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On the geometrically exact formulation of structural mechanics and its applications to dynamics, control and optimization

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Abstract

In this survey paper we re-examine the theoretical formulation of structural mechanics, introducing no restrictions with respect to the size of displacements, rotations or deformations, which is commonly referred to as geometrically exact. A special attention is given to clarifying the computational aspects of finite rotations as the key ingredient of any such formulation. We briefly discuss several novel applications of the geometrically exact formulation to dynamics, control and optimization. **To cite this article:** *A. Ibrahimbegovic, C. R. Mecanique 331 (2003).*

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Résumé

Sur la formulation géométriquement exacte de la mécanique des structures et ses applications en dynamique, contrôle et optimisation. Dans cet article nous réexaminons la formulation théorique de la mécanique des structures n'imposant aucune restriction sur la grandeur des déplacements, des rotations ou des déformations, qui est alors dite géométriquement exacte. Une attention particulière est portée aux aspects du calcul pertinents aux rotations finies, dont la maîtrise représente un élément clé pour toute formulation de ce type. Nous présentons brièvement quelques applications nouvelles de la théorie géométriquement exacte en dynamique, en contrôle et en optimisation. **Pour citer cet article :** *A. Ibrahimbegovic, C. R. Mecanique 331 (2003).*

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Mots-clés : Solides et structures ; Rotation finie ; Dynamique ; Contrôle ; Optimisation

1. Introduction

The model which was long considered the 'bread-and-butter' of a structural engineer – the Euler–Bernoulli beam theory – is even nowadays (e.g., [1]) introduced within the framework of geometrically linear theory, limited to small or rather infinitesimal displacements, rotations and deformations. It seems it is long forgotten in the mechanics community that the original developments of beam model of 'Euler elastica' were indeed presented in

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geometrically nonlinear setting with no restriction on the size of deformation, other than suppressing extension and shear. The same kind of impression on a very limited capabilities of developing the geometrically nonlinear structural mechanics theories is confirmed for other commonly used models, such as plates and shells, where one in general uses either an updated Lagrangian formulation (e.g., [2]) or co-rotational formulation (e.g., [3]), with either one limited to moderate rotations of a moving reference frame and only small strains with respect to such a frame.

It is only with a more recent work of Reissner [4], on beam theory capable of dealing with arbitrary large displacements and deformations and moderate rotations, that interest was spurred again in truly geometrically nonlinear models. First, a finite rotation extension of the model of this kind for initially straight beams was presented by Simo [5], who also coined the label ‘geometrically exact theory’. Subsequently, it was shown how to generalize a geometrically exact theory to space-curved beam [6], shells with drilling rotations [7] and 3D solids with independent rotation field [8], thus providing a unified basis for constructing the structural mechanics model for a structure of arbitrary complexity by using the model components which all share the same configuration space.

An essential ingredient of any such theoretical formulation in structural mechanics pertains to our ability to account for 3D finite (unrestricted-in-size) rotations. In this respect, although a number of pertinent theoretical results have been available ever since the pioneering works of Euler, Hamilton and Rodrigues, it is only with a seminal work of Argyris [9] that the computational aspects have been re-examined. Some of the recent developments of this kind, focusing on the optimal choice of rotation parameters and the related computational procedure, are given in [10–17], among others.

Current developments in geometrically exact formulations in structural mechanics are very much motivated by potential industrial applications. Some of the fields which benefitted considerably from developments of geometrically exact structural mechanics theories are: multibody system dynamics (e.g., [18]), where one can now easily account for flexible multibody systems or their components, control of motion stability (e.g., [19]), which allows one to apply the qualitative methods of Poincaré and Lyapunov to real engineering structures; or yet the shape optimization of structures undergoing finite rotations, which permits one to develop a novel and potentially more efficient solution procedure for optimization problems [20].

The outline of this survey paper is as follows. In the next section we briefly present a couple of model problems of developing the geometrically exact formulation, the first one for a 3D solid with independent rotation field and the second one for a 3D beam. Several important computational aspects of 3D finite rotations are also examined in that section. Some new developments in geometrically exact structural mechanics theories are presented in Section 3 in application to dynamics, control and optimization. Closing remarks are stated in Section 4.

2. Geometrically exact structural mechanics theory

In order to present the main features of the geometrically exact formulation in structural mechanics, we choose two model problems: a 3D solid with independent rotation field [8] and the 3D beam of [5,6]. Without loss of generality we can place the developments to follow in the framework of the Euclidean space, referring the reader to works of Marsden and Hughes [21] for a more general framework of manifolds, or a more traditional presentation of the same approach in classical works in [22–24]. However, our choice does not imply that we will ignore the difference between a tensor and its coordinate representation in terms of a matrix (as in [25]), nor shall we spare any effort to elaborate upon the very important difference between the material and the spatial representations of structural mechanics tensor fields.

2.1. 3D solid with independent rotation field

If we consider a 3D solid as an assembly of particles with each one identified by its position vector in either initial \mathcal{B} or deformed configuration \mathcal{S} , we denote deformation as

$$\mathbf{x} \mapsto \boldsymbol{\varphi}(\mathbf{x}) \in \mathcal{S}; \quad \forall \mathbf{x} \in \mathcal{B} \quad (1)$$

Remark 1. In the finite element incremental solution procedure, which is often the only one capable of solving a problem described by a geometrically exact formulation, one can choose so called isoparametric interpolations (e.g., [26]) by using the same shape functions to describe the element geometry and the incremental displacement field, which allows one to construct very easily any deformed configuration with

$$\left. \begin{aligned} \mathbf{x}(\xi) &= \sum_{a=1}^{n_{el}} N_a(\xi) \mathbf{x}_a \\ \mathbf{u}(\xi) &= \sum_{a=1}^{n_{el}} N_a(\xi) \mathbf{u}_a \end{aligned} \right\} \Rightarrow \boldsymbol{\varphi}(\xi) = \sum_{a=1}^{n_{el}} N_a(\xi) (\mathbf{x}_a + \mathbf{u}_a) \tag{2}$$

The isoparametric coordinates are thus equivalent to convected coordinates used in the classical works on finite elasticity (e.g., [22–24]).

