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High-order evolution equation for nonlinear wave-packet propagation with surface tension accounting

Équation d'évolution d'ordre élevé de groupes d'ondes non linéaires en présence de tension superficielle

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Abstract

The nonlinear problem for propagation of wave-packets along the interface of two semi-infinite fluids is solved on the basis of multiple scale asymptotic expansions. Unlike all previous investigations dealing only with third-order approximations, here fourth-order approximation is developed. The corresponding solvability condition is obtained and the evolution equation in the case away from the cut-off wave number is derived. As a result, the nonlinear higher-order Schrödinger equation is obtained which contains the nonlinear part in a compact form. This equation is valid for a wide range of wave numbers. The stability diagram shows regions of stability and instability of capillary-gravity wave-packets. ***To cite this article: I. Selezov et al., C. R. Mecanique 331 (2003).***

Résumé

Le problème non linéaire de la propagation de groupes d'ondes à l'interface de deux liquides semi-infinis est résolu en utilisant une méthode multi-échelles. Contrairement aux études antérieures développées qu'au troisième ordre, cet article consière une approximation au quatrième ordre. La condition de solvabilité correspondante est obtenue et l'équation d'évolution est formulée loin du nombre d'onde de coupure. Comme résultat on obtient une équation non linéaire de Schrödinger d'ordre élevé, dont la partie non linéaire est mise sous une forme compacte. Cette équation est utilisable pour un large intervalle de nombres d'onde. Le diagramme de stabilité met en évidence des domaines stables et instables de paquets d'ondes de gravité-capillarité. ***Pour citer cet article : I. Selezov et al., C. R. Mecanique 331 (2003).***

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1. Introduction

Recent investigations show that the occurence of wave train instabilities can be produced, among others, by even very small disturbances generated by capillary-gravity waves due to surface tension effect. Publication of more and more studies devoted to the water wave propagation taking into account the surface tension, is not accidental (see, for example, [1–3]).

At the same time, one can observe that there is a difference between the free surface which is precisely traced and the thermocline which is a rather spreading thin layer and surface tension can be considered not so strong. However, in laboratory experiments and industrial applications the interface can be clearly traced.

It should be noted that the stability of the system essentially depends on the ratio of densities $\rho = \rho_2/\rho_1$ (ρ_2 corresponds to upper fluid, ρ_1 to lower), as well as on the surface tension T. Hence, $\rho = 0$ corresponds to surface gravity waves while $\rho \neq 0$ corresponds to internal (interfacial) waves. We recover the Dysthe equation in the case of $\rho \to 0$ [4] and the Hogan equation [5] when $T \neq 0$. In some cases surface tension effect can be not necessarily important in oceanic situations but it can be important for experiments conducted in laboratories when lengthscales are shorter. The role of internal breaking waves in mixing processes in the upper layer of the ocean is also of great importance.

In this paper asymptotic expansions of fourth order are developed unlike most previous investigations restricted to third order expansions. As a result, an extended nonlinear Schrödinger equation is derived and the stability analysis of solutions is carried out.

2. Statement, asymptotic solutions and evolution equation for complex envelope

The mathematical statement of the problem includes the Laplace equations, kinematic and dynamic boundary conditions at the interface and regularity conditions

$$\nabla^2\varphi_j = 0 \qquad \text{in } \Omega_j \tag{1}$$

$$\eta_{,t} - \varphi_{j,z} = -\varphi_{j,x}\eta_{,x} \quad \text{at } z = \eta(x,t) \tag{2}$$

$$\varphi_{1,t} - \rho\varphi_{2,t} + (1-\rho)\eta + \frac{1}{2}(\nabla\varphi_1)^2 - \frac{1}{2}\rho(\nabla\varphi_2)^2 - T\left(1+\eta_{,x}^2\right)^{-3/2}\eta_{,xx} = 0 \quad \text{at } z = \eta(x,t) \tag{3}$$

$$|\nabla\varphi_j| \to 0 \quad \text{as } z \to \mp\infty \tag{4}$$

where φ_j $(j = 1,2)$ are the velocity potentials; η is the interface elevation; $\Omega_1 = \{(x,y,z)\colon -\infty < x < \infty, -\infty < y < \infty, z < 0\}$ and $\Omega_2 = \{(x,y,z)\colon -\infty < x < \infty,\ -\infty < y < \infty,\ z > 0\}$, $\rho = \rho_2/\rho_1$. Dimensionless values are introduced using the characteristic length L, characteristic time $(L/g)^{1/2}$ (g is the acceleration of the gravity) and density of the lower fluid is ρ_1. The characteristic length L can be the wavelength, for example. The dimensionless surface tension in this case is $T^* = T/(L^2\rho_1 g)$, the asterisk is then dropped.

