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C. R. Mecanique 331 (2003) 303–317

COMPTES RENDUS MECANIQUE

Le point sur.../Concise Review Paper

Local problems for vibrating systems with concentrated masses: a review

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Received 11 February 2003; accepted 28 February 2003

Article written at the invitation of the Editorial Board

Abstract

In this review we collect certain results obtained in the last decades on vibrating systems with concentrated masses. In particular, we show the connection of the eigenvalues and eigenfunctions of the local problem with the low and high frequency vibrations of the original problem. ***To cite this article: M. Lobo, E. Pérez, C. R. Mecanique 331 (2003).***

Résumé

Sur les problèmes locaux pour les systèmes vibratoires avec des masses concentrées. Ce rapport-ci contient quelques resultats obtenus tout au long des denières décades sur les systèmes vibratoires avec masses concentrées. Notamment, on met en evidence la connexion entre les éléments propres du problème local et les vibrations de basses fréquences et d'hautes fréquences du problème original. ***Pour citer cet article : M. Lobo, E. Pérez, C. R. Mecanique 331 (2003).***

Keywords: Vibrations; Spectral analysis; Concentrated masses; Low frequencies; High frequencies

Mots-clés : Vibrations ; Analyse spectrale ; Masses concentrées ; Fréquences basses ; Fréquences hautes

1. Introduction

The asymptotic behavior of the spectral problems associated with vibrating systems with concentrated masses has been addressed by many authors in the last two decades: [1–21]. Many different techniques have been used in these papers and a great variety of results have been obtained depending on the dimension of the space, the density of the concentrated masses and the number of concentrated masses. Nevertheless, it should be noticed that, even though the first papers date from 1984 (cf. [1,2]), and the last papers are as recent as 2003 (cf. [20,18]), there

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1631-0721/03/$ – see front matter
doi:10.1016/S1631-0721(03)00058-5

are still many open questions on the subject; we shall try to point out some of these questions throughout this paper.

We consider the vibrations of a body occupying a bounded domain Ω of $\mathbb{R}^n$, $n = 2,3$, that contains one, several or very many small regions of high density near the boundary $\partial\Omega$, the so-called *concentrated masses*. Each concentrated mass occupies a small domain $B^\varepsilon \subset \Omega$; B^ε has a diameter $\mathrm{O}(\varepsilon)$; the density takes the value $\mathrm{O}(\varepsilon^{-m})$ in B^ε and $\mathrm{O}(1)$ outside, m and ε are positive parameters, $\varepsilon \to 0$. Let $N(\varepsilon)$ denote the number of concentrated masses contained in Ω. We assume that the distance between two concentrated masses is of order of magnitude greater than ε and that $N(\varepsilon)$ can be either a fixed number or a ε dependent number as in boundary homogenization problems. Taking $m > 2$, we study the asymptotic behavior, when $\varepsilon \to 0$, of the eigenelements $(\lambda^\varepsilon, u^\varepsilon)$ of the corresponding spectral problem (cf. (1) and (5)).

Most of the above mentioned authors (cf. [1,3,5–9,16,18]) consider one single concentrated mass inside Ω, the Laplace operator, and a Dirichlet condition on the boundary $\partial\Omega$: that is, the spectral problem

$$\begin{cases} -\Delta u^\varepsilon = \lambda^\varepsilon \rho^\varepsilon(x) u^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\rho^\varepsilon(x)$ is the function defined by

$$\rho^\varepsilon(x) = \varepsilon^{-m} \quad \text{if } x \in B^\varepsilon \quad \text{and} \quad \rho^\varepsilon(x) = 1 \quad \text{if } x \in \Omega - \overline{B^\varepsilon}$$

Here, we assume that $m > 2$, $B^\varepsilon = \varepsilon B$, where B is a bounded open domain of $\mathbb{R}^n$, and both Ω and B contain the origin.

See [10] for a Neumann condition on the boundary $\partial\Omega$, [11] for the bi-harmonic operator, [2] for the elasticity operator, [20] for the operator associated with Reissner–Midlin plate model and [4] for the dimension 1 of the space. We refer to [12–14,17,21] for the case where $N(\varepsilon) \to +\infty$, the concentrated masses being periodically distributed on the boundary of Ω (cf. (5)). Let us observe that in all these papers, the cases where $m \leqslant 2$ have also been considered. Besides, we also note that the results in papers considering one single concentrated mass inside Ω (cf. (1)) can be extended to the case where the concentrated mass is placed at the boundary $\partial\Omega$ (cf. (5)) with minor modifications. For this reason, throughout the paper, we state the results for the concentrated masses placed at the boundary.

A common fact to all these problems is that for $m > 2$, the so-called *low frequencies*, that is the eigenvalues $\lambda^\varepsilon = \mathrm{O}(\varepsilon^{m-2})$, give rise to *local vibrations* of the concentrated masses, each one asymptotically independent from the others. Roughly speaking, the corresponding eigenfunctions associated with these frequencies are only significant in a small neighborhood of the concentrated masses and almost vanish at distance of order $\mathrm{O}(1)$ from the concentrated masses. Therefore, in general, in order to obtain vibrations affecting the whole structure, it is necessary to consider the eigenfunctions associated with the so-called *high frequencies*. These frequencies are of order of magnitude $\lambda^\varepsilon = \mathrm{O}(1)$ and the corresponding vibrations are referred to as the *global vibrations*. Of course, other kinds of vibrations associated with other different orders of magnitude of the frequencies could exist but the problem is to locate them (cf. [18]).

Let us note that the definition of local vibrations has been stated since the very beginning (cf. [1] when $n = 3$ and [5] when $n = 2$). The eigenvalues causing these vibrations, suitably re-scaled, are approached by the eigenvalues of the so-called *local problem* (see (8) and (10)). We also observe that among all the techniques, the classical asymptotic expansions are essential in order to describe this local behavior. Namely, the asymptotic expansions are useful for dimension $n = 2$ of the space, as a consequence of the behavior at infinity of the harmonic functions in outer domains as we outline here below (see also formulas (12) and (13)).

Indeed, considering (λ^0, U^0) an eigenelement of the local problem (10), for $\lambda^\varepsilon = \varepsilon^{m-2}\lambda^0 + \cdots$ in (1), and for a suitable normalization of the corresponding eigenfunctions u^ε (see Section 3), u^ε are approached in Ω by

$$u^\varepsilon \approx U^0\left(\frac{x}{\varepsilon}\right) - c + \frac{c}{\ln\varepsilon}\big(\ln|x| - f(x)\big), \quad \text{when } n = 2 \tag{2}$$

where f is a harmonic function in Ω, $f(x) = \ln|x|$ on $\partial\Omega$, and c is the constant in (10), and,

$$u^\varepsilon \approx U^0\left(\frac{x}{\varepsilon}\right), \quad \text{when } n = 3 \tag{3}$$

Formulas (2) and (3) show the local character of the corresponding vibrations (see [5] and Sections VII.10 in [7]): the eigenfunctions u^ε are of order $\mathrm{O}(1)$ only in a region near the concentrated mass (i.e., $|x| = \mathrm{O}(\varepsilon)$) and of order $\mathrm{o}(1)$ for $|x| = \mathrm{O}(1)$.

Asymptotic expansions are also useful for describing the asymptotic behavior of the eigenfunctions associated with the high frequencies of (1) ((5), resp.), inside each concentrated mass (cf. formulas (24)–(26)). They prove also to be essential when describing the behavior of the eigenfunctions that concentrate their support in certain other small regions, for instance, boundary layers near $\partial\Omega$ or near the interface of the concentrated masses: see [18] in connection with the so-called *whispering gallery eigenmodes*.