In a large overall motion any infinitesimal vector emanating from \mathbf{x} is taken by the deformation gradient tensor \mathbf{F} into its new position according to

$$d\mathbf{x} \mapsto \mathbf{F} d\mathbf{x} \in T_{\varphi(x)}\mathcal{S}; \quad \forall d\mathbf{x} \in T_x\mathcal{B}; \quad \mathbf{F} = \nabla\boldsymbol{\varphi}; \quad \det[\mathbf{F}] > 0 \tag{3}$$

where $T(\cdot)$ denotes the tangent space. The deformation gradient is thus a two-point tensor operating on a vector (in the tangent space) in the initial configuration to produce its image in the current configuration. For the case when only deformation (the change of the magnitude of $\|d\mathbf{x}\|$ in the deformed configuration) is of interest, one can appeal to the polar decomposition of the deformation gradient separating rotation, expressed with an orthogonal tensor \mathbf{R} , from deformation, represented with either the right stretch \mathbf{U} or the left stretch tensor $\boldsymbol{\Upsilon}$

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \boldsymbol{\Upsilon}\mathbf{R} \tag{4}$$

By considering that in each of two forms of the polar decomposition the rotation tensor is a two-point tensor, it thus follows that the right stretch tensor is a material strain measure ($\mathbf{U} \in T_x\mathcal{B}$), whereas the left stretch tensor is a spatial deformation measure ($\boldsymbol{\Upsilon} \in T_{\varphi(x)}\mathcal{S}$). By appealing to the orthogonality of the rotation tensor ($\mathbf{R}^{-1} = \mathbf{R}^T$) one can recover yet another strain measure in material representation in terms of the Biot strain [8] according to

$$\mathbf{H} = \mathbf{R}^T\mathbf{F} - \mathbf{I} \in T_x\mathcal{B} \tag{5}$$

The Biot strain measure provides the basis for constructing the geometrically exact formulation of a 3D solid with independent rotation field which share the same configuration space with geometrically exact theories of structures (see [6–8]).

The second ingredient of the geometrically exact theory of 3D elasticity concerns the set of equilibrium equations. In that sense, the spatial description of the strong form of equilibrium equations, featuring true or Cauchy stress tensor $\boldsymbol{\sigma}$ or its natural replacement the Kirchhoff stress tensor $\boldsymbol{\tau} = (J \circ \boldsymbol{\varphi}^{-1})\boldsymbol{\sigma}$, is abandoned in favor of more convenient, material description which is constructed by using either the first Piola–Kirchhoff stress tensor \mathbf{P} or yet the Biot stress tensor \mathbf{T} . Thus by comparing the spatial and the material description of the Cauchy principle and by exploiting the Nanson formula on mapping an infinitesimal surface element into its deformed configuration with $\mathbf{n} \mapsto J\mathbf{F}^{-T}\mathbf{n}$, we can obtain that

$$\boldsymbol{\tau} \circ \boldsymbol{\varphi} = \mathbf{P}\mathbf{F}^T = \mathbf{R}\mathbf{T}\mathbf{F}^T \tag{6}$$

The last result allows us to write the material representation of the strong form of equilibrium equation according to

$$\text{div}[\mathbf{RT}] + \mathbf{b} = \mathbf{0}; \quad \text{skew}[\mathbf{RTF}^T] = \mathbf{0} \tag{7}$$

where \mathbf{b} is the body force. In pursuing the finite element-based numerical solution procedure one replaces (7) by the weak form of the equilibrium equations by appealing to the virtual power principle (e.g., see [22]) to obtain

$$\dot{P}_{\text{ext}} = \dot{P}_{\text{int}} \Leftrightarrow \int_{\mathcal{B}} \dot{\mathbf{H}} \cdot \text{sym}[\mathbf{T}] dV = \int_{\mathcal{B}} \dot{\boldsymbol{\varphi}} \cdot \mathbf{b} dV + \int_{\partial\mathcal{B}_t} \dot{\boldsymbol{\varphi}} \cdot \bar{\mathbf{t}} dA \tag{8}$$

In a geometrically exact formulation of this kind all these equations are exact for any size of displacement or rotation. The approximations are introduced only at the later stage, either at the level of constitutive equations (e.g., by connecting the Biot stress and strain tensors by Hook’s law for so-called semi-linear material), or by constructing a finite element discrete approximation.

2.2. 3D beam and finite rotations

For a 3D solid body of a beam-like shape with one dimension (beam length) being considerably larger from the other two (beam cross-section), we can develop essentially 1D parameterization of the geometrically exact equilibrium equations. This kind of geometrically exact model is first proposed by Reissner [4], extended to handle finite rotations by Simo [5] and arbitrary space curved beams by Ibrahimbegovic [6]. A number of later works rederived the same model; see for example [3,10,12,27–30], among others. When seeking a beam-like solution one can further specialize the previously presented geometrically exact formulation by appealing to the hypothesis of non-deformable cross-sections. This allows us to express the motion of any point in the cross-section with respect to the moving frame which remains attached to the reference point on the beam axis, which provides a one-dimensional model of the beam (see Fig. 1).