The approximate solutions of the nonlinear problem (1)–(4) are developed by using the method of multiple scale expansions, so that desired functions are presented by the following expansions

$$\eta(x,t) = \sum_{n=1}^{4}\varepsilon^n\eta_n(x_0,x_1,x_2,x_3,t_0,t_1,t_2,t_3) + \mathrm{O}\left(\varepsilon^5\right) \tag{5}$$

$$\varphi_j(x,z,t) = \sum_{n=1}^{4}\varepsilon^n\varphi_{jn}(x_0,x_1,x_2,x_3,z,t_0,t_1,t_2,t_3) + \mathrm{O}\left(\varepsilon^5\right) \quad (j = 1,2) \tag{6}$$

where ε is a small dimensionless parameter characterizing the wave steepness, $x_n = \varepsilon^n x$, $t_n = \varepsilon^n t$.

Substituting (5) and (6) into (1)–(4) and equating coefficients of like powers of ε reduce the original nonlinear problem to four linear problems. The first-, second- and third-order problems were formulated, and the solutions of the first- and second-order problems and solvability conditions of the second- and third-order problems were derived in [6] for other dimensionless parameters, so that the dimensionless surface tension was $T = 1$.

As a result, we obtain the dispersion relationship

$$\omega^2 - (1+\rho)^{-1}\big(1-\rho+Tk^2\big)k = 0 \tag{7}$$

and the solvability conditions for the first-, second- and third-order problems

$$A_{,t_1} + \omega' A_{,x_1} = 0 \tag{8}$$

$$A_{,t_2} + \omega' A_{,x_2} - \frac{1}{2}\mathrm{i}\omega'' A_{,x_1x_1} = -\mathrm{i}k\omega^{-1}(1+\rho)^{-1} I A^2 \bar{A} \tag{9}$$

$$A_{,t_3} + \omega' A_{,x_3} - \mathrm{i}\omega'' A_{,x_1x_2} - \frac{\omega'''}{6}\cdot A_{,x_1x_1x_1} = k\omega^{-1}(1+\rho)^{-1}\big[JA\bar{A}A_{,x_1} - I\big(k\omega^{-1}\big)'\big(A^2\bar{A}\big)_{,x_1}\big], \tag{10}$$

where

$$I = k^2\frac{(1-6\rho+\rho^2)k^4T^2 + 0.5(1-31\rho+31\rho^2-\rho^3)k^2T + 4(1-2\rho+2\rho^2-2\rho^3+\rho^4)}{2(\rho+1)^2(1-\rho-2Tk^2)} \tag{11}$$

$$\begin{aligned} J = \mathrm{i}k\big[&4\big(\rho^2-6\rho+1\big)k^6T^3 + 2\big(\rho^3+5\rho^2-5\rho-1\big)k^4T^2 + \big(-\rho^4+32\rho^3-62\rho^2+32\rho-1\big)k^2T \\ &+ 4\big(\rho^5-3\rho^4+4\rho^3-4\rho^2+3\rho-1\big)\big]\big/\big[\big(1-\rho-2Tk^2\big)^2(1+\rho)^2\big] \end{aligned} \tag{12}$$

I and J are connected by a simple relationship

$$J = -\mathrm{i}\frac{\partial I}{\partial k} \tag{13}$$

where $I(T,k,\rho)$ and $J(T,k,\rho)$ are given by the forms (11) and (12), respectively.

Multiplying Eqs. (8)–(10) by ε, ε^2 and ε^3, respectively, adding all equations and taking into account formulae for derivatives $A_{,t}$, $A_{,x}$, $A_{,xx}$ and $A_{,xxx}$ and relationship (13), the following evolution equation can be derived

$$\begin{aligned} &A_{,t} + \omega' A_{,x} - \frac{\mathrm{i}\omega''}{2!}\cdot A_{,xx} - \frac{\omega'''}{3!}\cdot A_{,xxx} \\ &\quad = -\varepsilon^2(1+\rho)^{-1}\big\{\mathrm{i}k\omega^{-1}A\overline{A}[IA + I'A_{,x}] + \big(k\omega^{-1}\big)' I\big(A^2\bar{A}\big)_{,x}\big\} \end{aligned} \tag{14}$$

where $I' = \partial I/\partial k$. The left-hand side of Eq. (14) for the complex envelope A corresponds to the Shrödinger equation of third order. It contains one temporal derivative and three spatial derivatives. The nonlinear right part of the evolution equation (14) is of fourth-order approximation and it is expressed in a compact form using coefficients of the third-order approximation I and its derivative I' only.

3. Stability analysis

Changing x and t for new independent variables

$$\xi = x - \omega' t, \qquad \zeta = t$$

transforms Eq. (14) into the form

$$A_{,\zeta} - \frac{\mathrm{i}\omega''}{2!}\cdot A_{,\xi\xi} - \frac{\omega'''}{3!}\cdot A_{,\xi\xi\xi} = \frac{-\varepsilon^2}{1+\rho}\left\{\mathrm{i}\frac{k}{\omega}A\bar{A}[IA + I'A_{,\xi}] + \left(\frac{k}{\omega}\right)' I\big(A^2\bar{A}\big)_{,\xi}\right\} \tag{15}$$

The solution of Eq. (15) that varies with ζ only is written as follows

$$A = a\exp\left(-\frac{\mathrm{i}\varepsilon^2}{1+\rho}\cdot\frac{k}{\omega}Ia^2\zeta\right) \tag{16}$$

where a is constant. Following Hasimoto and Ono [7], investigation of stability analysis gives the stability condition for wave-packets on the fluid interface in the form

$$I\omega'' \leqslant 0 \tag{17}$$

In the case of gravity waves ($k \to 0$)

$$I \to 2\frac{k^2(1-\rho)(1+\rho^2)}{(1+\rho)^2}, \qquad \omega'' \to -\frac{(1-\rho)^{1/2}}{4k^{3/2}(1+\rho)^{1/2}} \tag{18}$$

so that gravity waves are stable when $\rho < 1$.

