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C. R. Mecanique 331 (2003) 183–188
COMPTES RENDUS MECANIQUE

Uniqueness in the problem of an obstacle in oblique waves

Unicité du problème d'un obstacle dans des ondes obliques

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Received and accepted 25 January 2003

Presented by Évariste Sanchez-Palencia

Abstract

A solution to the linearized water-wave problem involving a pair of surface-piercing cylinders in oblique waves and infinite water depth is proved to be unique for certain geometric arrangements and frequencies in some interval above the cut-off frequency. ***To cite this article: N. Kuznetsov, C. R. Mecanique 331 (2003).***

Résumé

Nous considérons le problème du mouvement sur la houle. À partir des conditions géométriques pour deux cylindres flottant dans une mer de profondeur infinie et dans des ondes obliques, nous obtenons un intervalle de fréquences d'unicité. ***Pour citer cet article : N. Kuznetsov, C. R. Mecanique 331 (2003).***

Keywords: Fluid mechanics; Oblique waves; Pair of floating cylinders; Nodal line; Conformal mapping; Uniqueness theorem

Mots-clés : Mécanique des fluides ; Ondes obliques ; Deux cylindres flottants ; Courbe nodale ; Transformation conforme ; Théorème d'unicité

1. Introduction: statement of the problem

In 1950, John [1] and Ursell [2] proved the first two results on uniqueness in the water-wave problem (WWP for short, in what follows). Not so long before that, Rellich [3] and Kupradze [4] (his book summarizes results obtained earlier and its Russian original was published in 1950 and English translation appeared in 1952), proved uniqueness theorems for boundary value problems describing acoustic fields outside a bounded obstacle. However, there is a great difference between the results in [1] and [2], on the one side, and those in [3] and [4], on the other side. In the acoustic case, there are no restrictions on the shape of an obstacle, whereas [2] deals only with totally

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1631-0721/03/$ – see front matter
doi:10.1016/S1631-0721(03)00042-1

submerged circular cylinders, and in [1], uniqueness is proved for surface-piercing bodies subjected to the condition now usually referred to as John's condition. It says that any vertical line through the horizontal free surface of water has no common points with the contour of the cylinder's cross-section. The subsequent developement of the linear theory of water waves (see the book [5] for an updated account of this theory) demonstrates that the difference between water and acoustic waves is an essential one. The most clear evidence of this was obtained in 1996, when McIver [6] constructed the first nontrivial solution to the homogeneous two-dimensional WWP. Various extensions of this work can be found in [5]; in particular, Section 5.4 contains an example of a nontrivial solution for the problem of oblique waves (this result was originally published in [7]). Implications which arise for hydrodynamic characteristics of floating structures at the frequencies of non-uniqueness were considered by Newman [8].

McIver's result has increased interest in determining conditions under which a solution is unique. Even before that, the importance of this question was emphasized by Ursell, who placed it first in his list [9] of unsolved and unfinished problems in the theory of water waves. In the passed few years, some progress was achieved concerning uniqueness in WWP in two and three dimensions, but still much less is known in the case of oblique waves (the known results are reviewed in [5] and [7]). The aim of this Note is to fill in this gap, at least partially. We will be mainly concerned with the most difficult case of two surface-piercing cylinders in water of infinite depth, but results for other geometries of obstacles will be also mentioned. Our method combines investigation of nodal lines similar to that in [10] with application of conformal mapping proposed by the author in 1988 (see [5], Subsection 4.2.4).