By taking into account that the placement of a local Cartesian frame is governed by the orthogonal tensor

$$\mathbf{a}_i = \mathbf{\Lambda} \mathbf{g}_i \Rightarrow \mathbf{\Lambda} = \mathbf{a}_i \otimes \mathbf{g}_i; \|\mathbf{a}_i\| = \|\mathbf{g}_i\| = 1 \Rightarrow \mathbf{\Lambda}^T \mathbf{\Lambda} = \mathbf{\Lambda} \mathbf{\Lambda}^T = \mathbf{I} \tag{9}$$

we can conclude that the configuration space is a set of position vectors and orthogonal tensors, parameterized over 1D domain

$$\xi \mapsto \{\varphi(\xi), \mathbf{\Lambda}(\xi)\} \tag{10}$$

The major difficulty in dealing with the finite element implementation of this kind of geometrically exact formulation pertains to the discrete approximation of the orthogonal tensor of finite rotations. Namely, by simply interpolating the nodal values of rotation tensor with finite element shape functions, it is not possible to preserve the orthogonality property of the rotation tensor everywhere along the beam, which is very important for assuring the frame-invariance requirements (e.g., see [28]). One can therefore choose either to satisfy the frame-invariance only at the Gauss quadrature points which are used for computing the finite element arrays (e.g., see [31]) or alternatively to appeal to a vector-like representation of finite rotations [14,15]; the latter implies first using the finite element interpolation or rotation vector $\boldsymbol{\theta}(\xi)$ and computing subsequently the corresponding orthogonal tensor at any point where it is needed by the applying the exponential mapping or Rodrigues formula (e.g., see [9]) to obtain

$$\mathbf{\Lambda} = \cos \theta \mathbf{I} + \frac{\sin \theta}{\theta} \boldsymbol{\Theta} + \frac{1 - \cos \theta}{\theta^2} \boldsymbol{\theta} \otimes \boldsymbol{\theta} =: \exp[\boldsymbol{\Theta}]; \boldsymbol{\Theta} \mathbf{v} = \boldsymbol{\theta} \times \mathbf{v}; \forall \mathbf{v} \in \mathbb{R}^3 \tag{11}$$

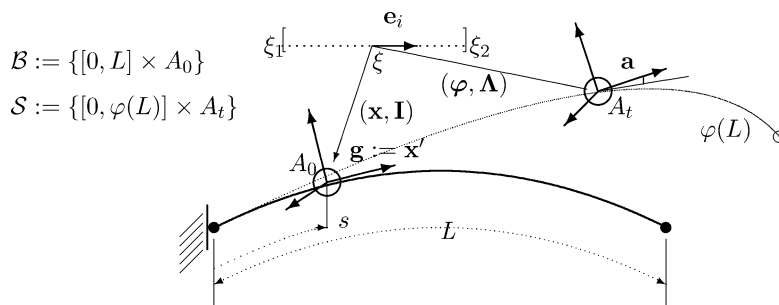


Fig. 1. Initial and deformed configuration of 3D beam.

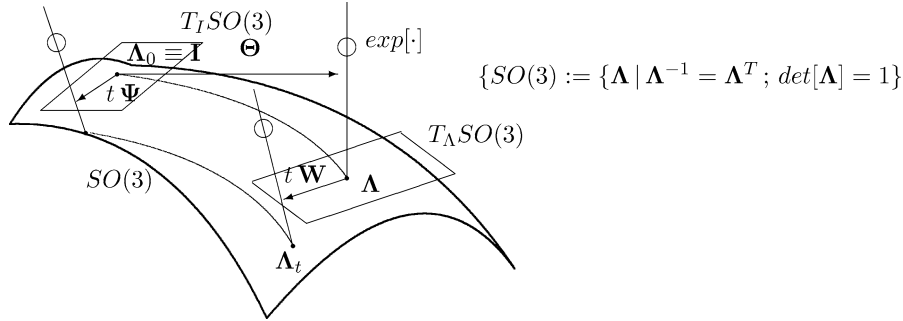


Fig. 2. Representation of $SO(3)$ manifold and its tangent spaces in initial and current configurations.

Another use of the exponential mapping, as illustrated in Fig. 2, pertains to constructing a kinematically admissible variation of the finite rotation tensor Λ_t , featuring in the virtual power principle, where an infinitesimal virtual rotation, represented by a skew-symmetric tensor ${}^t\mathbf{W}$, should be superposed on the existing finite rotation tensor Λ resulting with

$$\Lambda_t = \exp[{}^t\delta\mathbf{W}]\Lambda \Rightarrow \mathbf{W} = \frac{d}{dt}[\Lambda_t] \Big|_{t=0} \Lambda^T; \mathbf{W}\mathbf{v} = \mathbf{w} \times \mathbf{v}; \forall \mathbf{v} \in \mathbb{R}^3 \tag{12}$$

We note in passing that the finite rotation Λ is a two-point tensor, taking a vector from (the tangent space in) the initial configuration to (the tangent space in) the current configuration. One can thus also provide the material representation of the kinematically admissible variation of the finite rotation tensor in terms of the skew-symmetric tensor Ψ according to

$$\begin{aligned} \Psi &= \Lambda^T \mathbf{W} \Lambda \Leftrightarrow \mathbf{W} = \Lambda \Psi \Lambda^T \Rightarrow \Lambda_t = \Lambda \exp[{}^t\Psi] \\ \psi &= \Lambda^T \mathbf{w} \Leftrightarrow \mathbf{w} = \Lambda \psi; \Psi \mathbf{v} = \psi \times \mathbf{v}; \forall \mathbf{v} \in \mathbb{R}^3 \end{aligned} \tag{13}$$

Yet another form of the kinematically admissible finite rotation tensor can be obtained by making use of the admissible variation of rotation vector $\dot{\theta}$ leading to the following result

$$\mathbf{w} = \mathbf{T}\dot{\theta}; \quad \mathbf{T} = \frac{\sin\theta}{\theta} \mathbf{I} + \frac{1 - \cos\theta}{\theta^2} \boldsymbol{\Theta} + \frac{\theta - \sin\theta}{\theta^3} \boldsymbol{\theta} \otimes \boldsymbol{\theta} \tag{14}$$