In Section 2 we introduce the eigenvalue problem (5), the eigenvalue local problem (8), and some background on the subject. In Sections 3 and 4 we state the main results for the asymptotic behavior of the low frequencies, $\lambda^\varepsilon = \mathrm{O}(\varepsilon^{m-2})$, as $\varepsilon \to 0$, depending on the dimension of the space, on the multiplicity of the low frequencies of the local problem (8) and on the number of concentrated masses. We also provide results on the structure of the eigenfunctions associated with these frequencies. As regards the high frequencies, in Section 5 we introduce the *homogenized problems* (23) and we make clear that the computation of certain correcting terms for certain eigenfunctions u^ε is deeply involved with the study of the high frequencies of the local problem. These eigenfunctions u^ε are associated with the high frequencies, $\lambda^\varepsilon = \lambda^\varepsilon_{i(\varepsilon)} = \mathrm{O}(1)$, and, under certain restrictions on the geometry, the correcting terms improve the convergence of u^ε towards the eigenfunctions of (23) or towards zero (see [12–15,17] for convergence results).

The aim of this review can be summarized as follows: to describe different kinds of techniques to broach the asymptotic behavior of the eigenelements of the ε dependent spectral problem (5) ((1), resp.) and, at the same time, to show how this behavior strongly depends on the number of concentrated masses and the dimension of the space. Besides, we make it clear that the eigenelements of the local problem (8) ((10), resp.) are involved in the study of the low and high-frequency vibrations of (5) ((1), resp.), rather than being involved only with the low frequencies as one might think at a first stage (see [16,18]).

Finally, for further recent studies on vibrating systems with concentrated masses and for very different qualitative results let us mention: [19] for a non-periodical distribution of the concentrated masses when $m < 2$ and $N(\varepsilon) \to \infty$, [20] for a problem where the thickness h of the domain is a new small parameter which gives different limit behaviors of the spectrum, and [22,23] for very different geometries and spatial distribution of the regions of high density.

2. Setting of the problems

Let Ω be any bounded domain of $\mathbb{R}^n$, $n = 2, 3$, with a Lipschitz boundary $\partial\Omega$ and $\Omega \subset \{x_n < 0\}$. Let Σ and Γ_Ω be non-empty parts of the boundary, such that $\partial\Omega = \overline{\Sigma} \cup \overline{\Gamma_\Omega}$, and Σ is assumed to be in contact with $\{x_n = 0\}$. Let ε and η be two small parameters such that $\varepsilon \ll \eta$ and $\eta = \eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

For $n = 2$, let B be the half-circle $B = \{(y_1, y_2) \mid y_1^2 + y_2^2 < 1,\ y_2 < 0\}$ in the auxiliary space $\mathbb{R}^2$ with coordinates y_1, y_2. For $n = 3$, let B be the half-ball $B = \{(y_1, y_2, y_3) \mid y_1^2 + y_2^2 + y_3^2 < 1,\ y_3 < 0\}$ in the auxiliary space $\mathbb{R}^3$ with coordinates y_1, y_2, y_3. Let ∂B be the boundary of B, $\partial B = \overline{T} \cup \overline{\Gamma}$, where T is the part lying on $\{y_n = 0\}$. Let B^ε (and similarly T^ε, Γ^ε) denote its homothetic εB (εT, $\varepsilon\Gamma$). Let B_k^ε (and similarly T_k^ε, Γ_k^ε) denote the domain obtained by translation of the previous B^ε (T^ε, Γ^ε) centered at the point $\tilde{x}_k$ of Σ; $\tilde{x}_k$ are at distance η between them. k is a parameter ranging from 1 to $N(\varepsilon)$, $k \in \mathbb{N}$. $N(\varepsilon)$ denotes the number of B_k^ε contained in Ω; $N(\varepsilon)$ is of order $\mathrm{O}(\frac{1}{\eta})$ when $n=2$ and $\mathrm{O}(1/\eta^2)$ when $n=3$. The parameter α denotes the value

$$\alpha = \lim_{\varepsilon\to 0} \frac{-1}{\eta \ln \varepsilon} \quad \text{when } n=2 \quad \text{and} \quad \alpha = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\eta^2} \quad \text{when } n=3 \tag{4}$$

We consider the eigenvalue problem:

$$\begin{cases} -\Delta u^\varepsilon = \rho^\varepsilon \lambda^\varepsilon u^\varepsilon \quad \text{in } \Omega \\ u^\varepsilon = 0 \quad \text{on } \Gamma_\Omega \cup \bigcup T^\varepsilon \\ \dfrac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on } \Sigma - \overline{\bigcup T^\varepsilon} \end{cases} \tag{5}$$

where $\rho^\varepsilon = \rho^\varepsilon(x)$ is the function defined by

$$\rho^\varepsilon(x) = \begin{cases} \dfrac{1}{\varepsilon^m} & \text{if } x \in \bigcup B^\varepsilon \\ 1 & \text{if } x \in \Omega - \overline{\bigcup B^\varepsilon} \end{cases} \tag{6}$$

The symbol $\bigcup$ is extended, for fixed ε, to all the regions B_k^ε contained in Ω. The parameter m is a real number, $m > 2$ (see [12–14], for different values of the parameter m, boundary conditions and shapes of the domains Ω and B).

We also consider problem (5) in the case where there is a fixed number of concentrated masses placed at a fixed distance: that is $N(\varepsilon) = 1$ or $N(\varepsilon) = N$ with N independent of ε. In this case, $\bigcup B^\varepsilon$ (similarly, $\bigcup T^\varepsilon$) denote either B^ε (T^ε) or the union of the N regions B^ε (T^ε) contained in Ω ($\partial\Omega$).

As a matter of fact, let us note that here, and in previous papers [12–15,17], we have considered the case where the concentrated masses are placed on a part Σ of $\partial\Omega$, Σ lying on $\{x_n = 0\}$. In addition, rapidly alternating mixed boundary condition have been imposed on this part of the boundary. In order to fix ideas, we have chosen problem (5) as a model to state techniques and results throughout the paper. Nevertheless, sometimes, it can be more adequated to consider a Neumann boundary condition on the whole Σ or the case where the concentrated masses are inside Ω. In the case of one single concentrated mass inside Ω, problem (5) reads (1). We observe that the techniques in this paper apply to the study of the above problems with some minor modifications.

As is well known, problem (5) has a discrete spectrum. For fixed ε, let $\{\lambda_i^\varepsilon\}_{i=1}^\infty$ be the sequence of eigenvalues of (5), converging to ∞, with the classical convention of repeated eigenvalues. Using the minimax principle, it has been proved (cf. [12–14]) that for each fixed $i = 1, 2, \ldots,$ the eigenvalues satisfy the estimates

$$C\varepsilon^{m-2} \leqslant \lambda_i^\varepsilon \leqslant C_i \varepsilon^{m-2} \tag{7}$$

where C is a constant independent of ε and i and C_i is a constant independent of ε. Let $\{u_i^\varepsilon\}_{i=1}^\infty$ be the corresponding sequence of eigenfunctions which are assumed to be an orthonormal basis of the space $\mathbf{V}^\varepsilon$, where $\mathbf{V}^\varepsilon$ is the completion of $\{u \in \mathcal{C}^1(\overline{\Omega}) \mid u = 0 \text{ on } \Gamma_\Omega \cup \bigcup T^\varepsilon\}$ in $H^1(\Omega)$.