Let W denote the cross-section of a domain occupied by an inviscid, incompressible, heavy fluid (water). It is assumed that $W = \mathbb{R}^2_- \setminus (\overline{B_+} \cup \overline{B_-})$, where $\mathbb{R}^2_- = \{-\infty < x < +\infty,\ y < 0\}$ and B_+, B_- are simply connected domains in $\mathbb{R}^2_-$ attached to $\partial\mathbb{R}^2_-$ and such that: (a) $\overline{B_+} \cap \overline{B_-} = \emptyset$; (b) for $S_\pm = \partial B_\pm \cap \mathbb{R}^2_-$ the closure $\overline{S_\pm}$ is a C^2-curve (this condition may be weakened); (c) $\overline{S_\pm}$ is not tangent to the x-axis at its endpoints. Thus $\overline{B_+}$ ($\overline{B_-}$) is the cross-section of a right (left) infinitely long cylinder floating in the water surface. Let F_0 denote the part of the x-axis between $\overline{B_+}$ and $\overline{B_-}$, and $F_\infty = \partial\mathbb{R}^2_- \setminus (F_0 \cup \overline{B_+} \cup \overline{B_-})$. Neglecting the surface tension and assuming the water motion to be irrotational, we consider small-amplitude oscillations having the radian frequency ω. Mathematically one has to find a complex-valued velocity potential ϕ satisfying the following boundary value problem (it will be referred to as the oblique-wave problem, OWP for short, in what follows):

$$\nabla^2\phi = m^2\phi \quad \text{in } W, \qquad \phi_y = \nu\phi \quad \text{on } F_0 \cup F_\infty, \qquad \frac{\partial\phi}{\partial n} = f \quad \text{on } S_+ \cup S_- \tag{1}$$

$$\phi_{|x|} - \mathrm{i}\ell\phi = \mathrm{o}(1) \quad \text{uniformly in } y \in (-\infty, 0) \text{ as } |x| \to \infty \tag{2}$$

This problem must be complemented by the condition that the Dirichlet integral of ϕ is locally finite. In (1), m and f are a given nonnegative number and a given function on $S_+ \cup S_-$, respectively, and they depend on the type of problem (radiation or scattering); ν is a spectral parameter equal to ω^2/g, where g is the acceleration due to gravity. In the radiation condition (2), $\ell = (\nu^2 - m^2)^{1/2}$ and we assume that $\ell \in (0, \nu]$; that is $\nu > m$, and so ℓ/ν and m/ν are the sine and cosine, respectively, of the angle that the crests of the incident wave-train make with the plane normal to the generators of the cylinders. Hence for $m = 0$ OWP coincides with the two-dimensional WWP. The case $\nu < m$ is more simple and was studied in [7] (see also Section 4 below).

Since we are interested in the question of uniqueness, we put $f = 0$ in which case (cf. [5], Section 2.2):

$$\phi(x, y) = \mathrm{O}\big(\big[x^2 + y^2\big]^{-1/2}\big) \quad \text{and} \quad |\nabla\phi| = \mathrm{O}\big(\big[x^2 + y^2\big]^{-1}\big) \quad \text{as } x^2 + y^2 \to \infty \tag{3}$$

Therefore, the following conditions hold:

$$\int_W \big(|\nabla\phi|^2 + m^2\phi^2\big)\,\mathrm{d}x\,\mathrm{d}y < \infty, \qquad \int_{F_0 \cup F_\infty} \phi^2\,\mathrm{d}x < \infty \tag{4}$$

which mean that the kinetic and potential energy of waves per unit length of cylinder's generators is finite. For the homogeneous problem, conditions (4) can be imposed instead of (2), and so ϕ can be assumed to be real because if it were complex, then both the real and imaginary parts would separately satisfy the problem.

