Available online at www.sciencedirect.com
SCIENCE DIRECT
C. R. Mecanique 331 (2003) 99–112
COMPTES RENDUS MECANIQUE

Le Point Sur…/Review Article

Instability and friction

Instabilité et frottement

Quoc Son Nguyen

Laboratoire de mécanique des solides, CNRS-UMR7649, École polytechnique, 91128 Palaiseau cedex, France

Received 2 December 2002; accepted 18 December 2002

Article invited by the Editors

Abstract

A review on the stability analysis of solids in unilateral and frictional contact is given. The presentation is focussed on the stability of an equilibrium position of an elastic solid in frictional contact with a fixed or moving obstacle. The problem of divergence instability and the obtention of a criterion of static stability are discussed first for the case of a fixed obstacle. The possibility of flutter instability is then considered for a steady sliding equilibrium with a moving obstacle. The steady sliding solution is generically unstable by flutter and leads to a dynamic response which can be chaotic or periodic. This dynamic response leads to the generation of stick–slip–separation waves on the contact surface in a similar way as Schallamach waves in statics. Illustrating examples and principal results recently obtained in the literature are reported. Some problems of friction-induced vibration and noise emittence, such as brake squeal for example, can be interpreted in this spirit. ***To cite this article: Q.S. Nguyen, C. R. Mecanique 331 (2003).***

Résumé

On présente dans cette Note une synthèse des résultats de la littérature sur l'analyse de stabilité des solides sous contact unilatéral avec frottement de Coulomb. Le problème de contact frottant d'un solide élastique avec un obstacle fixe ou mobile est examiné. La possibilité d'instabilité par divergence et la recherche d'un critère de stabilité statique d'un équilibre sont examinées dans le cas d'un obstacle fixe. La possibilité d'instabilité par flottement est ensuite discutée pour un équilibre résultant d'un glissement stationnaire avec un obstacle mobile. La solution de glissement stationnaire est dynamiquement instable par flottement et conduit à une réponse dynamique qui peut être chaotique ou périodique. En particulier, la réponse dynamique génère des ondes d'adhérence–glissement–séparation sur la surface de contact d'une façon comparable aux ondes de Schallamach en statique. Des exemples et des résultats récents de la littérature sont rapportés. Quelques problèmes de vibration et d'émission acoustique induites par le frottement, comme le crissement des freins par exemple, peuvent être interprétés dans cet esprit. ***Pour citer cet article : Q.S. Nguyen, C. R. Mecanique 331 (2003).***

Keywords: Friction; Unilateral contact; Instability; Stick–slip–separation waves

Mots-clés : Frottement ; Contact unilatéral ; Instabilité ; Ondes adhérence–glissement–séparation

E-mail address: son@lms.polytechnique.fr (Q.S. Nguyen).

1631-0721/03/$ – see front matter
doi:10.1016/S1631-0721(03)00020-2

1. Solids in frictional contact with an obstacle

1.1. Coulomb's law of dry friction

At a contact point of a solid with an obstacle, the relative velocity $v = v_s - v_o$ is by definition the difference between the velocities of the material points of the solid and of the obstacle in contact. The relative velocity v and reaction R can be decomposed into normal and tangential components

$$v = v_T + v_N n, \qquad R = T + Nn \tag{1}$$

where n denotes the external normal vector to the obstacle at a contact point, v_T is the sliding velocity vector, T the tangential reaction vector and N the normal reaction, N and v_N are scalars. The unilateral condition of contact implies that the normal reaction must be non-negative $N \geqslant 0$. When there is contact, Coulomb's law of dry friction states that the friction criterion must be satisfied and that the friction force must have the opposite direction to the sliding velocity,

$$\phi = \|T\| - fN \leqslant 0 \quad \text{and} \quad \|T\| = fN, \quad T = -a v_T,\ a \geqslant 0,\ \text{if } v_T \neq 0 \tag{2}$$

where f denotes the friction coefficient. In particular, the dissipation by friction is $-T \cdot v_T = fN\|v_T\|$. Coulomb's law has often been interpreted in the literature as a non-associated law since the velocity (v_T, v_N) is not a normal to the domain of admissible forces. In particular, it has been discussed as a bi-potential law, cf. [1]. A more standard interpretation consists of saying that Coulomb's friction is a standard dissipative law with a state-dependent dissipation potential, cf. [2,3]. Indeed, normality law is satisfied by the flux v_T and the force T since T and v_T are related through a state-dependent dissipation potential $D(v_T, N)$

$$T = -D,_{v_T} \quad \text{with } D = fN\|v_T\| \tag{3}$$

where $D,_{v_T}$ is understood in the sense of sub-gradient. The set of admissible forces, which is a sphere of radius fN, depends on the present state through the present value of N.

1.2. Governing equations

The simple case of an elastic solid occupying a volume V in the undeformed position is considered. The solid is submitted to given forces and displacements $r^d = r^d(\lambda(t))$, $u^d = u^d(\lambda(t))$ respectively on the portions S_r, S_u of the boundary S, and $\lambda(t)$ denotes a control parameter defining the loading history. On the complementary part S_R, the solid may enter into contact with a moving obstacle $h(m,t) < 0$ and the non-penetration condition is

$$h\big(x + u(x,t), t\big) \geqslant 0 \quad \forall x \in S_R \tag{4}$$

The mechanical response of the solid are governed by unilateral contact conditions, Coulomb's law and classical equations of elastodynamics. In Lagrange description, the governing equations are

$$\begin{cases} \mathrm{Div}\, b = \rho \ddot{u}, \quad b = W,_{\nabla u} \quad \forall x \in V \\ b \cdot n_s = r^d \quad \forall x \in S_r, \quad b \cdot n_s = Nn + T \quad \forall x \in S_R, \quad u = u^d \quad \forall x \in S_u \end{cases} \tag{5}$$

where $W(\nabla u)$ and b denote respectively the elastic energy per unit volume and the unsymmetric Piola–Lagrange's stress. In particular, the unknowns (u, N) must satisfy the local equations

$$u = u^d \quad \forall x \in S_u, \qquad h \geqslant 0, \quad N \geqslant 0, \quad Nh = 0 \quad \forall x \in S_R \tag{6}$$

and the variational inequality

$$\begin{cases} \int_V (\nabla u^* - \nabla \dot{u}) : W,_{\nabla u}(\nabla u)\,\mathrm{d}V - \int_{S_r} r^d \cdot (u^* - \dot{u})\,\mathrm{d}a + \int_V \rho \ddot{u} \cdot (u^* - \dot{u})\,\mathrm{d}V \\ - \int_{S_R} (v_N^* - v_N) N\,\mathrm{d}a + \int_{S_R} fN(\|v_T^*\| - \|v_T\|)\,\mathrm{d}S \geqslant 0 \quad \forall u^* \in U_{\mathrm{ca}} \end{cases} \tag{7}$$