The geometrically exact formulation of 3D beam model makes use of the exact equilibrium equations (e.g., [4–6])

$$\mathbf{n}' + \mathbf{f} = \mathbf{0}; \quad \mathbf{m}' + \boldsymbol{\varphi}' \times \mathbf{n} = \mathbf{0}; \quad (\cdot)' = \frac{\partial}{\partial s}(\cdot) \tag{15}$$

where \mathbf{f} are external forces whereas \mathbf{n} and \mathbf{m} are internal forces and couples. The latter can be represented in terms of integrals of the Biot stress over the cross-section (see [7]). For starting point of the discrete approximation, we will rather choose the weak form of exact equilibrium equations or virtual power principle, which can be expressed as

$$G(\boldsymbol{\varphi}, \Lambda; \mathbf{v}, \mathbf{w}) := \int_L (\mathcal{L}_w(\boldsymbol{\varepsilon}) \cdot \mathbf{n} + \mathcal{L}_w(\boldsymbol{\kappa}) \cdot \mathbf{m}) ds - G_{\text{ext}}(\mathbf{v}, \mathbf{w}) = \mathbf{0} \tag{16}$$

where $\mathcal{L}_w(\cdot)$ denotes the Lie derivative formalism (e.g., see [21]), with pull-back and push-forward carried out by the operator which controls the motion of the cross-section, namely the rotation tensor Λ . For the exact finite strain measures of the geometrically exact beam theory one thus obtains

$$\begin{aligned} \boldsymbol{\varepsilon} = \boldsymbol{\varphi}' - \mathbf{a} &\Rightarrow \mathcal{L}_w(\boldsymbol{\varepsilon}) := \Lambda \frac{\partial}{\partial t} [\Lambda_t^T \boldsymbol{\varepsilon}_t] \Big|_{t=0} = \dot{\boldsymbol{\varphi}}' - \mathbf{w} \times \boldsymbol{\varphi}' \\ \boldsymbol{\kappa} = \boldsymbol{\omega}; \quad \boldsymbol{\Omega}\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}; \quad \forall \mathbf{v} \in \mathbb{R}^3 &\Rightarrow \mathcal{L}_w(\boldsymbol{\kappa}) := \Lambda \frac{\partial}{\partial t} [\Lambda_t^T \boldsymbol{\kappa}_t] \Big|_{t=0} = \dot{\boldsymbol{\omega}} - \mathbf{w} \times \boldsymbol{\omega} \equiv \mathbf{w}' \end{aligned} \tag{17}$$

3. Applications

In this section we briefly discuss several subsequent developments of geometrically exact formulation of structural mechanics in application to problems in dynamics, control and optimization. In each of these domains the geometrically exact formulations requires a novel solution approach but in return it provides extended modelling capabilities beyond the reach of the traditional methods.

3.1. Dynamics

The geometrically exact formulation of structural mechanics can easily be extended to dynamics, to deal with the cases when the rate of external loading is sufficiently important so that one is no longer allowed to ignore the inertia effects. In that respect, the main advantage of the geometrically exact formulation pertains to a simple, quadratic form one can retain for the kinetic energy even in a large overall motion when a fixed, inertia frame is chosen for that purpose. The latter is in sharp contrast to traditional approaches to flexible body dynamics (e.g., [32] or [33]), which are set in a moving frame resulting with the Coriolis acceleration term and elaborate expressions for the kinetic energy. This advantage comes at no expense regarding additional complexities in internal force computations, since the geometrically exact formulation is capable of extracting the strain measures from an arbitrarily large overall motion.

The weak form of the momentum balance can thus be written simply by extending the static equilibrium equations by linear inertia terms to obtain

$$G_{\text{dyn}}(\boldsymbol{\varphi}, \boldsymbol{\Lambda}; \mathbf{v}, \mathbf{w}) := \int_L (\mathbf{v} \cdot \dot{\mathbf{p}} + \mathbf{w} \cdot \dot{\boldsymbol{\pi}}) ds + G_{\text{stat}}(\boldsymbol{\varphi}, \boldsymbol{\Lambda}; \mathbf{v}, \mathbf{w}) \quad (18)$$

where G_{stat} are the static equilibrium equations in (16), whereas \mathbf{p} and $\boldsymbol{\pi}$ are linear and angular momenta, which can be written as

$$\mathbf{p} = A_\rho \mathbf{v}; \quad \boldsymbol{\pi} = \boldsymbol{\Lambda} \mathbf{I}_\rho \boldsymbol{\Lambda}^T \mathbf{w} \quad (19)$$

with A_ρ and \mathbf{I}_ρ as the section mass and inertia tensor. It can easily be shown from (18) above that any rigid body translation ($\mathbf{v} = \text{cst.}$) and rigid body rotation ($\mathbf{w} = \text{cst.}$) under self-equilibrated system of forces will preserve the linear and angular momenta, respectively. Moreover, for the case where the external loading is conservative, as for example the central force field which derives from a potential ensuring rotational invariance $\Pi_{\text{ext}}(\boldsymbol{\Lambda}^T \boldsymbol{\varphi})$, and the case where the constitutive behavior is hyper-elastic, which allows us to define the total potential energy functional,

$$\Pi(\boldsymbol{\varphi}, \boldsymbol{\Lambda}) := \Pi_{\text{int}}(\boldsymbol{\varphi}, \boldsymbol{\Lambda}) - \Pi_{\text{ext}}(\boldsymbol{\Lambda}^T \boldsymbol{\varphi}); \quad \Pi_{\text{int}} = \int_L W_{\text{int}}(\boldsymbol{\varphi}, \boldsymbol{\Lambda}) ds \quad (20)$$

one can easily show that the total energy remains conserved. The latter follows from (18) and the governing Hamiltonian functional corresponding to the total energy with

$$\begin{aligned} H(\boldsymbol{\varphi}, \boldsymbol{\Lambda}, \mathbf{v}, \mathbf{w}) &:= \Pi(\boldsymbol{\varphi}, \boldsymbol{\Lambda}) + T(\mathbf{v}, \underbrace{\boldsymbol{\Lambda}^T \mathbf{w}}_{\boldsymbol{\psi}}) = \text{cst.}; \quad T(\mathbf{v}, \boldsymbol{\psi}) = \frac{1}{2} \int_L (\mathbf{v} \cdot A_\rho \mathbf{v} + \boldsymbol{\psi} \cdot \mathbf{I}_\rho \boldsymbol{\psi}) ds \\ \Rightarrow \dot{H}(\boldsymbol{\varphi}, \boldsymbol{\Lambda}, \mathbf{v}, \mathbf{w}) &:= \dot{T}(\mathbf{v}, \boldsymbol{\Lambda}^T \mathbf{w}) + \dot{\Pi}(\boldsymbol{\varphi}, \boldsymbol{\Lambda}) \equiv G_{\text{dyn}}(\boldsymbol{\varphi}, \boldsymbol{\Lambda}; \mathbf{v}, \mathbf{w}) = 0 \end{aligned} \quad (21)$$