2. Existence of nodal lines

Let $(d,0) \in \overline{F_\infty^{(+)}}$, where $F_\infty^{(+)}$ is the part of F_∞ on the right of $\overline{S_+}$. The function $\psi = \mathrm{e}^{\nu y} \sin \ell(x-d)$ satisfies the same boundary condition on $F_\infty^{(+)}$ as ϕ and the same modified Helmholtz equation holds for ψ in $\mathbb{R}^2_-$. Let W satisfy John's condition, then taking into account (3) and the behaviour of ψ at infinity, one can apply the second Green's identity to ϕ and ψ in the domain $W \cap \{x > d\}$, thus obtaining

$$\int_{-\infty}^{0} \phi(d,y)\,\mathrm{e}^{\nu y}\,\mathrm{d}y = 0 \quad \text{for every } (d,0) \in \overline{F_\infty^{(+)}}.$$

Since $\mathrm{e}^{\nu y}$ is strictly positive and $\phi(d,y)$ is a continuous function of y, we get that $\phi(d,y_0) = 0$ for some $y_0 \in (-\infty, 0)$, and a value of y_0 may be found for every $(d,0) \in \overline{F_\infty^{(+)}}$. Then there exists at least one nodal line on which $\phi = 0$ in the quadrant that lies strictly below $F_\infty^{(+)}$. Similar considerations show that there exists at least one nodal line in the quadrant that lies strictly below $F_\infty^{(-)}$, which is the part of F_∞ on the left of $\overline{S_-}$. It is known (see, for example, [5], Section 4.1) that the nodal line cannot terminate in W. Finally, it follows from [11] that $\phi(x,0) \neq 0$ for sufficiently large values of $|x|$, and so there exists a nodal line going to infinity. Hence we arrive at

Proposition 1. *Let the contours S_+ and S_- satisfy John's condition and let ϕ be a solution to the homogeneous* OWP. *Then at least one of the following two configurations is present in the set of nodal lines of ϕ*:

(i) *a nodal line whose both ends go to infinity, thus separating a subdomain $W_\infty \subset W$ from $F_0 \cup F_\infty$;*
(ii) *a pair of non-intersecting nodal lines emanating from $\overline{F_0} \cup S_+ \cup S_-$ and such that one goes to infinity in the positive x-direction, whereas the other one goes to infinity in the negative x-direction.*

If there exists a subdomain W_∞ separated from $F_0 \cup F_\infty$ by a nodal line or by nodal lines and a part of $S_+ \cup S_-$, then properties (3) allow us to write the first Green's identity for ϕ in W_∞. Since ∂W_∞ consists of curves on which either $\phi = 0$ or $\partial\phi/\partial n = 0$, this identity immediately proves the following

Corollary 1. *Let ϕ be a solution to the homogeneous* OWP *in W. If ϕ has a nodal line such that it separates a subdomain $W_\infty \subset W$ from $F_0 \cup F_\infty$, then ϕ vanishes identically. The same is true if nodal lines of ϕ separate W_∞ from $F_0 \cup F_\infty$ together with a part of $S_+ \cup S_-$.*

It is observed in [10] on the basis of numerical computations that for both symmetric and antisymmetric nontrivial solutions of the homogeneous WWP, which are constructed with the help of procedure proposed in [6], there exists a pair nodal lines emanating from S_+ and S_- and going to the positive and negative x-infinity, respectively. Therefore, one can hardly expect that Corollary 1 is applicable in the general case. However, if there is a single floating body that satisfies John's condition, then Corollary 1 gives one more proof of the John's uniqueness theorem for OWP (one more proof based on Maz'ya's identity can be found in [5], Subsection 3.2.3).

3. An upper bound of uniqueness interval

In this section we deal with configurations of nodal lines which are permitted by Proposition 1, but such that Corollary 1 cannot be applied to these configurations. They consist of pairs of non-intersecting nodal lines emanating from $\overline{F_0} \cup S_+ \cup S_-$, but not from the same body's contour. For studying this case we impose a restriction on S_+ and S_- that will be referred to as the circle condition. Namely, let $S_\pm$ be contained within a circle having its centre on the x-axis and going through the point $\overline{F_0} \cap \overline{S_\pm}$. It is clear that there exists the smallest radius $r_\pm$ for which this property holds. Let us choose the origin of the (x,y)-coordinates so that the smallest circles containing

