Indeed the inextensibility condition is expressed with an equality constraint defining a nonconvex manifold even if the function h in the definition is itself convex.

2. A convex difference mixed formulation

We introduce an arbitrary admissible configuration $\mathbf{x}$ in such a way the displacement field $\mathbf{v} = \mathbf{y} - \mathbf{x}$ belongs to the vector sub-space $\mathcal{V}$ associated with $\mathcal{A}$ and we define then an affine differential operator D such that $D\mathbf{v} = \mathbf{x}' + \mathbf{v}'$. By this way, the relevant potential energy is expressed as a function of $\mathbf{v}$ and is then denoted φ. The local minimization problem is then formulated as a constrained problem on the $\mathbf{v}$ field,

$$\inf_{\mathbf{y}\in\mathcal{A}\cap\mathcal{C}}(loc)J(\mathbf{y}) \quad \Longleftrightarrow \quad \inf_{\mathbf{v}\in\mathcal{V},\ h(D\mathbf{v})=0}(loc)\varphi(\mathbf{v}) \tag{5}$$

Starting from (5) we introduce a scalar multiplier field in order to postulate an inf–sup problem for which the Lagrangian function is non convex with respect to the first variable $\mathbf{v}$ and linear with respect to the Lagrange multiplier μ,

$$\inf_{\mathbf{v}\in\mathcal{V}}(loc) \sup_{\mu\in \mathrm{L}^2(0,\ell)} L(\mathbf{v};\mu) \tag{6}$$

Table 2. Two steps algorithm DCalg1 (three steps algorithm DCalg2).
Tableau 2. *Algorithme DCalg1 à deux étapes* (*algorithme DCalg2 à trois étapes*).

- $(\mathbf{u}^0, \boldsymbol{\sigma}^0; \lambda^0)$ given,
- $(\mathbf{u}^{n-1}, \boldsymbol{\sigma}^{n-1}; \lambda^{n-1})$ known, determine $\mathbf{u}^n$, $\boldsymbol{\sigma}^n$; λ^n as follows,
  step 1: Determine $(\mathbf{u}^n, \boldsymbol{\sigma}^n)$ by the DC algorithm on $\mathrm{L_{II}}(\cdot, \cdot; \lambda^{n-1})$
  a: $\mathbf{u}^{n,i} = \arg\min \mathrm{L_{II}}(\cdot, \boldsymbol{\sigma}^{n,i-1}; \lambda^{n-1})$
  b: $\boldsymbol{\sigma}^{n,i} \in \arg\min \mathrm{L_{II}}(\mathbf{u}^{n,i}, \cdot; \lambda^{n-1})$
  step 2: Update the multiplier: $\lambda^n = \lambda^{n-1} + \rho h(D\mathbf{u}^n)$

where $L(\mathbf{v}; \mu) = \varphi(\mathbf{v}) + \int_0^\ell \mu(s)h(D\mathbf{v}(s))\,\mathrm{d}s$. Consequently, it is convenient to write the Euler–Lagrange equations characterizing a *stationary point* $(\mathbf{u}, \lambda)$ of the Lagrangian functional,

$$0 = \frac{\partial L}{\partial \mathbf{v}}(\mathbf{u}; \lambda), \qquad 0 = \frac{\partial L}{\partial \mu}(\mathbf{u}; \lambda) \tag{7}$$

A DC algorithm may deal with the first equation (7a) by introducing the following splitting of the Lagrangian into the difference of two convex functions,

$$L(\mathbf{v}; \mu) = \Phi_1(\mathbf{v}; \mu) - \Phi_2(\mathbf{v}; \mu) \tag{8}$$

$$\Phi_1(\mathbf{v}; \mu) = \varphi(\mathbf{v}) + \int_0^\ell \mu^+(s)h\big(D\mathbf{v}(s)\big)\,\mathrm{d}s, \qquad \Phi_2(\mathbf{v}; \mu) = \int_0^\ell \mu^-(s)h\big(D\mathbf{v}(s)\big)\,\mathrm{d}s \tag{9}$$

The functions μ^- and μ^+ are defined by, $\mu^+(s) = \max(0, \mu(s))$ and $\mu^-(s) = \max(0, -\mu(s))$. This decomposition uses at best the convexity of h, but a similar approach may be carried out when we have a DC splitting of h. According to the approach of Auchmuty [2] we define a type II Lagrangian depending on three fields by using the Fenchel transform of the second function with respect to the first variable,

$$\mathrm{L_{II}}(\mathbf{v}, \boldsymbol{\tau}; \mu) = \Phi_1(\mathbf{v}; \mu) + \Phi_2^*(\boldsymbol{\tau}; \mu) - \int_0^\ell \mathbf{v}' \cdot \boldsymbol{\tau}\,\mathrm{d}s \tag{10}$$