These findings can be exploited to design the time-integration schemes with enhanced performance obtained by preserving the salient features of the continuum problem by the discrete approximation, and in particular the energy and the angular momentum conserving schemes (e.g., see [30]). Another interesting development presented in [34] pertains to an optimal time-integration scheme which conserves the energy in low frequency modes and dissipates

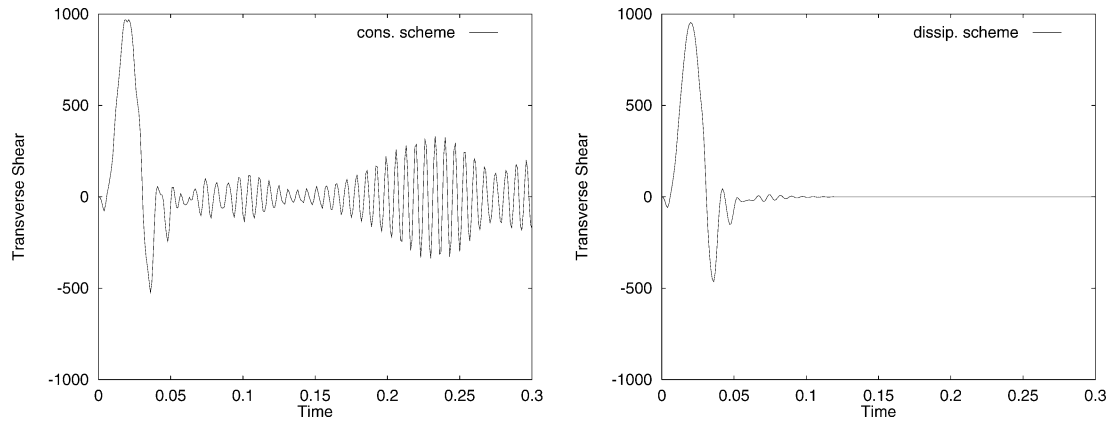


Fig. 3. Shear force time history for beam spin-up maneuver computed by energy conserving and energy decaying schemes.

the high frequency mode contribution, the latter often considered undesirable since it is not represented in a reliable manner by the finite element model. The challenge in that respect, which was successfully tackled in [34], is to design a scheme which dissipates the contribution of higher modes with no need to identify them explicitly or with no possibility when they keep changing due to finite motion. Moreover, the latter is achieved at no expense of verifying the geometrically exact equilibrium equations or kinematics, but with only modification of algorithmic constitutive equations. The energy decaying scheme has a very important advantage for providing an improved accuracy for stress computation with respect to the energy conserving scheme, which lead to comparable values of displacement but practically useless results for stress for any case where high frequency content is significant. This kind of finding is confirmed by the results obtained for a spin-up maneuver around a hinged end of flexible beam by a couple of free-end forces applied in a form of a triangular pulse, which start at zero at $t = 0$, peaks at $t = 0.025$ and goes back to zero at $t = 0.05$. Thereafter, the beam undergoes free vibrations. The numerical results shown in Fig. 3, obtained by the finite element model which consists of eight 2-node beam elements, indicate that in the forced vibration phase both energy conserving and energy decaying schemes yield practically the same result. However, in the free vibration phase, the result computed by the energy conserving scheme clearly displays spurious high frequency noise, contrary to the one obtained by the energy decaying scheme which correctly wipes out those vibrations.

The application of the geometrically exact formulation of structural mechanics in dynamics are numerous. For example, one currently very active research domain of multibody dynamics, where either high operating speed or extreme slenderness of a particular component require that the system flexibility be taken into account. One also finds benefits of the geometrically exact theory in application to currently very active research in microscale and mesoscale models of dynamic fracture (e.g., see [35]). The geometrically exact beam of this kind is used for modelling cohesive forces between two neighboring particles represented by Voronoi cells, which allows to study without difficulty the fragmentation phenomena where a group of connected cells splits from the main structure, as illustrated in Fig. 4.

3.2. Control

Modern structures are often designed to withstand very large displacements and rotations and remain fully operational. In that respect, the goal of a control problem pertains to bringing the structure directly into desired state $\bar{\phi}_d = (\bar{\varphi}_d, \bar{\mathbf{A}}_d)$. This can formally be presented as a minimization problem under constraint, with the constraint assuring that the computed state which the closest to the desired state is also an equilibrium state

$$\min_{G(\phi, v; \cdot) = 0} j(\hat{\phi}(v), \lambda); \quad j(\cdot) = \frac{1}{2} [\|\bar{\phi}_d - \hat{\phi}(\mathbf{F}_0 \lambda)\|^2 + \alpha \|v\|^2] \tag{22}$$

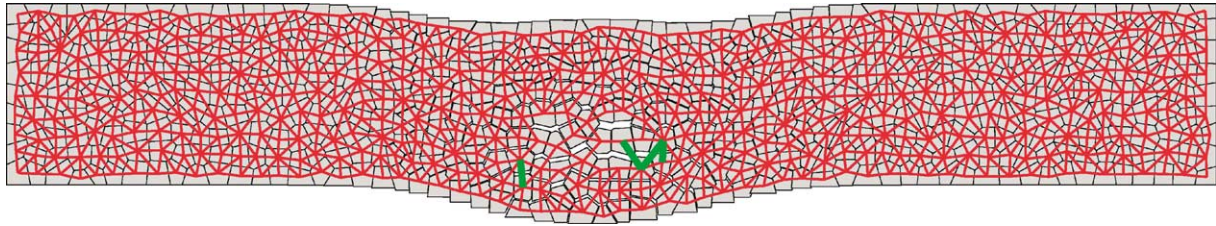


Fig. 4. Fragmentation for a beam-lattice model in a dynamics fracture test.