Because of (7), the low frequencies are the eigenvalues $\lambda_i^\varepsilon = \mathrm{O}(\varepsilon^{m-2})$ for fixed i, $i = 1, 2, \ldots$. Convergence results for the low frequencies can be found in [1–3,5,7–9] for one single concentrated mass and in [12–15,21] for many concentrated masses. As already noted in Section 1, in general, the low frequencies are associated with the local vibrations of the concentrated masses, each one asymptotically independent from the others. Nevertheless, we have found an exception: for $n = 3$ and $\alpha > 0$ these frequencies also give rise to global vibrations affecting the whole body. That is, only for a 3-dimensional body containing many concentrated masses near the boundary at mutual distances $\eta \approx \sqrt{\varepsilon}$ can the eigenvalues of order $\mathrm{O}(\varepsilon^{m-2})$ be approached by means of those for a problem obtained from the homogenization of the concentrated masses (cf. (21)). Apart from this exception, which we consider in Section 4, the low frequencies and the corresponding eigenfunctions are asymptotically described, in a certain way, by the local problem (8).

The local problem is an eigenvalue problem posed in an unbounded domain:

$$\begin{cases} -\Delta_y U = \lambda U \quad \text{in } B \\ -\Delta_y U = 0 \quad \text{in } \mathbb{R}^{n-} - \overline{B} \\ [U] = \left[\dfrac{\partial U}{\partial n_y}\right] = 0 \quad \text{on } \Gamma \\ U = 0 \quad \text{on } T \\ \dfrac{\partial U}{\partial y_n} = 0 \quad \text{on } \{y_n = 0\} - \overline{T} \\ U(y) \to c, \quad \text{as } |y| \to \infty,\ y_n < 0 \text{ when } n = 2 \\ U(y) \to 0, \quad \text{as } |y| \to \infty,\ y_n < 0 \text{ when } n = 3 \end{cases} \tag{8}$$

where the brackets denote the jump across Γ, $\bar{n}_y$ the unit outward normal to Γ and c some unknown but well determined constant. $\mathbb{R}^{n-}$ is the half-plane $\{(y_1, y_2) \mid y_2 < 0\}$ for $n = 2$ and the half-space $\{(y_1, y_2, y_3) \mid y_3 < 0\}$ for $n = 3$. The variable y is the *local variable*:

$$y = \frac{x - \tilde{x}_k}{\varepsilon} \tag{9}$$

which dilates the neighborhood of each point $\tilde{x}_k$ and transforms B_k^ε into B.

As it is known (cf. [12,14]), (8) can be written as a standard eigenvalue problem with a discrete spectrum in the space $\widetilde{\mathcal{V}}$, where $\widetilde{\mathcal{V}}$ is the completion of $\{U \in \mathcal{D}(\overline{\mathbb{R}^{n-}}) \mid U = 0 \text{ on } T\}$ for the Dirichlet norm $\|\nabla_y U\|_{L^2(\mathbb{R}^{n-})}$. See [5, 12–14] and Section IV.8 in [7] for the weak formulation of (8) in the space completion of $\{U \in \mathcal{C}^1(\overline{B}) \mid U = 0 \text{ on } T\}$ for the norm of $H^1(B)$ and the definition of the so-called *Neumann–Dirichlet operator* from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$.

Let $\{\lambda_i^0\}_{i=1}^\infty$ be the eigenvalues of (8), with the classical convention of repeated eigenvalues, and let $\{U_i^0\}_{i=1}^\infty$ be the corresponding eigenfunctions of norm 1 in $\widetilde{\mathcal{V}}$.

In Section 3 we summarize results on the relationship of the low frequencies of (5) with those of (8). In the same way, in Section 5 we show the connection of the high frequencies of (5) with the very large frequencies of (8) obtained in previous papers (cf. [16,18]). From results in Sections 3 and 5 we can assert that the low frequencies of the ε dependent problem (5) are always closely associated to the low frequencies of the local problem (8) while the high frequency vibrations of (5) are associated to the homogenized problems (23) and the very large frequencies of (8).

We emphasize that all the results and techniques in Section 3 and in the previously mentioned papers (cf. [12–15, 17,24]) can be extended, with minor modifications, to the case of other geometries for Ω and Σ, or other boundary conditions on $\partial\Omega$, or to the case where the masses are placed on a surface inside the domain Ω. For instance, when a Neumann condition in imposed on the whole Σ, the local problem is (8) where the Dirichlet condition on T is replaced by a Neumann one, $\partial U/\partial y_n = 0$ on T (cf. [12]). In the case where the concentrated masses are inside Ω (cf. (1)), the local problem reads:

$$\begin{cases} -\Delta_y U = \lambda U \quad \text{in } B \\ -\Delta_y U = 0 \quad \text{in } \mathbb{R}^{n} - \overline{B} \\ [U] = \left[\dfrac{\partial U}{\partial n_y}\right] = 0 \quad \text{on } \partial B \\ U(y) \to c, \quad \text{as } |y| \to \infty,\ \text{when } n = 2 \\ U(y) \to 0, \quad \text{as } |y| \to \infty,\ \text{when } n = 3 \end{cases} \tag{10}$$

In order to obtain the results in Section 3 for all these problems, we observe that it is easier to handle the local problem (10) (instead of (8)) and that the solution of (8) can be extended by a harmonic function outside a domain that contains B. See [25–27,12,14] for local problems posed on the half-space $\mathbb{R}^{n-}$ with mixed boundary conditions on the plane $\{x_n = 0\}$, n being $n = 2, 3$.

We refer to [12] for the results and techniques in Section 4 when a Neumann condition is imposed on the whole Σ and $N(\varepsilon) \to \infty$. As regards the case where the concentrated masses are placed on a surface Σ inside the domain Ω, $N(\varepsilon) \to \infty$, the normal derivative appearing in the equation on Σ in the homogenized problems (21) and (23) is likely to become a transmission condition (cf. [28,29]): it seems clear that all the statements can be rewritten with minor modifications.

Finally, let us also observe that, except for certain explicit computations in [16] and [18] (cf. Section 5), it is not essential for B to be a ball or half-ball of $\mathbb{R}^n$; B can be any other domain or $\mathbb{R}^n$ with a Lipschitz boundary.

3. Low frequencies and local vibrations

Let us change the variable in (5) by setting $y = x/\varepsilon$. We obtain the integral formulation:

$$\int_{\Omega_\varepsilon} \nabla_y U^\varepsilon \cdot \nabla_y V^\varepsilon \, \mathrm{d}y = \nu^\varepsilon \int_{\Omega_\varepsilon} \beta^\varepsilon(y) U^\varepsilon V^\varepsilon \, \mathrm{d}y, \quad \forall V^\varepsilon \in \widetilde{\mathbf{V}}^\varepsilon \tag{11}$$

where Ω_ε denotes the domain $\{y \mid \varepsilon y \in \Omega\}$, $\nu^\varepsilon = \lambda^\varepsilon/\varepsilon^{m-2}$, λ^ε the eigenvalues of (5), and $\beta^\varepsilon(y)$ is defined as:

$$\beta^\varepsilon(y) = 1 \quad \text{if } y \in \bigcup \tau_y B^\varepsilon \quad \text{and} \quad \beta^\varepsilon(y) = \varepsilon^m \quad \text{if } y \in \Omega_\varepsilon - \overline{\bigcup \tau_y B^\varepsilon}$$

where $\tau_y B^\varepsilon$ denote the transformed domains of the regions B^ε contained in Ω to the y variable (see (6)). $\widetilde{\mathbf{V}}^\varepsilon$ is the functional space $\{U = U(y) \mid U(\varepsilon y) \in \mathbf{V}^\varepsilon\}$.

