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Asymptotically sharp uniform estimates in a scalar spectral stiff problem

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Abstract

Estimates of convergence rates for rescaled eigenvalues of the stiff Neumann problem for the Laplacian are obtained. The bounds are expressed in terms of the stiffness ratio and properties of the limit spectrum both for low and middle frequency ranges. ***To cite this article: M. Lobo et al., C. R. Mecanique 331 (2003).***

Résumé

Estimations asymptotiques uniformes pour le spectre d'un problème scalaire raide. On obtienent des estimations de la vitesse de convergence des valeurs propres, convenablement mises a l'échelle, d'un problème de Neumann raide pour l'operateur de Laplace. Des bornes correspondantes a ces estimations sont exprimées en termes du rapport des raideurs et des propriétés du spectre limite dans les rangs des fréquences basses et moyennes. ***Pour citer cet article : M. Lobo et al., C. R. Mecanique 331 (2003).***

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Soient $\Omega^\pm \subset \mathbb{R}^n$ domaines bornées avec des frontières lipschitziennes $\partial\Omega^\pm$ and $\Omega^+ \cap \Omega^- = \emptyset$. Soit h le rapport de raideur des deux parties $\Omega^\pm$. On introduit les ensembles $(n-1)$-dimensionels $\Upsilon = \partial\Omega^+ \cap \partial\Omega^-$, $\Sigma^\pm = \partial\Omega^\pm \setminus \overline{\Upsilon}$. On considère la junction Ω des deux corps $\Omega^\pm$, oú Ω^+ est la partie plus raide du corps, et on

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suppose que la zone de contact $\varUpsilon$ est à mesure positive $\text{meas}_{n-1}\varUpsilon$. Les ensembles $\varUpsilon$ et $\varSigma^{\pm}$ sont aussi entourés par des frontières lipschitziennes $(n-2)$-dimensionelles. On impose une condition de Neumann sur la surface $\partial\varOmega$. On peut signaler les references [1–4] pour la condition de Dirichlet sur $\partial\varOmega$ et pour le cas où $\varSigma^{-}$ est l'ensemble vide, ainsi que [5] pour des operateurs differentiels et conditions aux limites plus generales.

Nous donc considèrons le problème spectral (2) pour l'operateur de Laplace Δ. La formulation faible du problème (2) nous ramène a celui de trouver $\varLambda(h)\in\mathbb{R}$ et une solution no triviale $u(h,\cdot)\in H^1(\varOmega)$ du problème (3). Comme il est bien connu, pour chaque h fixé, les valeurs propres du problème (3), forment la suite (4) avec la convention des valeurs propres repetées. Nous étudions le comportement asymptotique des éléments propres $(\varLambda(h),u(h,\cdot))$ de (3), lorsque $h\to 0$.

Une petite modification des aproximations obtenues dans [6] et [2] (cf. §5.7–5.10 dans [7]) permet d'établir le résultat de convergence (5), où la limite (6) représente le spectre du problème (7). La formulation faible de (7) est donée par (8). Nous remarquons que la condition aux limites $(7)_2$, qui est non locale de type Steklov, peut être obtenue en utilisant des developpements asymptotiques comme dans [5].

Par ailleurs, des approximations asymptotiques, developpées dans [2], pour les fréquences moyennes, présentent les proprietés suivantes : 1) Pour chaque $\lambda>0$ il y a une suite de valeurs propres $\varLambda_{N(h)}(h)$ dans (4) qui converge vers λ pour $h\to 0$. 2) Si $\beta_k>0$ est une valeur propre du problème spectral de Neumann (9), il existe une valeur propre $\varLambda_{N(h)}(h)$ du problème (2) vérifiant (10). On doit signaler que, en accord avec (5), dans l'un et l'autre cas, l'indice de la valeur propre $N(h)\to+\infty$ pour $h\to+0$.

L'objet principal de nôtre note est celui de déterminer la vitesse de convergence dans la relation (5), et mettre en evidence la dependence en k (l'indice de la valeur propre β_k de (9)) des bornes dans les estimations asymptotiques de type (10).