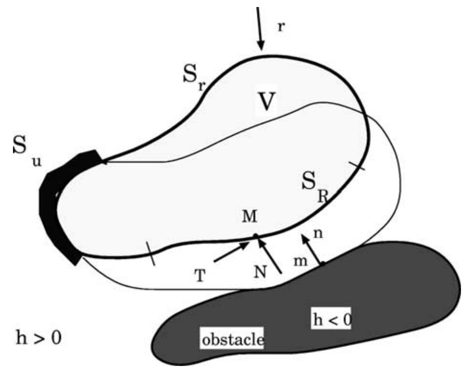

Fig. 1. A solid in unilateral contact with an obstacle.

where U_{ca} denotes the set of kinematically admissible rates

$$U_{\text{ca}} = \{u^* \mid u^* = \dot{u}^d \text{ on } S_u\} \tag{8}$$

and v^* and v are the relative rates

$$v_N = n \cdot \dot{u} - v_{oN}, \quad v_T = \dot{u} - (n \cdot \dot{u})n - v_{oT}, \quad v_N^* = n \cdot u^* - v_{oN}, \quad v_T^* = u^* - (n \cdot u^*)n - v_{oT}, \tag{9}$$

with $n = h_{,m} / \|h_{,m}\|$ and $v_{oN} = -h_{,t} / \|h_{,m}\|$. The system of solid in frictional contact under loads is associated with an energy potential and a dissipation potential:

$$\begin{cases} \mathbf{E}(u, \lambda, \mu) = \int_V W(\nabla u)\,\mathrm{d}V - \int_{S_r} r^d(\lambda) \cdot u\,\mathrm{d}a - \int_{S_R} \mu h(x+u)\,\mathrm{d}a \\ \mathbf{D}(N, v_T) = \int_{S_R} f N \|v_T\|\,\mathrm{d}a \end{cases} \tag{10}$$

where μ denotes the Lagrange multipler associated with the constraints (4) and $N = \mu \|h_{,m}\|$. The variational inequality (7) can be condensed as

$$(\mathbf{J} + \mathbf{E}_{,u}) \cdot (u^* - \dot{u}) + \mathbf{D}(N, v_T^*) - \mathbf{D}(N, v_T) \geqslant 0 \quad \forall u^* \in U_{\text{ca}} \tag{11}$$

where J denotes the inertial terms. Some regularizations of the frictional contact problem have been proposed in the literature:

- Non-local Coulomb's law, proposed by Duvaut [4], in which the local normal reaction N is replaced by its mean value $\overline{N}$ on an elementary representative surface S_{re}

  $$\overline{N} = \frac{1}{S_{\text{re}}} \int_{S_{\text{re}}} N(x)\,\mathrm{d}s$$

- Normal compliance law, discussed by Oden and Martins [5], Kikuchi and Oden [6], Andersson [7,8], Klarbring et al. [9], ... It consists of replacing Signorini's relations of unilateral contact by a relationship giving the normal reaction as a function of the gap $h(x + u(x))$. For example, nonlinear springs of energy $\varphi(h)$ per unit surface may be added to the system where $\varphi(h)$ is a regular function permitting an approximation of Signorini's conditions. In particular, with $\varphi(h) = \frac{k}{2}\langle h\rangle_-^2$, then the normal compliance law consists of writing that

  $$N = k\|h_{,m}\|\langle h\rangle_-$$

  General discussions on the existence of a solution of the quasi-static problem in small deformation have been given in the literature, when there is regularization by normal compliance cf. [7,9] and by nonlocal Coulomb's law cf. [10]. In these cases, it has been proved that the existence of at least one solution is ensured when the friction coefficient is small enough.

2. Divergence instability of an equilibrium

There is flutter or divergence instability if, under disturbances, the system will leave the equilibrium position with or without growing oscillations. As usual, the possibility of divergence instability can be discussed in a purely static approach [11,12,3]. The quasi-static problem has been much discussed, cf. [6,10,13,8,7,14]. For the sake of simplicity, the particular case of fixed obstacles is considered here. The obstacle is given by the time-independent domain $h(m) \leqslant 0$. Normal and tangential relative velocities are $v_N = \dot{u}_N$ and $v_T = \dot{u}_T$. The governing equations follow from (6), (7) by deleting the inertia terms. If the rigid motion of the solid is not excluded by the implied displacements, it is well known that the existence of an equilibrium position under applied loads is not always ensured.

2.1. Rate problem in quasi-statics

The analysis of the static stability of an equilibrium follows from the consideration of the rate response as in the theory of elastic and plastic buckling of solids. With the following notation

$$V_{\mathrm{ad}} = \left\{ u^* \;\middle|\; \begin{array}{l} u^* \in U_{\mathrm{ca}} \text{ and } \forall x \in S_{Rc}, \\ u_N^* = 0 \text{ if } N > 0 \text{ and } u_N^* \geqslant 0 \text{ if } N = 0, \\ u_T^* = 0 \text{ if } \phi < 0 \text{ and } u_T^* = -bT,\ b \geqslant 0 \text{ if } \phi = 0,\ N > 0 \end{array} \right\} \tag{12}$$

it is clear that $\dot{u} \in V_{\mathrm{ad}}$ which is the set of admissible rates. The virtual work equation holds in rate form for all $u^* \in V_{\mathrm{ad}}$

$$\int_V \nabla(u^* - \dot{u}) : \mathcal{L} : \nabla\dot{u}\,\mathrm{d}V - \int_{S_r} \dot{r} \cdot (u^* - \dot{u})\,\mathrm{d}a - \int_{S_R} \dot{R} \cdot (u^* - \dot{u})\,\mathrm{d}a = 0$$

where $\mathcal{L} = \mathcal{W}_{,\nabla u \nabla u}$ denotes the elastic modulus. Since $\dot{T} \cdot (u^* - \dot{u}) = \dot{T} \cdot (u_T^* - \dot{u}_T) + \dot{T} \cdot (u_N^* - \dot{u}_N)n$ and $\dot{T} \cdot n = 0$ when $T = 0$, the last term can also be written as $-\dot{R} \cdot (u^* - \dot{u}) = -N(u_T^* - \dot{u}_T) \cdot C \cdot \dot{u}_T - \dot{N}(v_N^* - v_N) - \dot{T} \cdot (v_T^* - v_T)$ where C denotes the curvature tensor of the obstacle at contact point. Taking account of the fact that:

$$\dot{T}(v_T - v_T^*) + f\dot{N}\big(\|v_T\| - \|v_T^*\|\big) \leqslant 0 \tag{13}$$

which is a consequence of Coulomb's law, cf. [13,15,3], the following statement is obtained

The rate $\dot{u} \in V_{\mathrm{ad}}$ and satisfies the variational inequality

$$\begin{aligned} &\int_V (\nabla u^* - \nabla\dot{u}) : \mathcal{L} : \nabla\dot{u}\,\mathrm{d}V - \int_{S_r} \dot{r}^d \cdot (u^* - \dot{u})\,\mathrm{d}a - \int_{S_{Rc}} N(u_T^* - \dot{u}_T) \cdot C \cdot \dot{u}_T\,\mathrm{d}a \\ &\quad - \int_{S_{Rc}} (u_N^* - \dot{u}_N)\dot{N}(\dot{u})\,\mathrm{d}a + \int_{S_{Rc}} f\dot{N}(\dot{u})\big(\|u_T^*\| - \|\dot{u}_T\|\big)\,\mathrm{d}a \geqslant 0 \quad \forall u^* \in V_{\mathrm{ad}} \end{aligned} \tag{14}$$

where

$$\dot{N}(\dot{u}) = \frac{\mathrm{d}}{\mathrm{d}t}(n \cdot b \cdot n_s) = (n_s \otimes n) : \mathcal{L} : \nabla\dot{u} - R \cdot C \cdot \dot{u} \tag{15}$$