We assume that the eigenfunctions of (11) satisfy $\|U^\varepsilon\|_{\widetilde{\mathbf{V}}^\varepsilon} = 1$. Thus, we can take (cf. (5)) $U^\varepsilon = u^\varepsilon$ when $n = 2$ and $U^\varepsilon = u^\varepsilon/\sqrt{\varepsilon}$ when $n = 3$. In addition, we observe that the elements of $\widetilde{\mathbf{V}}^\varepsilon$ extended by zero in $\mathbb{R}^{n-} - \overline{\Omega_\varepsilon}$ are elements of $\widetilde{\mathcal{V}}$. The first convergence result for the eigenelements of (11) in both cases $N(\varepsilon) = N$ or $N(\varepsilon) \to +\infty$ can be stated as follows:

Theorem 1. *If* $(\lambda^\varepsilon_{i(\varepsilon)}/\varepsilon^\beta) \xrightarrow{\varepsilon \to 0} \lambda^*$ *and the corresponding eigenfunctions* $U^\varepsilon_{i(\varepsilon)} \xrightarrow{\varepsilon \to 0} U^*$ *weakly in* $\widetilde{\mathcal{V}}$ *with* $U^* \neq 0$ *and* $\lambda^* \neq 0$, *then* $\beta = m - 2$ *and* (λ^*, U^*) *is an eigenelement of* (8). *In addition, each eigenvalue* λ^0 *of the local problem* (8) *is an accumulation point of values* $\lambda^\varepsilon_{i(\varepsilon)}/\varepsilon^{m-2}$; *the index* $i(\varepsilon)$ *can be fixed or dependent on* ε.

The first assertion in Theorem 1 is easily proved by taking limits in the variational formulation (11) for suitable V^ε. This technique, along with asymptotic expansions, proves to be very useful when identifying possible limits of the eigenelements of the ε dependent problem. Let us observe that in order to obtain this kind of results the normalization selected for the eigenfunctions is essential (see Theorem 7 in Section 5 to compare). Moreover, in the case where $\beta = m - 2$ and $N(\varepsilon) = 1$ or $N(\varepsilon) = N$, the U^* in the statement of Theorem 1 is different from zero, which proves useful to obtain stronger results of convergence.

The second assertion in Theorem 1 has been proved in [12–14] using Spectral Families and Fourier Transform. This technique is very general and can be used for very many spectral perturbation problems in which self-adjoint operators are involved: all ends up as a weak convergence of the corresponding spectral families operating on certain test functions. This allows us to obtain spectral convergence results when the limit spectral family is not a constant one. The Fourier Transform method provides information on the spectrum of the ε dependent problems when properties of the time dependent problems are known. We refer to [30,31,7] for an extensive theory and [24,32,7,31] for its application to different spectral perturbation problems. We also observe that, in general, the Fourier Transform does not provide information on other possible accumulation points of the spectrum or on the associated eigenfunctions.

A more precise structure of the eigenfunctions u^ε associated with the eigenvalues λ^ε such that $\lambda^\varepsilon/\varepsilon^{m-2} \approx \lambda^0$, where λ^0 is an eigenvalue of the local problem (8), is obtained by using asymptotic expansions. The technique of matched asymptotic expansions (cf. [5,15], and Section VII.10 in [7]) leads us to the composite expansion of the eigenfunctions U^ε in Ω:

$$U^\varepsilon \approx \sum_k a_k \left(U^0\left(\frac{x-\tilde{x}_k}{\varepsilon}\right) - c + \frac{c}{\ln\varepsilon}\big(\ln|x-\tilde{x}_k| - f(x-\tilde{x}_k)\big)\right), \quad \text{when } n=2, \tag{12}$$

and

$$U^\varepsilon \approx \sum_k a_k U^0\left(\frac{x-\tilde{x}_k}{\varepsilon}\right), \quad \text{when } n=3 \tag{13}$$

where the summation is extended, for fixed ε, to all the centres $\tilde{x}_k$ contained in Σ and a_k are constants. Moreover, when $n=2$, we have assumed that $\lim_{\varepsilon\to 0}(\ln\eta/\ln\varepsilon) = 0$, and, f is a harmonic function in Ω, satisfying homogeneous Neumann conditions on Σ and $f(x) = \ln|x|$ on Γ_Ω. c in (12) is the constant which appears in (8) for $\lambda = \lambda^0$ and $U = U^0$.

On the basis of (12) and (13), and since the constants a_k can take the value 0, we can assert that there are $N(\varepsilon)$ eigenfunctions associated with $\lambda^\varepsilon = \lambda_i^\varepsilon \approx \varepsilon^{m-2}\lambda^0$, $\{U_{i_k}^\varepsilon\}_{k=1}^{N(\varepsilon)}$ (see (2) and (3) to compare). Each $U_{i_k}^\varepsilon$ is approximately an eigenfunction associated with λ^0 in a small neighbourhood of the concentrated mass B_k^ε and it takes small values in points far from this mass. That is to say, if l_0 denotes the multiplicity of the eigenvalue λ^0 of (8), the multiplicity of λ^ε is likely to be equal to or greater than the product $l_0 N(\varepsilon)$. Obviously, in the case when λ^ε also originates global vibrations ($n=3$, $\eta \approx \sqrt{\varepsilon}$) other eigenfunctions can be associated with λ^ε (cf. Theorem 7 in Section 4).

In Sections 3.1 and 3.2 we provide some results that justify (12) and (13) and, at the same time, complement Theorem 1. In Section 3.1 we consider the case of one single concentrated mass or a fixed number of them N, while in Section 3.2 we consider the case of very many concentrated masses, i.e., $N(\varepsilon) \to \infty$.

3.1. The case of a fixed number of concentrated masses

On account of (7), for each fixed i we can extract converging subsequences $\lambda_i^{\varepsilon_n}/\varepsilon_n^{m-2}$ as $\varepsilon_n \to 0$. We note that in the best case we can prove the convergence of the spectrum with conservation of the multiplicity (cf. Sections VIII.1 and VIII.2 in [30] and Sections XI.1–XI.3 in [31]). If so, because of Theorem 1, one may think that it is possible to prove the convergence of the whole spectrum, $\{\lambda_i^\varepsilon/\varepsilon^{m-2}\}_{i=1}^\infty$, towards the eigenvalues of (8), $\{\lambda_i^0\}_{i=1}^\infty$, as well as the convergence of corresponding eigenfunctions in a certain topology to be stated. This is the case for one single concentrated mass as the following theorem states.

Theorem 2. *Let $N(\varepsilon)$ be $N(\varepsilon)=1$ and let m be $m>2$. Let λ_i^ε be the eigenvalues of* (5) *and U_i^ε the corresponding eigenfunctions with norm* 1 *in $\widetilde{\mathbf{V}}^\varepsilon$. Besides, for fixed i, the values $\lambda_i^\varepsilon/\varepsilon^{m-2}$ converge, when $\varepsilon \to 0$, towards the eigenvalues of* (8), *$\{\lambda_i^0\}_{i=1}^\infty$, with conservation of the multiplicity. For each sequence it is possible to extract a subsequence, still denoted by ε, such that the corresponding eigenfunctions, U_i^ε, converge towards U_i in $L^2(B)$* (*and weakly in $\widetilde{\mathcal{V}}$*), *$\varepsilon \to 0$, where U_i is an eigenfunction associated with the i-th eigenvalue of* (8), *and $\{U_i\}_{i=1}^\infty$ form an orthonormal basis of $\widetilde{\mathcal{V}}$.*

The proof of Theorem 2 is quite classical nowadays. It can be performed by using very different techniques. On the one hand, let us mention the spectral perturbation theory for implicit nonholomorphic eigenvalue problems: see [5] when $n=2$ and Section VII.11 in [7] when $n=3$ for the proof of Theorem 2, and, see Section V.10 in [7] for the general theory and references.