where \mathbf{v} is the vector of load parameters multiplying the fixed load patterns ordered in \mathbf{F}_0 . This problem can be transformed into an unconstrained minimization problem by using the classical method of Lagrange multipliers (e.g., see [36]), which allows that ϕ and λ can thus be considered as independent variables,

$$\begin{aligned} \max_{\forall \lambda} \min_{\forall \phi} L(\phi, \mathbf{v}; \lambda); \quad L(\phi, \mathbf{v}; \lambda) = J(\phi, \mathbf{v}) + G(\phi, \mathbf{v}; \lambda); \quad \phi = (\varphi, \mathbf{A}); \quad \lambda = (\mathbf{v}, \boldsymbol{\mu}) \\ J(\cdot) = \frac{1}{2} [\|\bar{\phi}_d - \phi\|^2 + \alpha \|\mathbf{v}\|^2]; \quad G(\phi, \mathbf{v}; \lambda) := \int_L (\dot{\mathbf{v}}' - \boldsymbol{\mu} \times \mathbf{v}' \cdot \mathbf{n} + \boldsymbol{\mu}' \cdot \mathbf{m}) ds - G_{\text{ext}}(\dot{\mathbf{v}}, \dot{\boldsymbol{\mu}}) = \mathbf{0} \end{aligned} \quad (23)$$

where \mathbf{v} and $\boldsymbol{\mu}$ denote, respectively, the set of Lagrange multipliers for both displacements and rotations. For this particular choice of control one can eliminate (see [19]) the Lagrange multipliers from the discrete approximation of this problem, resulting with

$$\begin{aligned} \mathbf{0} = \mathbf{r}(\phi) - \mathbf{F}_0 \mathbf{v} \\ \mathbf{0} = \alpha \mathbf{v} + \mathbf{F}_0^T \mathbf{K}^{-1}(\phi - \bar{\phi}_d); \quad \mathbf{K} := \frac{\partial \mathbf{r}}{\partial \phi} \end{aligned} \quad (24)$$

where $\mathbf{r}(\phi)$ is the internal force vector, and \mathbf{K}^{-1} is the inverse of the tangent stiffness matrix. Written in this form, the present control problem becomes fully equivalent to the well-known continuation problem commonly encountered in studies of nonlinear bifurcation (e.g., see [37] and reference therein), with a particular choice of the stabilization term. In that respect, it is important to note that the geometrically exact formulation of structural mechanics is capable of handling (e.g., see [29]) both the linear instability or so-called Euler buckling problems and nonlinear instability problems, where displacements and rotations can become quite large before reaching the critical point. The method of direct computation of the critical points can be placed [38] within the framework of continuation methods, where additional condition providing the critical point identification ought to be supplied. One has the choice between using the supplementary equation to (24) as

$$g(\lambda, \phi) := \det[\mathbf{K}] = 0 \quad (25)$$

which can directly identify only the value of load parameter corresponding to a critical point, or the supplementary equation which also provides the corresponding instability mode with

$$\begin{aligned} \mathbf{g}(\lambda, \phi, \boldsymbol{\psi}) := \mathbf{K} \boldsymbol{\psi} = \mathbf{0} \\ l(\boldsymbol{\psi}, \lambda) := \left\{ \begin{array}{l} \|\boldsymbol{\psi}\| - 1 \\ \boldsymbol{\psi}^T \mathbf{F}_0 \boldsymbol{\lambda} \end{array} \right\} = 0 \end{aligned} \quad (26)$$

The last condition is of particular interest for identifying a bifurcation point, in which case the instability mode can serve to perform the branch-switching explorations. The current works have turned towards the control problems in dynamics, where the main difficulty would concern the chaos [39] in the presence of instabilities. The first finding which is already made in that respect concerns the fact that a robust time-integration scheme, such as the energy conserving or decaying scheme in [34], plays a very important role in order to separate the true chaotic, instability

caused behavior from the numerical instabilities which may occur when integrating stiff differential equations (e.g., [40]).

3.3. Shape optimization

Shape optimization finds its place naturally in designing the structural systems. Ever increasing demands to achieve a more economical design of a structural system motivates the current trends to analyze and exploit the nonlinear behavior of such a system. The geometrically exact formulation of structural mechanics is thus of direct interest in optimization methods which are called upon to guide the optimal shape design of a structure allowed to undergo large displacement and rotations remaining fully operational. This task can formally be presented as constrained minimization of a chosen cost function, specifying the goal to achieve and the design variable to modify to that end, with the constraint which concerns satisfying the equilibrium equations.

Traditionally (e.g., see [41] for a recent review), shape optimization and structural mechanics are studied separately and then brought to bear on the same problem, by using a sequential approach where each task, the minimization of the cost function or solving the equilibrium equations, is carried out by a dedicated computer code. In this manner the communication requirements are reduced since each computer code gets only the minimum information from the other one, so-called design sensitivity (e.g., [42]) for optimization code, or design variables for the finite element code for structural mechanics. It is clear that the traditional approach to analysis and design will sacrifice the computational efficiency for the case of practical interest where both cost function and structural mechanics equilibrium problem are nonlinear and impose each an iterative solution procedure.