In the case of capillary waves ($k \to \infty$) the following conditions take place

$$I \to \frac{-Tk^4(1-6\rho+\rho^2)}{4(1+\rho)^2}, \qquad \omega'' \to \frac{3T^{1/2}}{4k^{1/2}(1+\rho)^{1/2}} \tag{19}$$

The capillary waves are stable only if $(\sqrt{2}-1)^2 < \rho < (\sqrt{2}+1)^2$.

Fig. 1 shows the stability diagram obtained on the basis of numerical analysis of the stability condition (17) for uniform travelling wave trains. Regions of stability and instability are separated by five curves marked by indices 0 to 5. Index 0 corresponds to the curve $\rho = 1 + Tk^2$ which separates the region of linear instability V_0; along curves 1 and 5 the second derivative of the frequency of the wave-packet center is equal to zero, $\omega'' = 0$; for curves 2 and 3 the value I changes its sign and $I = 0$; along the curve 4 I changes its sign too, but $I \to \infty$.

Thus, three regions of nonlinear stability V_6, V_4 and V_2 and three regions of nonlinear instability V_5, V_3, V_1 are discovered.

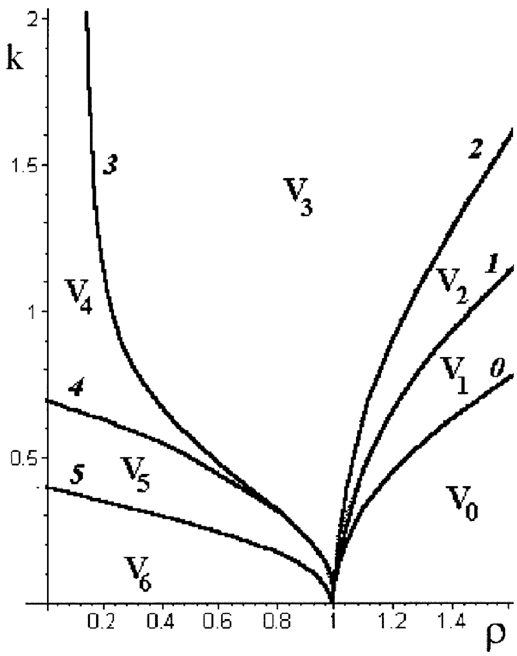

Fig. 1. Stability diagram.

The region V_6 ($k \to 0$) corresponds to long gravity waves, so that the conclusion about stability of gravity waves for $\rho < 1$ is verified. Also, the region V_5 of instability of capillary-gravity waves due to the action of forces of different nature (gravity and surface tension), exists.

The stability of capillary waves for $\rho < 1$ in the region V_4 takes place when $\rho < (\sqrt{2} - 1)^2$ or for sufficiently small density of the upper fluid. Increasing the density ratio leads to destabilization of small wavelengths. The next instability region V_3 is unbounded from above and it is located between two vertical asymptotes $\rho = (\sqrt{2} + 1)^2$ and $\rho = (\sqrt{2} - 1)^2$, as it follows from asymptotic analysis of (19).

Increasing surface tension T extends the regions of instability of capillary waves and, accordingly, narrows the regions of nonlinear stability of gravity waves.

4. Conclusion

The nonlinear propagation of wave packets at the interface between two semi-infinite fluids is investigated taking into account surface tension effect. The method of multiple scale expansions of fourth-order approximation is developed to derive the nonlinear third-order partial differential equation describing the evolution of two-dimensional wave-packets propagating along the interface.

The evolution equation obtained in the case away from cut-off wave number is the nonlinear Schrödinger equation. The evolution equation is valid for a wide range of wave numbers. It contains only a first derivative in time and three derivatives in space coordinate of the envelope with coefficients involving the first three derivatives of the frequency of the wave-packet center with respect to the wave number. All the coefficients of the nonlinear right part of evolution equation in the fourth-order approximation are expressed in a compact form using coefficients of third-order approximation and its derivative with respect to the wave number only.

Three regions of nonlinear stability and three regions of nonlinear instability are discovered. One of the stable regions corresponds to long gravity waves and one of the unstable regions of capillary-gravity waves is due to force actions of different nature (gravity and surface tension).

It is shown that increasing the intensity of surface tension extends regions of instability of capillary waves and narrows regions of nonlinear stability of gravity waves.

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