On the other hand, Theorem 2 can be proved by using the results on spectral convergence in Section III.1 of [9], namely, Theorems 1.4 and 1.7. These results state the convergence of the spectrum for a sequence of abstract operators acting on different Hilbert spaces, under certain restrictions for this sequence. Their proofs are based on Lemma 1 (cf. Section 3.2) and on particular properties of the sequence of operators. Let us refer to [8] for the proof of Theorem 2 when $n=2$ and to Section III.5 of [9] when $n=3$. Theorems 1.4 and 1.7 in Section III.1 of [9] also provide certain estimates for the difference between the eigenvalues and eigenfunctions of the ε dependent problem and the limit problem.

Uniform estimates for bounds of the convergence rate of the eigenelements of (5) towards those of (8), depending on ε and the eigenvalue number interest from many viewpoints, and some improvements of the results in Theorem 2 should be performed in this direction. For the time being, we refer to [33] for the technique used to get these bounds for other parameter dependent problems. This technique is based on Lemma 1 and results on *almost orthogonality* of the eigenfunctions (cf. Section 3.2).

Finally, let us also mention the technique of the G-convergence theory of elliptic operators in Section III.9 of [34], as an effective fast method to check the convergence of the eigenelements as stated in Theorem 2. This technique is based on the minimax principle along with orthonormalization processes.

It is worth observing that the proof of the first assertion in Theorem 2 can be extended with minor modifications to the case of N concentrated masses. Indeed, for simplicity let us take $N=2$ and the two masses centered at the fixed points (independent of ε) $\tilde{x}_1$ and $\tilde{x}_2$. Then, by denoting $\widetilde{\mathcal{V}}_1$ ($\widetilde{\mathcal{V}}_2$, resp.) the space $\widetilde{\mathcal{V}}$ with the variable y in (9) for $k=1$ ($k=2$, resp.), we consider the space product $\mathcal{W}=\widetilde{\mathcal{V}}_1\times\widetilde{\mathcal{V}}_2$ whose elements are pairs of functions (U,V), $U\in\widetilde{\mathcal{V}}_1$, $V\in\widetilde{\mathcal{V}}_2$; the scalar product in $\mathcal{W}$ is the sum of the scalar products in $\widetilde{\mathcal{V}}_1$ and $\widetilde{\mathcal{V}}_2$. In the same way, we consider an eigenvalue problem with the same eigenvalues of (8) and the double multiplicity of each eigenvalue: To find λ and $(U,V)\in\mathcal{W}$, $(U,V)\neq 0$, U and V satisfying Eqs. (8).

Then, the extension of the results in Theorems 2 allow us to assert that for each eigenvalue λ^0 of the local problem (8) with multiplicity l_0, the total multiplicity of the eigenvalues of (5) converging towards λ^0 is l_0N, N being the number of concentrated masses. Therefore, we observe that the limit eigenvalues of the sequences $\lambda^\varepsilon/\varepsilon^{m-2}$ are not influenced by the number of concentrated masses. The main difference in the limit behavior of the eigenelements of (5) lies in the approach for the eigenfunctions. See [10] for other very different results when a Neumann condition is imposed on the boundary of Ω.

As a matter of fact, we observe that the results on the structure of the eigenfunctions of (5) in Theorem 3 below also apply to the case here considered, $N(\varepsilon)=N$, but we cannot extend the results in Theorem 2 to the case where $N(\varepsilon)\to\infty$ as we shall show in Section 3.2.

3.2. The case of many concentrated masses: $N(\varepsilon)\to+\infty$

Let $N(\varepsilon)$ be $N(\varepsilon)\to\infty$ as stated in Section 2. For brevity we consider $\alpha\geqslant 0$ in (4). Let us point out that there is an important difference between the dimensions $n=2$ and $n=3$ of the space. More precisely, for $n=2$, Theorem 5 states that all the sequences $\lambda_i^\varepsilon/\varepsilon^{m-2}$ converge towards the first eigenvalue λ_1^0 of the local problem (8), $i=1,2,\ldots$. Instead, for $n=3$, there are other different limit accumulations points of sequences $\lambda_i^\varepsilon/\varepsilon^{m-2}$ for which the corresponding eigenfunctions are associated with global vibrations of the whole structure (cf. Theorems 6 and 7 in Section 4).

Theorem 3 justifies (12) and (13) since it shows that, for any fixed K, $K<N(\varepsilon)$, there are at least l_0K values $\lambda^\varepsilon/\varepsilon^{m-2}$ converging towards each eigenvalue λ^0 of (8), l_0 being the multiplicity of λ^0. More specifically, it is proved that, for sufficiently small ε, the sum of the multiplicities of the eigenvalues $\lambda^\varepsilon_{i(\varepsilon)}$ of (5), such that $\lambda^\varepsilon_{i(\varepsilon)}/\varepsilon^{m-2}$ approaches λ^0, increases as the number of the concentrated masses $N(\varepsilon)$ does, and converges to ∞ as $\varepsilon\to 0$. Besides, the corresponding eigenfunctions are approached in the space $\widetilde{\mathbf{V}}^\varepsilon$ by the eigenfunctions of (8) associated with λ^0, concentrating their support asymptotically in neighborhoods of the concentrated masses as stated in Theorem 3.

Let us consider an eigenvalue λ^0 of (8) with multiplicity l_0 and $U_1^0, U_2^0, \ldots, U_{l_0}^0$ the corresponding eigenfunctions, orthogonal in $\widetilde{\mathcal{V}}$, satisfying $\|\nabla_y U_i^0\|_{L^2(\mathbb{R}^{n-})} = 1$. Let us introduce certain cut-off functions $\tilde{\varphi}^\varepsilon$ such that $U_p^0 \tilde{\varphi}^\varepsilon \in \widetilde{\mathbf{V}}^\varepsilon$, and, $U_p^0 \tilde{\varphi}^\varepsilon \to U_p^0$ in $\widetilde{\mathcal{V}}$ as $\varepsilon \to 0$ (cf. [12,14]).

We define $\tilde{\varphi}^\varepsilon(y)$ as a function defined depending on the value of n. For $n = 2$, we consider $R_\varepsilon = \sqrt{(\varepsilon + \eta/4)/\varepsilon}$ and we define:

$$\begin{aligned} &\tilde{\varphi}^\varepsilon(y) = 1 \quad \text{if } |y| \leqslant R_\varepsilon, \qquad \tilde{\varphi}^\varepsilon(y) = 1 - \frac{\ln|y| - \ln R_\varepsilon}{\ln R_\varepsilon} \quad \text{if } R_\varepsilon \leqslant |y| \leqslant R_\varepsilon^2 \\ &\tilde{\varphi}^\varepsilon(y) = 0 \quad \text{if } |y| \geqslant R_\varepsilon^2 \end{aligned} \tag{14}$$

For $n = 3$, we consider $\tilde{\varphi}^\varepsilon$ as a smooth function which takes the value 1 in the half-ball of radius $(\varepsilon + \eta/8)/\varepsilon$, $B((\varepsilon + \eta/8)/\varepsilon)$, and zero outside the half-ball of radius $(\varepsilon + \eta/4)/\varepsilon$, $B((\varepsilon + \eta/4)/\varepsilon)$:

$$\tilde{\varphi}^\varepsilon(y) = \varphi\left(2\frac{|\varepsilon y| - \varepsilon}{\eta}\right) \tag{15}$$

where $\varphi \in C^\infty[0,1]$, $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ in $[0, 1/4]$ and $\mathrm{Supp}(\varphi) \subset [0, 1/2]$.