S_+ and S_- are bounded by coordinate lines of the bipolar coordinates with poles at $(+a, 0)$ and $(-a, 0)$. For this purpose one has to find $a, b_\pm, d_\pm > 0$ from the following system

$$a = r_+ \sinh d_+ = r_- \sinh d_-, \quad b_+ + b_- = h, \quad b_+ + r_+ = r_+ \cosh d_+, \quad b_- + r_- = r_- \cosh d_-$$

where h is the length of F_0. One can easily verify that this system has a unique solution such that

$$a^2 = (b_+ + r_+)^2 - r_+^2 = (b_- + r_-)^2 - r_-^2 \tag{5}$$

Then placing the origin of the (x, y)-coordinates on F_0 so that $(b_\pm, 0) = \overline{F_0} \cap \overline{S_\pm}$, one finds that the smallest circles containing S_+ and S_- are bounded by the coordinate lines $u = d_+$ and $u = -d_-$, respectively, of the bipolar coordinates (u, v) related to the above chosen Cartesian coordinates as follows (see, for example, [12], Section 2.01):

$$x = \frac{a \sinh u}{\cosh u - \cos v}, \qquad y = \frac{a \sin v}{\cosh u - \cos v} \tag{6}$$

The metric coefficients of this conformal mapping are $g_{11} = g_{22} = a^2/(\cosh u - \cos v)^2$.

Let us describe for some subsets of $\overline{\mathbb{R}^2_-}$ their images under the conformal mapping (6). It is clear that the image of the domain outside the circles containing S_+ and S_- is the rectangle $\mathcal{R} = \{-d_- < u < d_+,\ -\pi < v < 0\}$. Moreover, $\mathcal{F}_0 = \{-d_- < u < d_+,\ v = -\pi\}$ is the image of F_0 and $(u, v) = (0, 0)$ is the image of infinity on the (x, y)-plane. Let W_0 be an infinite subdomain of W bounded from above by one of the following curves permitted by Proposition 1:

(I) a part of F_0 and a pair of nodal lines emanating from F_0 and going to infinity;
(II) $\overline{F_0}$, parts of S_+ and S_-, and a pair of non-intersecting nodal lines such that one emanates from S_+ and the other one from S_- and both go to infinity;
(III) a part of F_0, a part of S_+ (S_-), and a pair of non-intersecting nodal lines such that one of them emanates from F_0 and the other one emanates from S_+ (S_-) and both go to infinity.

Then the image $\partial\mathcal{W}_0$ that corresponds to ∂W_0 is a curvilinear triangle in the case (I). This triangle has one side on $\mathcal{F}_0$, the opposite vertex at $(0, 0)$, and nodal lines are the lateral sides. A curvilinar pentagon is the image of $\partial\mathcal{W}_0$ in the case (II). This pentagon has $\overline{\mathcal{F}_0}$ as the side opposite to the vertex at $(0, 0)$ and two pairs of the lateral sides each formed by the images of a nodal line and of a part of $S_\pm$. In the case (III), one of two pairs of the lateral curves in the case (II) must be replaced by a single nodal line thus producing a quadrangle. It is important that the images of S_+ and S_- lie outside of $\mathcal{R}$ and this results from the circle condition.

Let $\varphi(u, v) = \phi(x(u, v), y(u, v))$, where $x(u, v)$ and $y(u, v)$ are given by (6) and $(u, v) \in \mathcal{W}_0$ because only this domain will be used below. It follows from (1) that

$$\nabla^2 \varphi = \frac{(ma)^2 \varphi}{(\cosh u - \cos v)^2} \quad \text{in } \mathcal{W}_0, \qquad \varphi_v + \frac{\nu a \varphi}{1 + \cosh u} = 0 \quad \text{on } \mathcal{F}_0' \tag{7}$$