Remark that $\mathrm{L_{II}}$ is only separately convex with respect to $\mathbf{v}$ and $\boldsymbol{\tau}$. A "saddle point" type problem may be associated to this Lagrangian,

$$\begin{aligned}&\text{Find } (\mathbf{u}, \boldsymbol{\sigma}, \lambda) \in \mathcal{V} \times \mathrm{L}^2(0,\ell) \times \mathrm{L}^2(0,\ell) \text{ such that}\\ &\quad \mathrm{L_{II}}(\mathbf{u}, \boldsymbol{\sigma}; \mu) \leqslant \mathrm{L_{II}}(\mathbf{u}, \boldsymbol{\sigma}; \lambda) \leqslant \mathrm{L_{II}}(\mathbf{v}, \boldsymbol{\tau}; \lambda), \quad \forall (\mathbf{v}, \boldsymbol{\tau}; \mu) \in \mathcal{V} \times \mathrm{L}^2(0,\ell) \times \mathrm{L}^2(0,\ell)\end{aligned} \tag{11}$$

This treatment of the inextensibility condition leads to a three fields problem (or five scalar fields). Naturally, we can derive from this formulation different solution methods based on the Uzawa algorithm to solve Eqs. (7a), (7b). The first choice consists in solving fully Eq. (7a) by a DC algorithm (iteration index i in Table 2) before updating the Lagrange multiplier according to the Uzawa iteration (iteration index n in Table 2). The DC algorithm consists in minimizing successively $\mathrm{L_{II}}$ with respect to the first variables $\mathbf{v}$ and $\boldsymbol{\tau}$ (steps (1a) and (1b)); the step (1a) is a global quadratic minimisation problem, the step (1b) is a local one. Such a method converges to a ∂-*critical point* of $\mathrm{L_{II}}(\cdot, \cdot; \lambda)$ which minimizes separately $\mathrm{L_{II}}$ with respect to the two variables [4,1]. The second strategy refers to a block relaxation procedure by limiting the DC algorithm to a single iteration (DCalg2).

3. Comparison with the augmented Lagrangian approach

It is interesting to compare this approach to the augmented Lagrangian techniques developed in [5], specially in the context of flexible and inextensible rods. Lets recall the expression of the augmented

Lagrangian for our problem by introducing an additional field $\mathbf{q}$, a *vector multiplier* field $\boldsymbol{\mu}$ to enforce the constraint between the primal variable and the additional field $\mathbf{q}$ of the problem, and a penalty factor r,

$$\mathcal{L}_r(\mathbf{v}, \mathbf{q}; \boldsymbol{\mu}) = \varphi(\mathbf{v}) + G(\mathbf{q}) + \frac{r}{2}\|D\mathbf{v} - \mathbf{q}\|^2 + \int_0^\ell \boldsymbol{\mu} \cdot (D\mathbf{v} - \mathbf{q})\,\mathrm{d}s \tag{12}$$

where the perturbation function G is defined by, $G(\mathbf{q}) = 0$ if $\|\mathbf{q}\| = 1$ a.e. s, $+\infty$ otherwise. The associated saddle-point problem can be solved numerically by an algorithm similar to DCalg2 consisting then of three steps per iteration n [5]. The updating of the Lagrange multiplier is in this case: $\boldsymbol{\lambda}^n = \boldsymbol{\lambda}^{n-1} + \rho(D\mathbf{u}^n - \mathbf{p}^n)$. The AL algorithm (ALalg) needs two parameters, r and ρ since only ρ has to be chosen with DC. The local problem may be solved analytically in both cases, even if this local problem is not convex for the AL algorithm contrary to DC one of course. The initial minimization problem (5) has to be understood in a local sense, but it seems to us that the ability to reach a local minimum is theoretically better with the DC algorithm. Indeed in all the numerical simulations, we get a local minimum of J on $\mathcal{A} \cap \mathcal{C}$ and never a local maximum.

To compare the performance of the different algorithms we study the problem of the large bending of a rod, of length ℓ, clamped at the origin and submitted to a terminal loading force $\mathbf{F}$ with different directions with nonnull vertical component (cf. Fig. 1). The data are: $\ell = 10$ m, $EI = 1000$ N/m^2. Six finite elements are performed; the parameters $\rho = 20$ for DCalg, $\rho = 20$ or 200 and $r = 200$ for ALalg. The DC algorithms converge generally faster than the AL method with the same factor; the r factor is chosen the best, i.e., to keep convergence for the five cases, by numerical experiments for the AL algorithm (cf. Table 1). The internal loop of DCAlg1 does not improve the global convergence of the algorithm, but this conclusion is limited to this numerical test. The influence of the nonconvexity is well underlined: the number of iterations increases as the number of Gauss–Legendre points where λ is negative (i.e., the nonconvexity is active); ALalg may not converge.