For each $k = 1, 2, \ldots, N(\varepsilon)$, $p = 1, 2, \ldots, l_0$, we introduce the function $Z_{k,p}^\varepsilon$,

$$Z_{k,p}^\varepsilon(y) = \frac{U_p^0(y - \tilde{x}_k/\varepsilon)\tilde{\varphi}^\varepsilon(y - \tilde{x}_k/\varepsilon)}{\|\nabla_y(U_p^0 \tilde{\varphi}^\varepsilon)\|_{L^2(\mathbb{R}^{n-})}} \tag{16}$$

where $y = x/\varepsilon$. They satisfy the *almost orthogonality* codition:

$$\langle Z_{k_1,p}^\varepsilon, Z_{k_2,q}^\varepsilon \rangle_{\widetilde{\mathbf{V}}^\varepsilon} = 0 \quad \text{for } k_1 \neq k_2, \qquad \left| \langle Z_{k,p}^\varepsilon, Z_{k,q}^\varepsilon \rangle_{\widetilde{\mathbf{V}}^\varepsilon} - \delta_{p,q} \right| = \mathrm{o}_{1,\varepsilon} \quad \forall k, p, q \tag{17}$$

where $\mathrm{o}_{1,\varepsilon} = C(\ln(1 + \eta/4\varepsilon))^{-1/2}$ when $n = 2$, and $\mathrm{o}_{1,\varepsilon} = C\varepsilon/\eta$ when $n = 3$, with constant C independent of ε.

Theorem 3. *Let us consider an eigenvalue* λ^0 *of* (8) *with multiplicity* l_0 *and let* $U_1^0, U_2^0, \ldots, U_{l_0}^0$ *be the corresponding eigenfunctions which are assumed to be orthonormal in* $\mathcal{V}$. *For any* $K > 0$ *there is* $\varepsilon^*(K)$ *such that, for* $\varepsilon < \varepsilon^*(K)$, $K < l_0 N(\varepsilon)$ *and the interval* $[\lambda^0 - d^\varepsilon, \lambda^0 + d^\varepsilon]$ *contains eigenvalues of* (11) $\lambda_{i(\varepsilon)}^\varepsilon/\varepsilon^{m-2}$ *with total multiplicity greater than or equal to* K; d^ε *is a certain sequence,* $d^\varepsilon \to 0$ *as* $\varepsilon \to 0$ *and the interval* $[\lambda^0 - d^\varepsilon, \lambda^0 + d^\varepsilon]$ *does not contain other eigenvalues of* (8) *different from* λ^0.

In addition, there are $l_0 N(\varepsilon)$ *functions,* $\{U_{k,p}^\varepsilon\}_{k=1,N(\varepsilon)}^{p=1,l_0}$, $U_{k,p}^\varepsilon \in \widetilde{\mathbf{V}}^\varepsilon$, *such that* $\|U_{k,p}^\varepsilon\|_{\widetilde{\mathbf{V}}^\varepsilon} = 1$, $U_{k,p}^\varepsilon$ *belongs to the eigenspace associated with all the eigenvalues in* $[\lambda^0 - d^\varepsilon, \lambda^0 + d^\varepsilon]$, *and*

$$\left\| U_{k,p}^\varepsilon - Z_{k,p}^\varepsilon \right\|_{\widetilde{\mathbf{V}}^\varepsilon} \leqslant 2(\mathrm{o}_{2,\varepsilon})^{1-\beta} \tag{18}$$

In (18), β *is a constant* $0 < \beta < 1$, $\mathrm{o}_{2,\varepsilon} \xrightarrow{\varepsilon \to 0} 0$, $\mathrm{o}_{2,\varepsilon}$ *is given by* $\mathrm{o}_{2,\varepsilon} = \mathrm{o}_{1,\varepsilon}$ *in* (17) *when* $n = 2$ *and by* $\mathrm{o}_{2,\varepsilon} = C\max\{(\varepsilon/\eta)^{1/2}, \varepsilon^{m-2}\}$ *when* $n = 3$, $Z_{k,p}^\varepsilon$ *is defined by* (16) *and* $\tilde{\varphi}^\varepsilon(y)$ *is defined by* (14) *when* $n = 2$ *and by* (15) *when* $n = 3$. *These functions,* $\{U_{k,p}^\varepsilon\}_{k=1,N(\varepsilon)}^{p=1,l_0}$, *are such that for any extracted subset of* K *functions* $\{U_{j_1}^\varepsilon, U_{j_2}^\varepsilon, \ldots, U_{j_K}^\varepsilon\}$, *they are linearly independent.*

We refer to [15,21] for the the proof of Theorem 3, as well as for certain restrictions on ε and η when $\alpha = +\infty$. Here, we just outline that the proof of Theorem 3 is based on the application of Lemma 1 below and the orthogonality conditions (17) for the eigenfunctions.

It is worth stating the following basic result in spectral perturbation theory (cf. [35] and Section III.1 in [9]) on *almost eigenfunctions*:

Lemma 1. *Let $\mathcal{A}:\mathbf{H}\to\mathbf{H}$ be a self-adjoint positive and compact operator in a Hilbert space $\mathbf{H}$. Let $u\in\mathbf{H}$, with $\|u\|_{\mathbf{H}}=1$ and $\lambda, r>0$ such that $\|\mathcal{A}u-\lambda u\|_{\mathbf{H}}<r$. Then, there is an eigenvalue λ_i of $\mathcal{A}$ satisfying $|\lambda-\lambda_i|<r$. Moreover, for any $r^*>r$ there is $u^*\in\mathbf{H}$ with $\|u^*\|_{\mathbf{H}}=1$ such that*

$$\|u-u^*\|_{\mathbf{H}}<\frac{2r}{r^*}$$

u^* belonging to the eigenspace associated with all the eigenvalues of the operator $\mathcal{A}$ lying on the segment $[\lambda-r^*,\lambda+r^*]$.

Let us observe that, in general, even though the quantities r and $2r/r^*$ in the statement of Lemma 1 are sufficiently small, we cannot assert that the function u approaches a true eigenfunction of operator $\mathcal{A}$ associated with λ, but a linear combination of eigenfunctions u^* associated with all the eigenvalues in $[\lambda-r^*,\lambda+r^*]$. The function u is the so-called *almost eigenfunction* or *quasimode* (cf. [36,37]). In order to obtain more precise results on the approach of the eigenfunctions, other spectral properties for operator $\mathcal{A}$ should be known (cf. [37,24]). This is the reason why Lemma 1 is used as an intermediate step to prove the stronger spectral perturbation theorem that provide the convergence of the whole spectrum as mentioned in Section 3.1 (cf. Section III.1 of [9]). In general, Lemma 1 provides useful information on the spectrum of ε dependent operators $\mathcal{A}^\varepsilon$ and on the corresponding eigenfunctions in the case where spectral theorems that guarantee the convergence of the whole spectrum do not work. Besides, the approach to the eigenfunctions is usually stronger than the approach provided by the convergence of the ε dependent corresponding spectral families (cf. [32,24,17]).

Let us also notice that the construction of almost eigenfunctions has been widely used in the literature either to approach true eigenfunctions or to detect points of the essential spectrum of a self-adjoint operator: let us mention Section IV in [38] and Section IV.3 in [7] as general references.

On the other hand, the result in Lemma 1 should also be completed with other results which provide information on the total number of eigenvalues of operator $\mathcal{A}$ in the interval $[\lambda^*-r,\lambda^*+r]$. We refer to Section IV.2.3 of [37] for general results on *orthogonal families of quasimodes*, as well as for references, and to Section VII.1 of [39] for certain useful algebraic results.

As a matter of fact, we observe that the assertion in Theorem 3 on the total multiplicity of the eigenvalues in the interval $[\lambda^0-d^\varepsilon,\lambda^0+d^\varepsilon]$ cannot be improved by using the general results in [37,39]. Instead the minimax principle and certain properties of the harmonic functions in a half-plane with a finite energy allow the results to be improved as stated in the following theorems (cf. [21] and Sections II.2 in [40]).