If a regularization of unilateral contact by normal compliances is introduced, there are no more unilateral constraints. For some particular cases of elastic structures in frictional contact with normal compliances, the framework of standard dissipative systems can be directly applied, cf. [16,17,3,18]. Such a system is defined by an energy potential $\mathbf{E}(q, \lambda)$ and a dissipation potential $\mathbf{D}(q, \dot{q})$ which is convex and positively homogenous of degree 1 with respect to $\dot{q}$. The quasi-static evolution is governed by

$$\mathbf{E}_{,q} + \mathbf{D}_{,\dot{q}} = 0 \tag{16}$$

where $\mathbf{D}_{,\dot q}$ denotes the sub-gradient of $\mathbf{D}$ with respect to $\dot q$. The associated rate equations are

$$\mathbf{E}_{,qq} \cdot \dot q + \mathbf{E}_{,q\lambda} \dot\lambda + \mathbf{D}_{,q\dot q} \cdot \dot q = 0 \tag{17}$$

These equations can also be written in variational form as

$$\begin{cases} \mathbf{E}_{,q} \cdot (q^* - \dot q) + \mathbf{D}(q, q^*) - \mathbf{D}(q, \dot q) = 0 \quad \forall q^* \\ (q^* - \dot q) \cdot (\mathbf{E}_{,qq} \cdot \dot q + \mathbf{E}_{,q\lambda} \dot\lambda) + \dot q \cdot \big(\mathbf{D}_{,q}(q, q^*) - \mathbf{D}_{,q}(q, \dot q)\big) = 0 \quad \forall q^* \in V_{\mathrm{ad}} \end{cases}$$

Rate equations (14) are written in the same spirit, with some additional difficulties due to the presence of constraints and to the state-dependent expressions of tangential and normal components in terms of $\dot q$.

2.2. Static stability criterion

It is assumed that under the action of a given dead load, an equilibrium position of a solid in frictional contact with a fixed obstacle exists. The stability of this equilibrium is the subject of interest here.

The classical concept of static stability consists of defining the stability as the absence of additional displacement when the load does not vary $\dot\lambda = 0$. The equilibrium is thus statically stable if the rate problem admits only the trivial solution $\dot u = 0$. The following statement is then straightforward [15,19,17,14,20,18]:

The condition of positivity

$$\mathbf{I}(u^*) = \int_V \nabla u^* : \mathcal{L} : \nabla u^* \,\mathrm{d}V + \int_{S_{Rc}} N u_T^* \cdot C \cdot u_T^* \,\mathrm{d}a + \int_{S_{Rc}} f \dot N(u^*) \, \|u_T^*\| \,\mathrm{d}a > 0 \quad \text{for all } u^* \neq 0 \in V_{\mathrm{ad}}^0 \tag{18}$$

is a criterion of stability since it ensures the static stability of the considered equilibrium.

The significance of this criterion can be better understood from its energy interpretation in the same spirit as Hill's criterion of stability in plasticity, cf. [11,12,15,17]. A perturbation of the equilibrium by perturbation forces is introduced for $t \geqslant 0$. Let $u(t)$ denotes the perturbed motion starting from the equilibrium $u(0) = u_{\mathrm{eq}}$. The energy balance at time t is

$$W_{\mathrm{per}}(t) = \int_V \big(W(\nabla u(t)) - W(\nabla u_{\mathrm{eq}})\big) \,\mathrm{d}V - \int_{S_r} r^d(\lambda) \cdot \big(u(t) - u_{\mathrm{eq}}\big) \,\mathrm{d}a - \int_0^t \int_{S_R} R(\tau) \cdot \dot u \,\mathrm{d}a \,\mathrm{d}t + C(t)$$

where W_{per} (t) is the energy supplied by perturbation forces and $C(t)$ is the kinetic energy of the solid at time t. At the begining of the perturbation, i.e., for small t, if the perturbed motion and the energy balance can be expanded as

$$u(t) = u_{\mathrm{eq}} + u_1 t + u_2 \frac{t^2}{2} + \cdots, \qquad W_{\mathrm{per}}(t) = W_0 + W_1 t + W_2 \frac{t^2}{2} + \cdots + C(t)$$

then, the following expressions are obtained $W_0 = 0$, $W_1 = 0$, $W_2 = I(u_1)$, cf. [15,17]. Thus, condition (18) implies that the external world must supply energy at early times in order to perturbe the system from equilibrium. This criterion ensures a certain stability in the energy sense, also called directional stability [12]. Its violation in a direction of displacement leads to a divergence instability of the considered equilibrium since the system will leave this position with a growing kinetic energy.

3. Flutter instability of the steady sliding equilibrium

For the sake of clarity, the stability of the steady sliding equilibrium of an elastic solid in contact with a moving rigid half-space, in translation motion at a constant velocity w parallel to the free surface, is considered here in small deformation.

3.1. Steady sliding equilibrium

The steady sliding equilibrium u of the solid must satisfy

$$\int_V \nabla\delta u : L : \nabla u \, \mathrm{d}V - \int_{S_r} r^d \cdot \delta u \, \mathrm{d}a - \int_{S_R} (\delta u_N N + f N\tau \cdot \delta u_T) \, \mathrm{d}a = 0$$

This equation leads formally to a system of reduced equations of the form

$$N = k_{NN}[u_N] + k_{NT}[u_T] + N^d, \qquad T = f N\tau = k_{TN}[u_N] + k_{TT}[u_T] + T^d$$

The principal unknown u_N must satisfy

$$u_N = \mathbf{A}[N] + \mathbf{B}, \quad N \geqslant 0,\ u_N \geqslant 0,\ N \cdot u_N = 0 \tag{19}$$

with

$$\begin{cases} \mathbf{A} = \left(k_{NN} - k_{NT} k_{TT}^{-1} k_{TN}\right)^{-1} (I - f k_{NT} h_{TN}) \\ h_{TN}[N] = k_{TT}^{-1}[N\tau], \quad \mathbf{B} = \left(k_{NN} - k_{NT} k_{TT}^{-1} k_{TN}\right)^{-1} \left[N^d - k_{NT} k_{TT}^{-1} \left[T^d\right]\right] \end{cases} \tag{20}$$

It is clear that the linear operator $\mathbf{A}$ is not symmetric if $f \neq 0$:

$$\left(N^*, \mathbf{A}[N]\right) = \int_{S_R} N^*(x) \mathbf{A}[N](x) \, \mathrm{d}S \neq \left(N, \mathbf{A}[N^*]\right) \tag{21}$$

Thus, a linear complementary problem (LCP) must be considered. In particular, the existence and uniqueness of a steady sliding solution are ensured if $\mathbf{A}$ is positive-definite or P-positive [21,22,3].