4. Improvement of the algorithm for buckling

Taking into account the inextensibility condition becomes specially crucial as soon as large displacements occur. The two first algorithms (DCalg1 and DCalg2) fail to converge toward a postbuckling shape when no force occurs and the distance between the extremities is imposed smaller than the length ℓ. Because the problem of the first step is uncoupled in term of the vertical and horizontal components of the displacement, the algorithms tend to reach the trivial solution. To overcome this difficulty, we propose to modify the initial minimization problem by adding a quadratic term to the functional Φ_1. The matrix $\mathbf{A}$ of the quadratic form has to be symmetric positive definite and overall *nondiagonal*; in the numerical simulations we chose $\mathbf{A} = a\binom{2\ 1}{1\ 2}$ where a is a strictly positive coefficient. The Φ_1 function is still strictly convex and coercive in order to assure a well-posed problem. It is convenient to see a solution of the initial problem as a limit of a sequence, in such a way the additional term vanishes at the convergence. Consequently we consider the iterative scheme, based on the DCalg1 algorithm, where the first step is performed by replacing the type II Lagrangian $\mathrm{L_{II}}$ by the new functional depending on the previous iteration,

$$\mathrm{L}_{\mathrm{II}}^{n-1}(\mathbf{v}, \boldsymbol{\tau}; \mu) = \mathrm{L}_{\mathrm{II}}(\mathbf{v}, \boldsymbol{\tau}; \mu) + \int_0^\ell \left(\mathbf{v} - \mathbf{u}^{n-1}\right) \cdot \mathbf{A}\left(\mathbf{v} - \mathbf{u}^{n-1}\right) \mathrm{d}s \tag{13}$$

Using rough initializations, we can obtain several branches (Fig. 2); for the fourth branch the final shape reveals that the extrema in the middle of the shape are higher than the others.

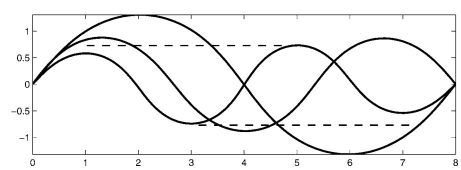

Figure 2. Secondary branches (20 finite elements).

Figure 2. *Branches secondaires* (20 *éléments finis*).

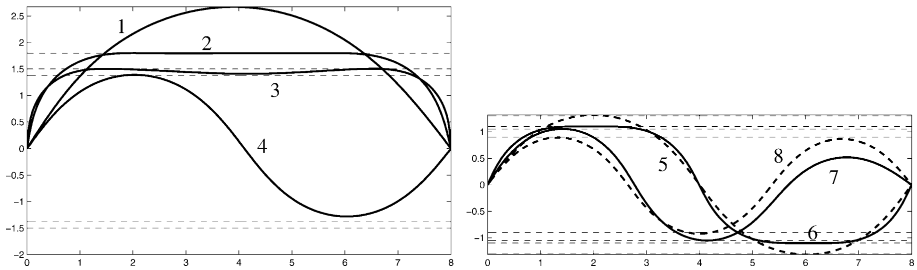

Figure 3. Some confined buckling shapes (20 finite elements).

Figure 3. *Quelques formes de flambement confinés* (20 *éléments finis*).

5. Unilateral contact and confined buckling

If a flexible rod can enter in frictionless contact with two flat obstacles located below and above the rod at the distances d_a and d_b, we just have to introduce two inequalities in the definition of the admissible configurations. The sub space $\mathcal{V}$ is replaced by the *convex set* defined as follows,

$$\mathcal{K} = \left\{ \mathbf{v} \in \mathcal{V};\, -d_a \leqslant (x_2 + v_2)(s) \leqslant d_b \;\forall s \in [0, \ell] \right\} \tag{14}$$

This convex constraint is added to the step (1a) which is then a convex constrained optimisation problem (the DC structure of the algorithm is preserved). Some classical numerical techniques may be used to solve this problem (over relaxation algorithm, Generalized Newton method, ...). With these numerical modifications, some confined buckling configurations for which the contact zones are unknown a priori (cf. Fig. 3) may be then determined according to different values of the distances d_a (d_b is taken equal to d_a). The previous values of ℓ and EI are used again, the coefficient r of the contact operator is taken around the value of EI. The two extremities of the rods are just supported and 8 m distant (cf. Fig. 3). With d_a equal to 2.7 m we have just a grazing contact associated with the first buckling branch (curve 1). Until 1.8 m the contact area increases and reaches 5 m (curve 2). After that the contact area on the upper side is splitted into two parts (example with d_a equal to 1.5 m, curve 3). For 1.4 m, the rod jumps to a second branch like configuration with a single top point contact (curve 4). The second point contact on the lower side occurs since d_a equals 1.3 m (curve 5). A same process between the first and second branches may be described from the second to the third branch (curve 6). Specially for d_a equals to 1.05 m we recover a third branch like configuration with two point contacts, and the second maximum is far from the upper wall (curve 7). These behaviours have been observed on experiments at least for the first and second branches in [6].

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