Theorem 4. *Let n be $n=2$ or $n=3$. Let λ_1^ε and λ_1^0 be the first eigenvalues of* (5) *and* (8) *respectively. Then, there exist a constant $\lambda^*\leqslant\lambda_1^0$ and a sequence $\mathrm{o}_{3,\varepsilon}\to 0$, as $\varepsilon\to 0$, such that $\lambda^*\leqslant\lambda_1^\varepsilon/\varepsilon^{m-2}\leqslant\lambda_1^0+\mathrm{o}_{3,\varepsilon}$.*

Theorem 5. *Let n be $n=2$. Then, for each fixed $i=1,2,3,\ldots,$ the sequence $\lambda_i^\varepsilon/\varepsilon^{m-2}$ converge towards the first eigenvalue λ_1^0 of the local problem* (8)*.*

Theorem 4 shows that for $n=2,3$ the limit of any converging subsequence of $\lambda_1^\varepsilon/\varepsilon^{m-2}$ is bounded by the first eigenvalue λ_1^0 of the local problem (8). Therefore, in the case where $\lambda_1^\varepsilon/\varepsilon^{m-2}\to\lambda_1^0$, as $\varepsilon\to 0$, Theorem 2 ensures the convergence $\lambda_i^\varepsilon/\varepsilon^{m-2}\to\lambda_1^0$, as $\varepsilon\to 0$, for any fixed $i=1,2,\ldots.$ But this result, stated in Theorem 5, can only be proved for the dimension $n=2$. For the dimension $n=3$ there are other accumulation points of subsequences of $\lambda_1^\varepsilon/\varepsilon^{m-2}$ which are smaller than λ_1^0 as we outline in Section 4.

4. The case of many concentrated masses: global vibrations

Throughout this section we consider $N(\varepsilon) \to \infty$, α in (4) strictly positive and $n = 3$ and we make it clear that the results for the local vibrations in Section 3.2 are sharp. Indeed, Theorems 6 and 7 in this section show that for the dimension $n = 3$ other accumulation points of $\lambda_i^\varepsilon/\varepsilon^{m-2}$ smaller than λ_1^0, the first eigenvalue of the local problem (8), can exist. In order to state this, we introduce both a new function F and a Steklov eigenvalue problem.

When λ is not an eigenvalue of the local problem (8), we define the function $F(\lambda)$ as

$$F(\lambda) = -\left\langle \frac{\partial V^\lambda}{\partial n_y}\bigg|_\Gamma, 1 \right\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \tag{19}$$

where $V^\lambda - W$ is the solution of the nonhomogeneous problem associated with (8) when the equation in B is replaced by $-\Delta V = \lambda V + \lambda W$ in B, and W is the solution of the following local problem (cf. [25] for this solution):

$$\begin{cases} -\Delta_y W = 0 \quad \text{in } \mathbb{R}^{3-} \\ W = 0 \quad \text{on } T \\ \dfrac{\partial W}{\partial y_3} = 0 \quad \text{on } \{y_3 = 0\} - \overline{T} \\ W(y) \to 1, \quad \text{as } |y| \to \infty,\ y_3 < 0 \end{cases} \tag{20}$$

Let us also consider the Steklov eigenvalue problem, obtained from the homogenization of the concentrated masses:

$$\begin{cases} -\Delta u = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \Gamma_\Omega \\ \dfrac{\partial u}{\partial n} = \alpha\mu u \quad \text{on } \Sigma \end{cases} \tag{21}$$

with eigenvalues $\{\mu_k\}_{k=1}^\infty$, $0 < \mu_k \to \infty$ as $k \to \infty$ (cf. [12]).

Theorem 6. *The function $F(\lambda)$ defined by* (19) *is a meromorphic function with positive real poles $\{\lambda_i^0\}_{i=1}^\infty$, the eigenvalues of* (8). *Moreover, $F(\lambda)$ is negative for negative λ; and for each $i = 1, 2, \ldots,$ and real λ, it satisfies*:

$$\lim_{\lambda\to\lambda_i^{0+}} F(\lambda) = -\infty \quad \textit{and} \quad \lim_{\lambda\to\lambda_i^{0-}} F(\lambda) = +\infty$$

Consequently, for each eigenvalue μ_k of the Steklov problem (21), *the equation*

$$F(\lambda) = \mu_k \tag{22}$$

has infinitely many positive roots. Besides, in the case where the solution of (20), *W, is not orthogonal in $L^2(B)$ to the eigenspace associated with λ_1^0, there are infinitely many roots of* (22) *strictly smaller than λ_1^0.*

Theorem 6 has been proved in [12] on account of the properties of the resolvent operator associated with (8).

Theorem 7. *Each root λ^* of* (22), *λ^* such that $F'(\lambda^*) \neq 0$, is an accumulation point of eigenvalues $\lambda_{i(\varepsilon)}^\varepsilon/\varepsilon^{m-2}$. Besides if $(\lambda^\varepsilon/\varepsilon^{m-2}) \to \lambda^*$, as $\varepsilon \to 0$, and the corresponding eigenfunctions u^ε converge towards u^* in $H^1(\Omega)$ weakly, once assumed that λ^* is not an eigenvalue of* (8) *and $u^* \neq 0$, then $(u^*, F(\lambda^*))$ is an eigenelement of* (21).

The proof of Theorem 7 is in [12]: this proof involves the Energy Method for stationary boundary homogenization problems, the Laplace Transform, the Fourier Transform and results on boundary values of analytic functions to connect both transforms. We refer to [26,27] and [12–14] for the application of the Energy Method to this kind of boundary homogenization problems; see [41] for useful results on boundary values of analytic functions, when applying convergence properties for the perturbed spectral families (cf. [24] for a different problem).

Let us observe that for $n=2$ or $n=3$, if we consider the eigenfunctions u^ε of (5) of norm 1 in $H^1(\Omega)$, and assume that certain sequences $\lambda^\varepsilon/\varepsilon^{m-2}$ converge towards some $\lambda^*>0$ and the corresponding eigenfunctions converge towards some u^* weakly in $H^1(\Omega)$, as $\varepsilon\to 0$, then $u^*\equiv 0$ except for the case where $n=3$ and $\alpha>0$ (see (4)). In that case, some results on the approach to the eigenfunctions u^* of (21) that complement Theorem 7 should be obtained (see Theorems 1–3 to compare). As a matter of fact, we also notice that among all the cases considered, this case, $n=3$ and $\alpha>0$, is the only where the total weight of the concentrated masses is of order $\mathrm{O}(\varepsilon^{2-m})$.

5. High frequencies and local problem

Global vibrations for the dimensions $n=2$ and $n=3$ have been found associated with the eigenvalues λ^ε of order $\mathrm{O}(1)$ of (5) (see [12–14] for the *extreme cases* $\alpha=0$ and $\alpha=\infty$, and [17] for $\alpha>0$). On account of (7), the frequencies $\lambda^\varepsilon_{i(\varepsilon)}=\mathrm{O}(1)$, for $i(\varepsilon)\to\infty$ as $\varepsilon\to 0$, are referred to as the *high frequencies*; they are related with the *homogenized problems* which depend on the relation (4) between ε and η:

- The Robin type problem, for the *critical size* of the masses B^ε, $\alpha>0$,

$$\begin{cases} -\Delta u=\lambda u \quad \text{in } \Omega\\ u=0 \quad \text{on } \Gamma_\Omega\\ \dfrac{\partial u}{\partial n}=-\alpha C u \quad \text{on } \Sigma \end{cases} \tag{23}$$

  where constant C takes the value $C=S_n/2$ with S_n the surface of the unit sphere in $\mathbb{R}^n$: $C=\pi$ when $n=2$, and $C=2\pi$ when $n=3$.
- The mixed problem (23) for the extreme case $\alpha=0$; the condition on Σ reads: $\partial u/\partial n=0$.
- The Dirichlet problem for the extreme case $\alpha=\infty$; the condition on Σ being $u=0$.