3.2. Instability of the steady sliding equilibrium

The stability of the steady sliding position can be obtained from the study of small perturbed motions. However, the equations of motion cannot be linearized without the assumption of effective contact. Indeed, in the presence of a loose contact, a small perturbed motion is not necessarily governed by linear equations because of the possibility of separation. Under the assumption of an effective contact, if the sliding speed is never zero, the dynamic equations can be written as

$$\int_V \delta u \cdot \rho \ddot{u} \, \mathrm{d}V + \int_V \nabla\delta u : L : \nabla u \, \mathrm{d}V + \int_{S_R} N \delta u_N \, \mathrm{d}S + \int_{S_R} f N \frac{\dot{u}_T - w}{\|\dot{u}_T - w\|} \cdot \delta u_T \, \mathrm{d}S = 0 \quad \forall \delta u, \delta N \tag{22}$$

The linearization is then possible for sliding motions. The nature of this particular problem can be better understood in the discretized form. After discretization, the equations of motion are

$$\begin{cases} U_N = 0 \\ M_{YY} \ddot{Y} + K_{YY} Y = f \Phi(\dot{Y}) N + F_Y \\ M_{NY} \ddot{Y} + K_{NY} Y = N + F_N \end{cases} \tag{23}$$

where $u = (U_N, Y)$ and $\Phi(\dot{Y})$ is a matrix dependent on the direction of sliding. The linearized equations for sliding motions are

$$\begin{cases} U_N^* = 0 \\ M_{YY} \ddot{Y}^* + K_{YY} Y^* = f \Phi \ N^* + f \Phi_{\dot{Y}} \dot{Y}^* \\ M_{NY} \ddot{Y}^* + K_{NY} Y^* = N^* \end{cases} \tag{24}$$

The general expression $u^* = \mathrm{e}^{st} U$ with $U = (U_N = 0, X)$ then leads to

$$s^2 M_{YY} X + K_{YY} X = f s \Phi_{\dot{Y}} X + f \Phi \mathcal{N}, \qquad s^2 M_{NY} X + K_{NY} X = \mathcal{N} \tag{25}$$

i.e., to the generalized eigenvalue problem

$$s^2(M_{YY} - f\Phi M_{NY})X - sf\Phi_{\dot{Y}}X + (K_{YY} - f\Phi K_{NY})X = 0 \tag{26}$$

Thus, the considered equilibrium is asymptotically stable (with respect to sliding motions) if $\Re(s) < 0$ for all s and unstable if there exists at least one value s such that $\Re(s) > 0$. This generalized eigenvalue problem can be written as $(s^2\overline{M} + s\overline{C} + \overline{K})X = 0$ with non-symmetric matrices $\overline{M}$, $\overline{K}$ and complex eigenvalues and eigenvectors. This analysis leads to the definition of a critical value $f_d \geqslant 0$ such that the considered equilibrium is unstable when $f > f_d$.

3.3. *Example on the sliding contact of two elastic layers*

The simple example of the frictional contact of two elastic infinite layers is considered here as an illustrating example. This problem was discussed analytically by Adams [23], by Martins et al. [24] and by Renardy [28].

The contact in plane strain with friction of two infinite elastic layers, of thickness h and h^* respectively as shown in Fig. 2, is considered. The lower face of the bottom layer is maintained fixed in the axes $Oxyz$. The upper face of the top layer, assumed to be in translation of velocity w in the direction Ox, is compressed to the bottom layer by an implied displacement $\delta < 0$. At the interface $y = 0$, the contact is assumed to obey Coulomb's law of friction with a constant friction coefficient. The celerities of dilatation and shear waves are first introduced:

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \qquad c_2 = \sqrt{\frac{\mu}{\rho}}, \qquad \tau = \frac{c_2}{c_1}$$

for the top layer and for the bottom layer (superscript *), to write the governing equations for the displacements of the top layer $wte_1 + u(x - wt, y, t)$ and of the bottom layer $u^*(x, y, t)$ under the form:

$$\begin{cases} \left(1 - \tau^2\left(\dfrac{w}{c_2}\right)^2\right)u_{x,xx} + \tau^2 u_{x,yy} + (1 - \tau^2)u_{y,xy} = \tau^2\left(u_{x,tt} - 2\dfrac{w}{c_2}u_{x,tx}\right) \\ u_{y,yy} + \tau^2\left(1 - \left(\dfrac{w}{c_2}\right)^2\right)u_{y,xx} + (1 - \tau^2)u_{x,xy} = \tau^2\left(u_{y,tt} - 2\dfrac{w}{c_2}u_{y,tx}\right) \\ u^*_{x,xx} + \tau^{*2}u^*_{x,yy} + (1 - \tau^{*2})u^*_{y,xy} = \tau^{*2}u^*_{x,tt} \\ u^*_{y,yy} + \tau^{*2}u^*_{y,xx} + (1 - \tau^{*2})u^*_{x,xy} = \tau^{*2}u^*_{y,tt} \end{cases} \tag{27}$$

Boundary and interface conditions are

$$u(x - wt, h, t) = 0, \quad u^*(x, -h^*, t) = 0, \quad u_y(x - wt, 0, t) = u^*_y(x, 0, t)$$
$$\sigma_{yy}(x, 0, t) = \sigma^*_{yy}(x, 0, t), \quad \sigma_{xy}(x, 0, t) = \sigma^*_{xy}(x, 0, t), \quad f\sigma_{yy}(x, 0, t) = -\sigma_{xy}(x, 0, t)$$

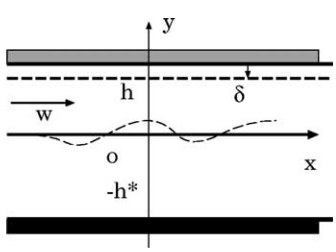

Fig. 2. Sliding contact of two elastic layers.

The stability of the steady sliding solution can be obtained by a linearization of the dynamic equation under the assumption of sliding perturbed motions near the steady sliding state. These motions are searched for under the form of slip waves of wave-length $L = 1/k$:

$$u(x - wt, y, t) = \mathrm{e}^{2\pi kct}\,\mathrm{e}^{2\mathrm{i}k\pi(x-wt)}X(y), \qquad u^*(x, y, t) = \mathrm{e}^{2\pi kct}\,\mathrm{e}^{2\mathrm{i}k\pi x}X^*(y)$$

The condition of existence of non null displacement modes $(X,\ X^*)$ requires that the pair c and k must be a root of the following equation:

$$\begin{aligned}\mathcal{F}(c,k) \;=\; &\rho c_2^2\big(A(p,q,kh)\big)\big(\mathrm{i}B(p^*,q^*,kh^*) + fC(p^*,q^*,kh^*)\big)\\ &+ \rho^* c_2^{*2}\big(A(p^*,q^*,kh^*)\big)\big(\mathrm{i}B(p,q,kh) - fC(p,q,kh)\big) = 0\end{aligned} \tag{28}$$

where $p,\ q,\ A,\ B,\ C$ are appropriate functions [25]

$$\begin{cases} p^2 = 1 + \left(\dfrac{c - \mathrm{i}w}{c_2}\right)^2, \quad q^2 = 1 + \tau^2\left(\dfrac{c - \mathrm{i}w}{c_2}\right)^2 \\ A(p,q,kh) = -4pq\big(1+p^2\big) + pq\big(4 + \big(1+p^2\big)^2\big)\cosh(2\pi pkh)\cosh(2\pi qkh) \\ \quad - \big(\big(1+p^2\big)^2 + 4p^2q^2\big)\sinh(2\pi pkh)\sinh(2\pi qkh) \\ B(p,q,kh) = q\big(1-p^2\big)\big(\sinh(2\pi pkh)\cosh(2\pi qkh) - pq\cosh(2\pi pkh)\sinh(2\pi qkh)\big) \\ C(p,q,kh) = pq\big(3+p^2\big) - pq\big(3+p^2\big)\cosh(2\pi pkh)\cosh(2\pi qkh) \\ \quad + \big(2p^2q^2 + \big(1+p^2\big)\big)\sinh(2\pi pkh)\sinh(2\pi qkh) \end{cases}$$