The main idea elaborated in [20] relates to an alternative method of analysis and design where those two phases are formulated and solved simultaneously. In that respect, the interdependence of analysis and design variables is no longer assumed so that one iterate simultaneously on both of them, which provides a drastic reduction of the computational cost. In particular, the design sensitivity analysis need not be performed separately, but is obtained as a part of the global solution procedure. An illustration of the proposed procedure is given for the chosen model problem of geometrically exact 3D beam. The traditional approach for shape optimization for this kind of problem can be formulated as the constrained minimization of the cost function $j(\cdot)$

$$\min_{G(\hat{\phi}(\lambda); \hat{\phi})=0} j(\hat{\phi}(\mathbf{x})) \tag{27}$$

where \mathbf{x} are the chosen design variables, and $G(\cdot) = 0$ is the given constraint in terms of the weak form of equilibrium equations. It is indicated in (27) by using $\hat{\phi}(\mathbf{x})$ that each new design can change the equilibrium state, and the state variables are considered as dependent variables. One can eliminate this dependence of the state variables on design variables by appealing to the method of Lagrange multipliers in order to replace the constrained minimization problem in (27) by the unconstrained minimization by introducing the Lagrange multipliers $\lambda = (\mathbf{v}, \boldsymbol{\mu})$ to obtain

$$L(\boldsymbol{\phi}, \mathbf{x}, \lambda) := J(\boldsymbol{\phi}, \mathbf{x}) + G(\boldsymbol{\phi}, \mathbf{x}; \lambda); \quad G(\boldsymbol{\phi}; \lambda) := \int_{\xi_1}^{\xi_2} [(\dot{\mathbf{v}}' - \boldsymbol{\mu} \times \mathbf{v}') \cdot \mathbf{n} + \boldsymbol{\mu}' \cdot \mathbf{m}] j(\xi) d\xi - G_{\text{ext}}; \tag{28}$$

$$j(\xi) = \left\| \frac{\partial \mathbf{x}}{\partial \xi} \right\|$$

We note that the weak form of equilibrium equations is now expressed in the reference configuration, which in the finite element setting can be chosen as the parent element of each isoparametric element. Although such a choice would allow a systematic development using the finite element assembly-like processing of design variables, it is in general rejected for the lack of efficiency and robustness. Increasing the latter is a very big challenge which concerns an adequate choice of the design variables and the corresponding geometry description.

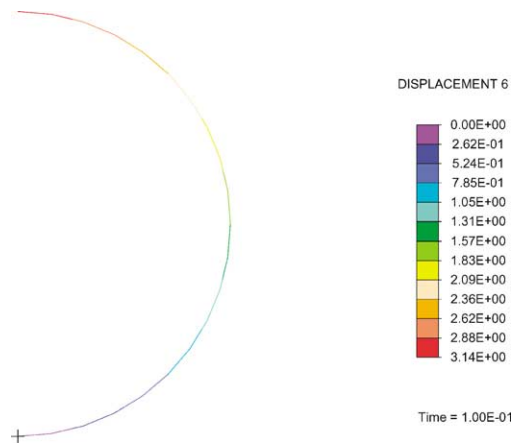


Fig. 5. Shape optimization of a cantilever in a large motion.

A simple example is given to illustrate the type of results one can obtain. In Fig. 5 we present the solution of the shape optimization problem for a pure bending of a cantilever, where only a few iterations are necessary to obtain the desired objective of a constant stress state and the deformed shape in the form of a semi-circle.

4. Conclusions

The geometrically exact theoretical formulation in structural mechanics provides the two main ingredients, kinematics and equilibrium (or momentum balance) in the exact form regardless of the size of displacements, rotations and deformations. It is both intellectually pleasing and practically useful that the theory of this kind is capable of handling the fundamental equations exactly, with all approximations allowed only at a later stage, such as for describing the constitutive behavior, constructing a discrete model, etc.

The geometrically exact formulations can be provided for basically all different models employed in structural mechanics. In particular, the developments presented herein, which concern geometrically exact 3D beams and solids, can easily be extended to shells both in statics [43] and in dynamics [44].

The geometrically exact formulation of structural mechanics requires in general a fresh approach for solving the problems in dynamics, control of motion stability or shape optimization, but in exchange provides the solution to the problems which are beyond the reach of traditional methods. The cases in point discussed herein of nonlinear instability as opposed to (linear) buckling, simultaneous solution procedure of the optimization problem as opposed to sequential approach or time-integration schemes for reliable stress computations in flexible multibody dynamics, are only several of a number of potential applications.

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References

- [1] K. Hjelmstad, *Fundamentals of Structural Mechanics*, Prentice-Hall, New York, 1997.
- [2] K.J. Bathe, *Finite Element Procedures*, Prentice-Hall, New Jersey, 1996.