Also, for $\alpha=0$, problem (23) is associated with the global vibrations of a vibrating system with only one single concentrated mass or a fixed number, i.e., $N(\varepsilon)=N$. For problem (1), with one single concentrated mass inside Ω, the boundary conditions in (23) read $u=0$ on $\partial\Omega$ (see Section VII.11 in [7], Section III.5 in [9], [8,16]). As is well known, problem (23) has a pure point spectrum.

It has been proved in [17] (cf. [16] for $N=1$) that *the high frequencies accumulate in* $(0,\infty)$ and not only at the points of the spectrum of the homogenized problem as one might think from the results in [12–14]. Moreover, these values, the eigenvalues of the homogenized problem, are singled out from the others depending on the asymptotic behavior of the corresponding eigenfunctions. Roughly speaking, we can assert that *only the eigenfunctions u^ε associated with eigenvalues λ^ε asymptotically near an eigenvalue of the homogenized problem* (23) *are asymptotically different from zero*; these eigenfunctions u^ε are approached by the eigenfunctions of (23). Of course, the suitable normalization of u^ε in $H^1(\Omega)$ has been chosen in order to prove these results (see Theorems 1 and 7 to compare).

It is worth mentioning that the homogenized problem (23) is obtained from the boundary homogenization as if the concentrated masses do not exist. Indeed, the boundary condition on Σ in (23) is obtained from the homogenization of a Dirichlet condition on $\bigcup\Gamma^\varepsilon$, which would happen if the eigenfunctions u^ε vanished inside B^ε. Bearing this in mind, some correcting terms for the eigenfunctions associated with the high frequencies are given in [17] (cf. [16] for $N=1$ and B^ε inside Ω). These correcting terms, $u^0(w^\varepsilon-1)$, are obtained by means of classical asymptotic expansions in boundary homogenization problems, and allow us to assert that the approach of u^ε by 0 inside the concentrated masses is better than the approach by u^0:

$$\|u^\varepsilon - u^0 w^\varepsilon\|_{H^1(\Omega)} \xrightarrow{\varepsilon\to 0} 0$$

Here u^0 is the eigenfunction of (23) associated with λ^0, $\lambda^\varepsilon \to \lambda^0$ as $\varepsilon \to 0$, and w^ε are the classical test functions appearing in boundary homogenization problems: they are positive smooth functions which take the value 0 in B^ε and the value 1 in $|x| \geqslant \eta/2$; they are extended by periodicity to to all the centers of the half-circles $\tilde{x}_k$. Let us refer to [28,34,26,27,29] for the explicit construction of w^ε when B is a ball or half-ball of $\mathbb{R}^n$, or it has another shape, as well as for the homogenized problems for other geometries of B and Ω and for other references.

Nevertheless, instead of vanishing in B^ε, u^ε can be strongly oscillating functions inside B^ε (cf. Section VII.10 in [7] and [16]) or concentrate in a neighborhood of the interface Γ^ε (cf. [18]). In fact, in [16,18], for B a circle and $n=2$, the concentrated mass being inside Ω (see (1) and (10)), we provide correcting terms for the eigenfunctions u^ε associated with the high frequencies which take into account the wavelength of the corresponding vibrations. The computations can also be performed with minor modifications for B a half-circle, the concentrated mass being near the boundary. These correcting terms are constructed from the eigenfunctions of the local problem (8) ((10), resp.) associated with the high frequencies of (8) ((10), resp.), as we outline here below.

We consider $\lambda^\varepsilon = \lambda^\varepsilon_{i(\varepsilon)} = \mathrm{O}(1)$, λ^ε converging towards λ^0 and the corresponding eigenfunctions u^ε converging towards u^0 weakly in $H^1(\Omega)$ as $\varepsilon \to 0$. It is known (cf. [17]) that λ^0 can be any positive number and, in the case where $u^0 \neq 0$, then, (λ^0, u^0) is an eigenelement of (23).

Using the asymptotic expansions in [16,18], we show that an alternative approach to $u^0 w^\varepsilon$ for the eigenfunctions u^ε inside each concentrated mass B^ε_k is given by:

$$u^\varepsilon \approx V^\varepsilon\left(\frac{x-\tilde{x}_k}{\varepsilon}\right)u^0(x) \quad \text{when } u^0(\tilde{x}_k) \neq 0, \qquad u^\varepsilon \approx V^\varepsilon\left(\frac{x-\tilde{x}_k}{\varepsilon}\right) \quad \text{when } u^0(\tilde{x}_k) = 0,\ u^0 \not\equiv 0 \tag{24}$$

and

$$u^\varepsilon \approx V^\varepsilon\left(\frac{x-\tilde{x}_k}{\varepsilon}\right) \quad \text{when } u^0 \equiv 0 \tag{25}$$

where u^0 in (24) is an eigenfunction associated with the eigenvalue λ^0 of problem (23) and V^ε satisfies (8) for $\lambda = \lambda^0/\varepsilon^{m-2}$.

More precisely, in (24), (25), V^ε is a solution of (8) ((10), resp.) where the equation in B is replaced by $-\Delta_y V^\varepsilon = (\lambda^0/\varepsilon^{m-2})V^\varepsilon$ in B, and V^ε converging towards some constant c when $|y| \to \infty$ which proves to be 0 or 1 depending on whether $u^0(\tilde{x}_k)$ be equal to zero or different from zero. Therefore, for $\lambda = \lambda^0/\varepsilon^{m-2}$, we have the eigenvalue problem (8) ((10), resp.) where we chose a certain normalization of the eigenfunctions in the case where $c = 1$. Besides, the sequence ε must be chosen such that $\lambda = \lambda^0/\varepsilon^{m-2}$ be an eigenvalue (8): that is, ε varies in very particular subsequences.

That the approaches (24), when $u^0(0) \neq 0$, and (25) improve the approaches through 0 inside the concentrated masses has been proved for the case of problem (1), where $\tilde{x}_k = 0$, $n = 2$ and B is a circle (cf. [16,18]). This shows the oscillatory behavior of the eigenfunctions inside B^ε. The proofs are performed by means of explicit computations and using Lemma 1. Explicit computations can also be extended to the case of B a half-circle and (25) can be justified in this way. Instead, it is still an open problem that (24), when $u^0(0) = 0$, provide a true correcting term for the case of one single concentrated mass in (1). Also, for $N = 1$ and $n = 3$, they are open problems justifying (24), (25).

Finally, let us observe that all must be done for the case where $n = 2, 3$ and $N(\varepsilon) \to \infty$. In this case, it seems as if global correcting terms in Ω are obtained from (25); formally:

$$u^\varepsilon \approx V^\varepsilon\left(\frac{x}{\varepsilon}\right)\big(1 - w^\varepsilon(x)\big) \quad \text{in } \Omega,\ \text{when } u^0 \equiv 0 \tag{26}$$

for $\lambda^\varepsilon \approx \lambda^0$ and λ^0 not an eigenvalue of (23). w^ε, $V^\varepsilon(y)$ and ε as stated above, and $(1 - w^\varepsilon(x))V^\varepsilon(x/\varepsilon)$ has been extended by periodicity to all the centers $\tilde{x}_k$ of the half-circles.

Thus, we emphasize that an analysis of the structure of the eigenfunctions of (8) ((10), resp.) associated with very large eigenvalues λ must be performed in order to obtain more information on the structure of u^ε in (24)–(26).

Acknowledgements

This work has been partially supported by the DGES: BMF 2001-1266.

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