The case of a rigid top layer is obtained when $c_2 \Rightarrow +\infty$

$$\mathcal{F}(c,k) = \mathrm{i}B(p^*q^*, kh^*) + fC(p^*,q^*,kh^*) = 0 \tag{29}$$

For an elastic half-plane compressed into a moving rigid half-plane, cf. Martins et al. [24], the results are:

$$\mathcal{F}(c) = \mathrm{i}q^*\big(1 - p^{*2}\big) + f\big(1 + p^{*2} - 2p^*q^*\big) = 0 \tag{30}$$

In the case of two elastic half-planes, cf. Adams [23], this equation can be written as:

$$\begin{aligned}\mathcal{F}(c) \;=\; &\rho c_2^2\big(\big(1+p^2\big)^2 - 4pq\big)\big(\mathrm{i}q^*\big(1-p^{*2}\big) + f\big(1+p^{*2} - 2p^*q^*\big)\big)\\ &+ \rho^* c_2^{*2}\big(\big(1+p^{*2}\big)^2 - 4p^*q^*\big)\big(\mathrm{i}q\big(1-p^2\big) - f\big(1+p^2-2pq\big)\big) = 0\end{aligned} \tag{31}$$

It has been established in each case that there exists a critical value $f_d \geqslant 0$ such that the steady sliding solution is unstable for $f \geqslant f_d$. For example, $f_d = 0$ occurs for the system of two layers of finite depths while the possibility $f_d > 0$ may happen in the sliding contact of half-spaces, cf. [24,26,20,27,29].

4. Stick–slip-separation waves

The fact that the steady sliding solution is unstable leads to the study of possible dynamic bifurcations of the sliding contact of solids. In the spirit of Hopf bifurcation [30], a periodic response can be expected as an alternative stable response. This possibility has been explored in the example of two coaxial cylinders [27,31,32].

The mechanical response in plane strain of a brake-like system composed of an elastic tube, of internal radius R and external radius R^*, in frictional contact on its inner surface with a rotating rigid cylinder of radius $R + \Delta$ and of angular rotation Ω has been discussed, cf. Fig. 3. The mismatch $\Delta \geqslant 0$ is a load parameter controlling the normal contact pressures. This model problem enables us to exhibit the existence of nontrivial periodic solutions in the form of stick–slip or stick–slip–separation waves propagating on the contact surface.

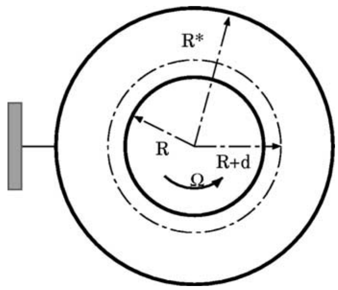

Fig. 3. The problem of coaxial cylinders in frictional contact.

The governing equations of the system follow from the kinetic relations, the fundamental law, the linear elastic constitutive equations, and the boundary unilateral contact conditions with Coulomb's friction:

$$\begin{cases} \bar{\varepsilon} = (\nabla \bar{u})_s \\ \operatorname{Div} \bar{\sigma} = \gamma \ddot{\bar{u}} \\ \bar{\sigma} = \dfrac{\nu}{(1+\nu)(1-2\nu)} \operatorname{Tr}(\bar{\varepsilon}) I + \dfrac{1}{1+\nu} \bar{\varepsilon} \\ \bar{u}_r(\xi, \theta, t) = \bar{u}_\theta(\xi, \theta, t) = 0 \\ \bar{\sigma}_{rr}(1, \theta, t) = -p(\theta, t), \quad \bar{\sigma}_{r\theta}(1, \theta, t) = -q(\theta, t) \\ \bar{u}_r \geqslant \delta, \quad p \geqslant 0, \quad p(\bar{u}_r - \delta) = 0 \\ |q| \leqslant fp, \quad q(1 - \dot{\bar{u}}_\theta) - fp|1 - \dot{\bar{u}}_\theta| = 0 \end{cases} \tag{32}$$

where non-dimensional variables are introduced,

$$\bar{u} = \frac{u}{R}, \quad \bar{\sigma} = \frac{\sigma}{E}, \quad \bar{r} = \frac{r}{R}, \quad \gamma = \frac{\rho R^2 \Omega^2}{E}, \quad \xi = \frac{R^*}{R}, \quad \delta = \frac{\Delta}{R}, \quad \bar{t} = \Omega t, \quad \dot{\bar{u}} = \frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{t}}$$

The steady sliding solution is given by,

$$\begin{cases} \bar{u}_{er} = \delta \dfrac{1}{\xi^2 - 1} \left(\dfrac{\xi^2}{\bar{r}} - \bar{r} \right), \quad \bar{u}_{e\theta} = \delta f \dfrac{1}{\xi^2 - 1} \left(\dfrac{\xi^2}{\bar{r}} - \bar{r} \right) \left(1 + \dfrac{1}{\xi^2 (1 - 2\nu)} \right) \\ p_e = \delta \dfrac{1}{\xi^2 - 1} \dfrac{1}{1+\nu} \left(\xi^2 + \dfrac{1}{1 - 2\nu} \right) > 0, \quad q_e = f p_e \end{cases} \tag{33}$$

Since closed form dynamical solutions cannot be generated, two complementary approaches has been followed. The first approach is semi-analytical after a reduction to a simpler system of equations. The second approach consists of a numerical simulation by the finite element method and appropriate time-integrations.

An interesting simplification to the problem is obtained when the displacement is sought in the form

$$\bar{u}_r = U(\theta, \bar{t}) F(\bar{r}), \qquad \bar{u}_\theta = V(\theta, \bar{t}) F(\bar{r}), \qquad F(\bar{r}) = \frac{1}{\xi^2 - 1} \left(\frac{\xi^2}{\bar{r}} - \bar{r} \right) \tag{34}$$

In this approximation, the following local equations are obtained from the virtual work equation

$$\begin{cases} \ddot{U} - bU'' - dV' + gU = P \\ \ddot{V} - aV'' + dU' + hV = Q \\ P \geqslant 0, \quad U - \delta \geqslant 0, \quad P(U - \delta) = 0 \\ |Q| \leqslant fP, \quad Q(1 - \dot{V}) - fP|1 - \dot{V}| = 0 \end{cases} \tag{35}$$

where $'$ denotes the derivative with respect to θ and $a,\ b,\ g,\ h,\ d$ are material and geometry constants. All of them are positive except for the coupling coefficient d. Finally, only the non-dimensional displacements on the contact surface $U(\theta,t)$ and $V(\theta,t)$ and the non-dimensional reactions $P(\theta,t)$ and $Q(\theta,t)$ remain as unknowns in the reduced equations.