- [3] M.A. Crisfield, *Nonlinear Finite Element Analysis of Solids and Structures*, Wiley, 1997.
- [4] E. Riessner, On finite deformations of space-curved beams, *J. Appl. Math. Phys.* 32 (1981) 734–744.
- [5] J.C. Simo, A finite-strain beam formulation. The three-dimensional dynamic problem. Part I, *Comput. Methods Appl. Mech. Engrg.* 49 (1985) 55–70.
- [6] A. Ibrahimbegovic, Finite element implementation of Reissner's geometrically nonlinear beam theory: three-dimensional curved beam finite elements, *Comput. Methods Appl. Mech. Engrg.* 122 (1995) 10–26.
- [7] A. Ibrahimbegovic, Stress resultant geometrically nonlinear shell theory with drilling rotations. Part 1: A consistent formulation, *Comput. Methods Appl. Mech. Engrg.* 118 (1994) 265–284.
- [8] A. Ibrahimbegovic, Finite elastic deformations and finite rotations of 3d continuum with independent rotation field, *Revue européenne élém. finis* 4 (1995) 555–576.
- [9] J. Argyris, An excursion into large rotations, *Comput. Methods Appl. Mech. Engrg.* 32 (1982) 85–155.
- [10] S.N. Atluri, A. Cazzani, Rotation in computational solid mechanics, *Arch. Comp. Mech.* 2 (1995) 49–138.
- [11] N. Buechter, E. Ramm, Shell theory versus degeneration – a comparison in large rotation finite element analysis, *Int. J. Numer. Methods Engrg.* 34 (1992) 39–59.
- [12] M. Geradin, D. Rixen, Parametrization of finite rotations in computational dynamics, *Revue européenne élém. finis* 4 (1995) 497–554.
- [13] W.C. Hassenplug, Rotation angles, *Comput. Methods Appl. Mech. Engrg.* 105 (1993) 111–124.
- [14] A. Ibrahimbegovic, F. Frey, I. Kozar, Computational aspects of vector-like parameterization of three-dimensional finite rotations, *Int. J. Numer. Methods Engrg.* 38 (1995) 3653–3673.
- [15] A. Ibrahimbegovic, On the choice of finite rotation parameters, *Comput. Methods Appl. Mech. Engrg.* 149 (1997) 49–71.
- [16] J.C. Simo, L. Vu-Quoc, Three-dimensional finite strain model. Part II: Computational aspects, *Comput. Methods Appl. Mech. Engrg.* 38 (1986) 79–118.
- [17] J.C. Simo, The (symmetric) Hessian for geometrically nonlinear models in solid mechanics: intrinsic definition and geometric interpretation, *Comput. Methods Appl. Mech. Engrg.* 96 (1992) 183–200.
- [18] A. Ibrahimbegovic, R.L. Taylor, Nonlinear dynamics of flexible multibody systems, *Comp. Structures* 81 (2003) 1113–1132.
- [19] A. Ibrahimbegovic, P. Villon, C. Knopf-Lenoir, Optimal control of structures undergoing finite rotations, *Int. J. Numer. Methods Engrg.* 58 (2003), in press.
- [20] A. Ibrahimbegovic, C. Knopf-Lenoir, Shape optimization of elastic structural systems undergoing large rotations: Simultaneous solution procedure, *Int. J. Comput. Model. Engrg. Sci.* 5 (2003), in press.
- [21] J.E. Marsden, T.J.R. Hughes, *Mathematical Foundations of Elasticity*, Dover, New York, 1994.
- [22] P. Germain, *Cours de mécanique de l'École Polytechnique*, Tome 1, Ellipses, Paris, 1986.
- [23] J. Green, H. Zerna, *Theoretical Elasticity*, Dover, New York, 1992.
- [24] C. Truesdell, W. Noll, *The Non-Linear Field Theories*, Springer, Berlin, 1965.
- [25] P.G. Ciarlet, *Mathematical Elasticity*, North-Holland, 1988.
- [26] O.C. Zienkiewicz, R.L. Taylor, *The Finite Element Method*. Vols. I–III, 5th edition, Butterworth–Heinemann, Oxford, 2000.
- [27] J.F. Bourgat, P. Le Tallec, S. Mani, Modélisation et calcul des grands déplacements de tuyaux élastiques en flexion–torsion, *J. Méc.* 7 (1988) 379–408.
- [28] M.A. Crisfield, G. Jelenic, Objectivity of strain measures in the geometrically exact three-dimensional beam theory and its finite-element implementation, *Proc. Roy. Soc. London Ser. A* 455 (1999) 1125–1147.
- [29] A. Ibrahimbegovic, H. Shakourzadeh, J.L. Batoz, M. Almkhdad, Y.Q. Guo, On the role of geometrically exact and second order theories in buckling and post-buckling analysis of three-dimensional beam structures, *Comp. Structures* 61 (1996) 1101–1114.
- [30] J.C. Simo, N. Tarnow, M. Doblare, Nonlinear dynamics of three-dimensional rods: Exact energy and momentum conserving algorithm, *Int. J. Numer. Methods Engrg.* 38 (1995) 1431–1473.
- [31] A. Ibrahimbegovic, R.L. Taylor, On the role of frame-invariance of structural mechanics models at finite rotations, *Comput. Methods Appl. Mech. Engrg.* 191 (2002) 5159–5176.
- [32] B.N. Fraeijs de Veubeke, The dynamics of flexible bodies, *Int. J. Engrg. Sci.* 14 (1976) 895–913.
- [33] T.R. Kane, D.A. Levinson, *Dynamics: Theory and Applications*, McGraw-Hill, New York, 1985.
- [34] A. Ibrahimbegovic, S. Mamouri, Energy conserving/decaying implicit time-stepping scheme for nonlinear dynamics of 3d beams undergoing finite rotations, *Comput. Methods Appl. Mech. Engrg.* 191 (2002) 4241–4258.
- [35] A. Ibrahimbegovic, A. Delaplace, Microscale and mesoscale discrete models for dynamic fracture of structures built of brittle materials, *Comp. Structures* 81 (2003) 1255–1265.
- [36] P.G. Ciarlet, *Introduction à l'analyse numérique matricielle et à l'optimisation*, Masson, 1990.
- [37] W.C. Rheinboldt, *Numerical Analysis of Parameterized Nonlinear Equations*, Wiley, London, 1986.
- [38] A. Ibrahimbegovic, M. Almkhdad, Quadratically convergent direct calculation of critical points for 3d structures undergoing finite rotations, *Comput. Methods Appl. Mech. Engrg.* 189 (2000) 107–120.
- [39] J. Argyris, G. Faust, M. Haase, *An Exploration of Chaos*, North-Holland, Amsterdam, 1994.
- [40] E. Hairier, G. Wanner, *Solving Ordinary Differential Equations: Stiff and Differential-Algebraic Problems*, Springer, Berlin, 1991.

- [41] M. Kleiber, H. Antunez, T.D. Hein, P. Kowalczyk, *Parameter Sensitivity in Nonlinear Mechanics; Theory and Finite Element Computations*, Wiley, 1997.
- [42] B. Rousselet, A finite strain rod model and its design sensitivity, *Mech. Struct. Mach.* 20 (1992) 413–432.
- [43] B. Brank, J. Korelc, A. Ibrahimbegovic, Nonlinear shell problem formulation accounting for through-the-thickness stretching and its finite element implementation, *Comput. Structures* 80 (2002) 699–717.
- [44] B. Brank, J. Korelc, A. Ibrahimbegovic, Dynamics and time-integration for elastic shells undergoing finite rotations, *Comput. Structures* 81 (2003) 1193–1210.