where $\mathcal{F}_0'$ is the part of $\mathcal{F}_0$ belonging to $\partial\mathcal{W}_0$. On the rest of $\partial\mathcal{W}_0$ either the homogeneous Dirichlet or the homogeneous Dirichlet and Neumann conditions are fulfilled. Moreover, conditions (4) imply that

$$\int_{\mathcal{W}_0} \left[|\nabla\varphi|^2 + \frac{(ma)^2 \varphi^2}{(\cosh u - \cos v)^2} \right] \mathrm{d}u\, \mathrm{d}v < \infty, \qquad \int_{\mathcal{F}_0} \frac{\varphi^2}{1 + \cosh u}\, \mathrm{d}u < \infty$$

and so the first Green's identity can be applied in $\mathcal{W}_0$. In view of (7) and the homogeneous boundary conditions on the rest of $\partial\mathcal{W}_0$ we get:

$$\int_{\mathcal{W}_0} \left[|\nabla\varphi|^2 + \frac{(ma)^2 \varphi^2}{(\cosh u - \cos v)^2} \right] \mathrm{d}u\, \mathrm{d}v = \nu a \int_{\mathcal{F}_0'} \frac{\varphi^2}{1 + \cosh u}\, \mathrm{d}u \tag{8}$$

Now the last integral can be estimated as follows. Since the images of S_+ and S_- lie outside of $\mathcal{R}$, for any $(u, -\pi) \in \mathcal{F}_0'$ we have the obvious equality:

$$-\varphi(u, -\pi) = \int_{-\pi}^{v_0(u)} \varphi_v(u, v)\,\mathrm{d}v \tag{9}$$

where $-v_0(u)$ is the largest value of $-v$ such that the point $(u, v_0(u))$ lies on the nodal line of φ. Both sides of (9) are squared and the Schwarz inequality gives

$$\varphi^2(u, -\pi) \leqslant \pi \int_{-\pi}^{v_0(u)} \left|\varphi_v(u, v)\right|^2 \mathrm{d}v \tag{10}$$

because $|\pi + v_0(u)| \leqslant \pi$. Multiplying both sides of (10) by $\nu a/(1 + \cosh u)$ and integrating over $\mathcal{F}_0'$, we obtain:

$$\nu a \int_{\mathcal{F}_0'} \frac{\varphi^2}{1 + \cosh u}\,\mathrm{d}u \leqslant \nu a \pi \int_{\mathcal{F}_0'} \frac{\mathrm{d}u}{1 + \cosh u} \int_{-\pi}^{v_0(u)} \left|\varphi_v(u, v)\right|^2 \mathrm{d}v \leqslant \nu a \frac{\pi}{2} \int_{\mathcal{W}_0} \varphi_v^2\,\mathrm{d}u\,\mathrm{d}v$$

Combining this and equality (8), one arrives at

$$\int_{\mathcal{W}_0} \varphi_u^2\,\mathrm{d}u\,\mathrm{d}v + \left(1 - \nu a \frac{\pi}{2}\right) \int_{\mathcal{W}_0} \varphi_v^2\,\mathrm{d}u\,\mathrm{d}v + (ma)^2 \int_{\mathcal{W}_0} \frac{\varphi^2\,\mathrm{d}u\,\mathrm{d}v}{(\cosh u - \cos v)^2} \leqslant 0$$

From Proposition 1, Corollary 1, and this inequality, it is easy to derive

Proposition 2. *Let John's condition and the circle condition hold for S_+ and S_-. If $m \geqslant 0$ and $ma < \nu a \leqslant 2/\pi$, where a is defined by* (5)*, then the homogeneous* OWP *has only a trivial solution.*

4. Discussion

Uniqueness has been established in OWP at frequencies such that $\nu a \in (ma, 2/\pi]$ when $ma < 2/\pi$, water has infinite depth, and the cross-sections of two surface-piercing cylinders satisfy John's condition and the circle condition. This extends previous results in two directions. First, uniqueness is proved for OWP instead of the two-dimensional WWP. Second, the upper bound in Proposition 2 is better than that in [5]. This can be demonstrated for the simple geometry when both cylinders have the semicircular cross-section of the same radius r and the distance between them is equal to $2b$. In this case $a = (b^2 + 2br)^{1/2}$ and the inequality providing uniqueness in [5] is as follows: $\nu a \leqslant (r - b)/(2\pi r)$, thus being valid only if r is larger than b. In Proposition 2, upper bound of the uniqueness interval is obtaind without any restriction on r and b and this bound is more than four times better than in the latter inequality.