The steady sliding solution, given by $U_e=\delta$, $V_e=\delta fg/h$, $P=P_e$ and $Q_e=fP_e$, is unstable for the reduced system. Indeed, under the assumption of sliding motions, a small perturbed motion is described by $U=U_e$, $V=V_e+V_*$, $P=P_e+P_*$ and $Q=Q_e+Q_*$. It follows that

$$\ddot{V}_*-aV_*''+fdV_*'+hV_*=0 \tag{36}$$

If a general solution is sought in the form $V_*=\mathrm{e}^{s\bar{t}}\mathrm{e}^{\mathrm{i}k\theta}$, then $-s^2=ak^2+h+\mathrm{i}kfd$. When $f=0$, it follows that $s=\pm\mathrm{i}\omega_k$ with $\omega_k^2=ak^2+h$. Thus two harmonic waves propagating in opposite senses of the form $\cos(k\theta\pm\omega_k\bar{t}+\varphi)$ are obtained as in classical elasticity. When $f>0$ and $d>0$, then $s=\pm(s_{rk}+\mathrm{i}s_{ik})$, $s_{rk}>0$, $s_{ik}<0$, thus a general solution of the difference V_* of the form $V_*=\mathrm{e}^{\pm s_{rk}\bar{t}}\cos(k\theta\pm s_{ik}\bar{t}+\varphi)$ is obtained and represents two waves propagating in opposite senses: an exploding wave in the sense of the implied rotation, and a damping wave propagating in the opposite direction. If $f>0$ and $d<0$, the exploding wave propagates in the opposite sense since the previous expression of s is still valid with $s_{rk}>0$ and $s_{ik}>0$.

It is expected that in some particular situations, there is a dynamic bifurcation of Poincaré–Andronov–Hopf's type. This means that the perturbed motion may evolve to a periodic response. This transition has been observed numerically in many examples of the literature, cf., for example, [33] or [34]. To explore this idea, a periodic solution has been sought in the form of a wave propagating at constant velocity:

$$U=U(\phi),\quad V=V(\phi),\quad \phi=\theta-\bar{c}\bar{t} \tag{37}$$

where $\bar{c}$ is the non-dimensional wave velocity, U and V are periodic functions of period $T=2\pi/k$. The physical velocity of the wave is thus $c=|\bar{c}|R\Omega$ and the associated dynamic response is periodic of frequency $|\bar{c}|k\Omega$. The propagation occurs in the sense of the rotation when $c>0$. According to the regime of contact, a slip wave, a stick–slip wave, a slip-separation wave or a stick–slip–separation wave can be discussed. The governing equations of such a wave follow from (32):

$$\begin{cases}(\bar{c}^2-b)U''-dV'+gU=P\\ (\bar{c}^2-a)V''+dU'+hV=Q\\ P\geqslant 0,\quad U\geqslant\delta,\quad P(U-\delta)=0\\ |Q|\leqslant fP,\quad Q(1-\dot{V})-fP|1-\dot{V}|=0\end{cases} \tag{38}$$

The existence of stick–slip waves is obtained when the load is sufficiently strong or when the rotation is slow. For example, for $\xi=1.25$ and $f=1$, stick-positive slip solutions are obtained for $8\leqslant k\leqslant 12$. It is found that c must have the sign of d. These waves propagate in the sense of the previous exploding perturbed motions, thus opposite to the rotation of the cylinder when $d<0$, with a frequency and a celerity independent of the rotation velocity Ω. The celerity is close to the celerities of dilatation and shear waves in the solid while the frequency is inversely proportional to the radius R. For example, for $\sqrt{E/\rho}=1000$ m/s, $\xi=1.25$, $f=1$, $R=1$ m and $\Omega=100$ rad/s, the results obtained concerning the mode-8 wave are $\Psi=0.839$, $c=1255$ m/s and the associated frequency is 10045 Hz. If $\xi=2$, $f=0.3$, $R=0.5$ m, $\Omega=10$ rad/s, a frequency 8240 Hz and a celerity 1030 m/s are obtained for $k=4$ as shown in Fig. 4. For $\xi=1.15$, for example, d is positive and the propagation goes in the rotation direction. The limiting case $\xi\Rightarrow 1$ can be interpreted as the sliding motion of a rigid plate on an elastic layer or of a rigid half-space on an elastic half-space [23,24]. The obtained solution is a wave with an oscillation about the steady sliding response. The amplitude of the wave is linearly proportional to the rotation Ω. It also increases with the friction coefficient f and decreases with the mismatch. Thus, for vanishing rotations, the steady sliding solution is recovered as the limit of the dynamic response. The stick–slip solution can no longer be available if the rotation is strong enough since the associated pressure may become negative. In the same spirit, for a small mismatch, the pressure may become negative under the assumption of a stick–slip regime everywhere. This means that the possibility of separation is not excluded when the mismatch is not strong enough or if the rotation or the friction coefficient is sufficiently high.

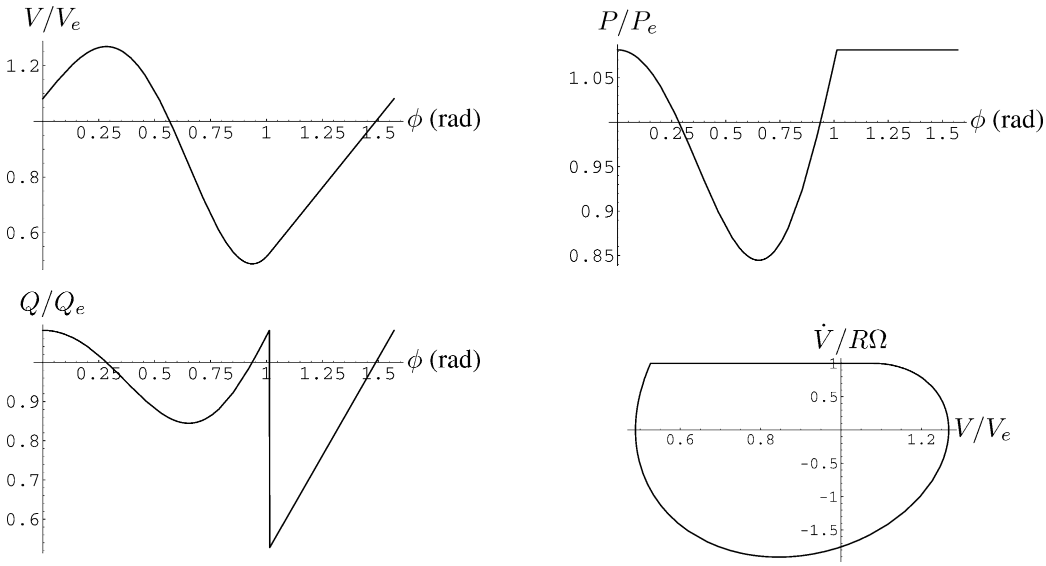

Fig. 4. Semi-analytical approach: an example of stick–slip wave in mode 4 with $\xi = 2$, $f = 0.3$, $\Omega = 10$ rad/s, $R = 0.25$ m and $\delta = 0.005$. It is found that $\Psi = 0.644$, $c = 1030$ m/s. Phase diagram and variations of V/V_e, P/P_e and Q/Q_e in $[0, 2\pi/k]$.

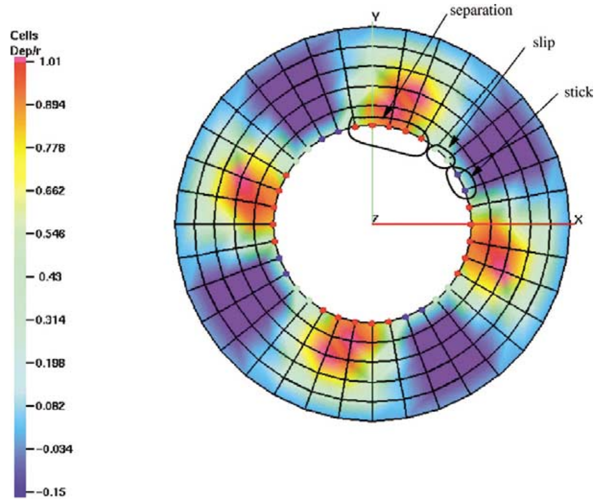

Fig. 5. A mode-4 stick–slip–separation wave, obtained by numerical simulations for $\xi = 2$, $\Omega = 50$ rad/s, $f = 0.7$, $\delta = 0.001$. The isovalues of the radial displacement (mm): separation, slip and stick nodes on the contact surface are given respectively in red, green, blue.

A numerical simulation with an explicit scheme using Lagrange multipliers [35,36] has been performed. The case $\xi = 2$ and $f = 0.3$ has been considered. Starting from a motionless initial state, the mismatch displacement is then increased linearly from 0 to its final value. A cyclic limit response is then obtained for large time. The numerical simulation leads to a stick–slip wave in mode 4 without forcing and the obtained response is close to the analytical solution of the reduced approach.

It was also checked that a stick–slip–separation wave is effectively obtained when the mismatch is small enough or when the friction is high enough. For example, when $\Omega = 50$ rad/s, $\delta = 0.001$ and $f = 0.7$, the limit cycle results as a stick–slip–separation wave. The result for radial displacements is shown in Fig. 5.

In fact, it is well known that a periodic response does not systematically result from the flutter instability of the steady sliding solution. It has been observed in various examples of discrete or continuous systems that the response may be quasi-periodic or non-periodic or chaotic [37,5,38,3]. Periodic responses prevail in the example of coaxial cylinders because of the special geometry of the system. Periodic solutions under the form of stick–slip waves have been also obtained by Adams in the sliding contact of two elastic half-spaces [39]. The generation of dynamic stick–slip–separation waves on the contact surface can be compared to Shallamach waves in the sliding contact of rubber in statics, cf. [40,36,27,41,32,42,43,39,34,20,44].

5. Friction-induced vibrations and noises

It is well known that the presence of friction induces mechanical vibrations and noise emittences in the sliding contact of solids. For example, the creaking noise of a door, the unsteady motion with fits and starts of a windscreenwiper can be interpreted as stick–slip motions resulting from the instability of the steady sliding solution. In particular, the phenomena of squeal [45] have been interpreted in the literature in this way. The squeals of band brakes in washing machines have been discussed in [46]. The brake squeals of an automotive disk brake has been examined in [27,47], cf. Fig. 6. The squeal of a system glass–rubber in finite deformation is considered in [34].

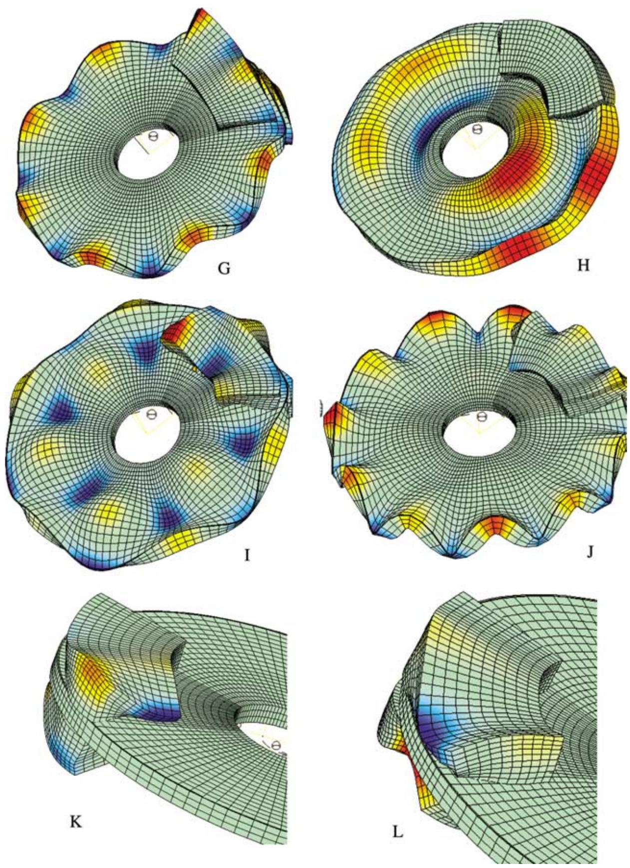

Fig. 6. Some unstable modes of the system pad-disk in an automotive disk brake [27,25]