The method described in Section 3 allows us to replace (6) by a more general conformal mapping of $\mathbb{R}^2_-$ onto a strip $\{-\infty < u < +\infty,\ -s < v < 0\}$ for formulating geometric restrictions on $S_\pm$ and for obtaining an upper bound of uniqueness interval. Instead of doing this we mention two more examples of conformal mappings from [12], Section 2.01, which can be used in he present context. They have the egg-shaped inverse Cassinian ovals and the so-called sn curves as their level lines (see Figs. 2.08 and 2.16, respectively, in [12], where coordinate lines of the corresponding curvilinear coordinates are shown). The latter curves have their foci at $x = \pm a$ and $x = \pm a/k$, $0 < k < 1$, and are the u level lines of the conformal mapping:

$$x = a\Lambda^{-1}\mathrm{sn}u\,\mathrm{dn}v, \quad y = a\Lambda^{-1}\mathrm{cn}u\,\mathrm{dn}u\,\mathrm{sn}v\,\mathrm{cn}v, \quad \text{where } \Lambda = 1 - \mathrm{dn}^2u\,\mathrm{sn}^2v$$

and sn, cn, and dn is the copolar trio of Jacobian elliptic functions (see, for example, Gradshteyn and Ryzhik [13], Chapter 8); the metric coefficients of this conformal mapping are

$$g_{11} = g_{22} = a^2\Omega^2/\Lambda^2, \quad \text{where } \Omega^2 = \left(1 - \mathrm{sn}^2 u\, \mathrm{dn}^2 v\right)\left(\mathrm{dn}^2 v - k^2 \mathrm{sn}^2 u\right)$$

Using John's condition and the analogue of the circle condidion for sn curves, one arrives at the inequalities $ma < \nu a \leqslant 1/K'(k)$ for frequencies at which uniqueness holds. Here K' is complete elliptic integral of the first kind (see [13], Chapter 8). Note that $1/K'(k) \to 2/\pi$ as $k \to 1$ (cf. Proposition 2), but $1/K'(k) \to 0$ as $k \to 0$.

It should be noted that the result obtained by McIver [10] for the two-dimensional WWP is valid for OWP as well. She proved uniqueness when water is of finite but non-uniform depth and $\nu H_{\max} \leqslant 1$, where $H_{\max}$ is the maximum depth of the water layer. For OWP the last inequality must be complemented by the estimate $\nu > m \tanh mH$, where H is the constant depth of water at infinity and $m \tanh mH$ is the lower bound of the continuous spectrum of OWP.

In conclusion we mention two results which complement Proposition 2. Firstly, until now the only geometry, for which uniqueness in OWP is proved for all ν belonging to the continuous spectrum, is a pair of vertical barriers as was proved in [14] (see also [5], Subsection 4.2.1). Secondly, if $\nu < m$, then John's condition provides uniqueness in problem (1), (4) for any finite number of surface-piercing cylinders as is shown in [7] (see also [5], Subsection 5.4.1).

Acknowledgements

The author dedicates this work to Fritz Ursell on the occasion of his 80th birthday thus acknowledging the influence of Ursell's works and personality on the author himself and on the development of the water-wave theory during the past 50 years. The work was carried out during author's visits to Linköping University (Sweden). The financial support from the Wenner–Gren Foundations is gratefully acknowledged.

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