References

[1] G. de Saxcé, A generalization of Fenchel's inequality and its application to constitutive laws, C. R. Acad. Sci. Paris II 314 (1992) 125–129.
[2] J.J. Moreau, On unilateral constraints, friction and plasticity, in: New Variational Techniques in Mathematical Physics, in: CIME Course, Springer-Verlag, 1974.
[3] Q.S. Nguyen, Stability and Nonlinear Solid Mechanics, Wiley, Chichester, 2000.
[4] G. Duvaut, J.-L. Lions, Les inéquations en mécanique et en physique, Dunod, Paris, 1972.
[5] J.T. Oden, J.A.C. Martins, Models and computational methods for dynamic friction phenomena, Comput. Methods Appl. Mech. Engrg. 52 (1985) 527–634.
[6] N. Kikuchi, J.T. Oden, Contact Problem in Elasticity: A Study of Variational Inequalities and Finite Element Methods, SIAM, Philadelphia, 1988.
[7] L.E. Andersson, A quasistatic frictional problem with normal compliance, Nonlinear Anal. 16 (1991) 347–370.
[8] L.E. Andersson, A. Klarbring, Existence and uniqueness for quasistatic contact problems with friction, in: J.A.C. Martins, M.D.P. Montero Marques (Eds.), Contact Mechanics, Kluwer Academic, London, 2001, pp. 245–260.
[9] A. Klarbring, A. Mikelic, M. Shillor, A global existence result for the quasistatic frictional contact problem with normal compliance, in: Del Piero, Maceri (Eds.), Unilateral Problems in Structural Mechanics IV, Birkhäuser, Boston, 1991, pp. 85–111.
[10] M. Cocu, E. Pratt, M. Raous, Analysis of an incremental formulation for frictional contact problems, in: Contact Mechanics Marseille, 1995, Plenum Press, New York, 1996.
[11] R. Hill, A general theory of uniqueness and stability in elastic/plastic solids, J. Mech. Phys. Solids 6 (1958) 236–249.
[12] H. Petryk, A consistent approach to defining stability of plastic deformed processes, in: IUTAM Symp. Stability in the Mechanics of Continua, Springer-Verlag, Berlin, 1982, pp. 262–272.
[13] A. Klarbring, Derivation and analysis of rate boundary value problems of frictional contact, Eur. J. Mech. A Solids 9 (1990) 53–85.
[14] A. Klarbring, Stability and critical points in large displacement frictionless contact problems, in: J.A.C. Martins, M. Raous (Eds.), Friction and Instabilities, Udine 2000, in: CISM Courses and Lectures, Vol. 457, Springer, Wien, 2002, pp. 39–64.
[15] X. Chateau, Q.S. Nguyen, Buckling of elastic structures in unilateral contact with or without friction, Eur. J. Mech. A Solids 10 (1991) 71–89.
[16] M.A. Biot, Mechanics of Incremental Deformation, Wiley, New York, 1965.
[17] Q.S. Nguyen, Bifurcation and stability in dissipative media (plasticity, friction, fracture), Appl. Mech. Rev. 47 (1994) 1–31.
[18] Z. Mroz, Contact friction models and stability problems, in: J.A.C. Martins, M. Raous (Eds.), Friction and Instabilities, Udine 2000, in: CISM Courses and Lectures, Vol. 457, Springer, Wien, 2002, pp. 179–232.
[19] Q.S. Nguyen, Bifurcation et stabilité des systèmes irréversibles obéissant au principe de dissipation maximale, J. Méc. Théor. Appl. 3 (1984) 41–61.
[20] J.A.C. Martins, A. Pinto da Costa, F.M.F. Simoes, Some notes on friction and instabilities, in: J.A.C. Martins, M. Raous (Eds.), Friction and Instabilities, Udine 2000, in: CISM Courses and Lectures, Vol. 457, Springer, Wien, 2002, pp. 65–136.
[21] R. Cottle, J.S. Pang, R.E. Stone, The Linear Complementarity Problem, Academic Press, New York, 1992.
[22] A. Klarbring, A mathematical programming approach to three-dimensional contact problems with friction, Comput. Methods Appl. Mech. Engrg. 58 (1986) 175–200.
[23] G.G. Adams, Self-excited oscillations of two elastic half-spaces sliding with a constant coefficient of friction, J. Appl. Mech. 62 (1995) 867–872.
[24] J.A.C. Martins, J. Guimaraes, L.O. Faria, Dynamic surface solutions in linear elasticity and viscoelasticity with frictional boundary conditions, J. Vib. Acoustics 117 (1995) 445–451.
[25] F. Moirot, Q.S. Nguyen, Some problems of friction-induced vibrations and instabilities, in: J.A.C. Martins, M. Raous (Eds.), Friction and Instabilities, Udine 2000, in: CISM Courses and Lectures, Vol. 457, Springer, Wien, 2002, pp. 137–178.
[26] G.G. Adams, Dynamic instabilities in the sliding of two layered elastic half-spaces, J. Tribology 120 (1998) 289–295.
[27] F. Moirot, Etude de la stabilité d'un équilibre en présence du frottement de Coulomb. Application au crissement des freins à disque, Thèse, École Polytechnique, Paris, 1998.
[28] M. Renardy, Ill-posedness at the boundary for elastic solids sliding under Coulomb friction, J. Elasticity 27 (1992) 281–287.
[29] K. Ranjith, J.R. Rice, Slip dynamics at an interface between dissimilar materials, J. Mech. Phys. Solids 49 (2001) 341–361.
[30] H. Troger, A. Steindl, Nonlinear Stability and Bifurcation Theory, Springer-Verlag, Wien, 1991.
[31] F. Moirot, Q.S. Nguyen, An example of stick–slip waves, C. R. Acad. Sci. Paris IIb 328 (2000) 663–669.
[32] A. Oueslati, L. Baillet, Q.S. Nguyen, Transition vers une onde glissement-adhérence-décollement sous contact frottant de Coulomb, in: Journées Européennes sur les freins, Hermès, Paris, 2002, pp. 155–162.
[33] M. Oestreich, N. Hinrichs, K. Popp, Bifurcation and stability analysis for a non-smooth friction oscillator, Arch. Appl. Mech. 66 (1996) 301–314.
[34] D. Vola, M. Raous, J.A.C. Martins, Friction and instability of steady sliding squeal of a glass/rubber contact, Int. J. Numer. Methods Engrg. 45 (1999) 301–314.
[35] N.J. Carpenter, R.L. Taylor, M.G. Katona, Lagrange constraints for transient finite element surface contact, Int. J. Numer. Methods Engrg. 32 (1991) 103–128.

[36] L. Baillet, Y. Berthier, O. Bontemps, M. Brunet, Tribologie de l'interface fibre/matrice. Approche théorique et expérimentale, Rev. Composites et Matériaux Avancés 7 (1997) 89–105.
[37] B. van der Pol, On relaxation-oscillations, Philos. Mag. 7 (1926) 978–992.
[38] K. Popp, P. Stelter, Stick–slip vibrations and chaos, Philos. Trans. Roy. Soc. London Ser. A 332 (1990) 89–105.
[39] G.G. Adams, Steady sliding of two elastic half-spaces with friction reduction due to interface stick–slip, J. Appl. Mech. 65 (1998) 470–475.
[40] O.Y. Zharii, Frictional contact between the surface wave and a rigid strip, J. Appl. Mech. 63 (1996) 15–20.
[41] J.A.C. Martins, S. Barbarin, M. Raous, A. Pinto da Costa, Dynamic stability of finite dimensional linear elastic system with unilateral contact and Coulomb's friction, Comput. Methods Appl. Mech. Engrg. 177 (1999) 298–328.
[42] D. Vola, E. Pratt, M. Jean, M. Raous, Consistent time discretization for a dynamical frictional contact problem and complementary techniques, Rev. Eur. Elements Finis 7 (1998) 149–162.
[43] Y. Renard, Modélisation des instabilités liées au frottement sec des solides, aspects théoriques et numériques, Thèse de doctorat, Université de Grenoble, 1998.
[44] M. Raous, S. Barbarin, D. Vola, Numerical characterization and computation of dynamic instabilities for frictional contact problems, in: J.A.C. Martins, M. Raous (Eds.), Friction and Instabilities, Udine 2000, in: CISM Courses and Lectures, Vol. 457, Springer, Wien, 2002, pp. 233–291.
[45] D.A. Crolla, A.M. Lang, Brake noise and vibration: state of art, Vehicle Tribologie 18 (1991) 165–174.
[46] M. Nakai, M. Yokoi, Band brake squeal, J. Vib. Acoustics 118 (1996) 187–197.
[47] F. Moirot, Q.S. Nguyen, Brake squeal: a problem of flutter instability of the steady sliding solution? Arch. Mech. 52 (2000) 645–662.