Nous avons utilisé le procedure de reduction directe et inverse, décrite dans [8,9], pour obtenir les estimations des Propopositions 1 et 2 et des Téorèmes 1 et 2 dans la Section 1. Ces estimations permettent d'obtenir des bornes pour la vitesse de convergence en améliorant les résultats (5) et (10). Nous remarquons que les constantes $\mathbf{h}_p$ et $\mathbf{c}_p$, qui apparaîtrent dans les Propositions et les Téorèmes 1 et 2, ne dependent du paramètre h, ni de l'indice $k=1,2,\ldots$ de la valeur propre. Les bornes dépendent de la constante de raideur h et du numéro k de la valeur propre des problèmes limites et sont plus précises dans le cas de dimension 1 (cf. Section 3) où l'on connait les valeurs propres des problèmes aux limites (voir (24) et (25)) ainsi que la distance entre elles. En fait, pour $n=1$, les valeurs propres λ_k et β_k dans (5) et (10) sont données par (25), et pour $hk^3=\mathrm{O}(1)$ l'on écrit (5) comme $|h^{-1}\varLambda_k(h)-\lambda_k|=\mathrm{O}(hk^4)$, tandis que pour $hk^4=\mathrm{O}(1)$ la borne $C_kh^{1/4}$ dans (10) est d'ordre $\mathrm{O}(k^3h^{1/4})$.

1. The stiff spectral problem and the limit problems

Let $\varOmega^{\pm}\subset\mathbb{R}^n$ be bounded domains with Lipschitz boundaries $\partial\varOmega^{\pm}$ and $\varOmega^+\cap\varOmega^-=\emptyset$. Let h denote the ratio of stiffness of the two parts $\varOmega^{\pm}$. We introduce the $(n-1)$-dimensional sets

$$\varUpsilon=\partial\varOmega^+\cap\partial\varOmega^-,\qquad \varSigma^{\pm}=\partial\varOmega^{\pm}\setminus\overline{\varUpsilon} \tag{1}$$

Considering the junction $\varOmega$ of two bodies $\varOmega^{\pm}$, $\varOmega^{+}$ the stiffer part of the body, we assume that the contact zone $\varUpsilon$ has a positive measure $\text{meas}_{n-1}\varUpsilon$. The sets (1) are surrounded by Lipschitz $(n-2)$-dimensional surfaces. We impose a Neumann condition on the boundary $\partial\varOmega$. We refer to [1,2] for a Dirichlet condition on $\partial\varOmega$, to [3,4] for the Dirichlet condition and the case $\partial\varOmega\cap\partial\varOmega^+=\emptyset$, and to [5] for more general boundary conditions and differential operators. We consider the following spectral problem involving the Laplace operator Δ,

$$\begin{aligned}
&-h\Delta u^-(h,x)=\varLambda(h)u^-(h,x),\quad x\in\varOmega^-\\
&-\Delta u^+(h,x)=\varLambda(h)u^+(h,x),\quad x\in\varOmega^+\\
&\partial_n u^{\pm}(h,x):=n^{\pm}(x)^{\top}\nabla u^{\pm}(h,x)=0,\quad x\in\varSigma^{\pm}\\
&u^-(h,x)=u^+(h,x),\quad h\partial_n u^-(h,x)=\partial_n u^+(h,x),\quad x\in\varUpsilon
\end{aligned} \tag{2}$$

The weak formulation of problem (2) is: to find $\Lambda(h) \in \mathbb{R}$ and a nontrivial function $u(h,\cdot) \in H^1(\Omega)$ such that

$$h\big(\nabla u^-, \nabla v^-\big)_{\Omega^-} + \big(\nabla u^+, \nabla v^+\big)_{\Omega^+} = \Lambda(h)(u,v)_\Omega \quad \forall v = \big\{v^-, v^+\big\} \in H^1(\Omega) \tag{3}$$

As is known, for fixed h, the eigenvalues of problem (3) form the sequence

$$0 = \Lambda_1(h) < \Lambda_2(h) \leqslant \cdots \leqslant \Lambda_j(h) \leqslant \cdots \to +\infty \tag{4}$$

where we adopt the convection of repeated eigenvalues. The problem that we address in the paper consists in studying the asymptotic behavior of the spectral pairs $(\Lambda(h), u(h,\cdot))$ of problem (3), as $h \to 0$.

A slight modification of the approaches in [6] and [2] (cf. §5.7–5.10 in [7]) establishes the convergence

$$h^{-1}\Lambda_k(h) \to \lambda_k \quad \text{as } h \to +0 \tag{5}$$

with conservation of the multiplicity, where

$$0 = \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots \to +\infty \tag{6}$$

stands for the spectrum of the resulting problem

$$\begin{aligned} &-\Delta U(x) = \lambda U(x), \quad x \in \Omega^-; \qquad \partial_n U(x) = 0, \quad x \in \Sigma^- \\ &\lambda U(x) = \big(\operatorname{meas}_n \Omega^+\big)^{-1} \int_{\Upsilon} \partial_n U(x)\,\mathrm{d}s, \quad x \in \Upsilon \end{aligned} \tag{7}$$

The boundary condition $(7)_2$, which is nonlocal and of Steklov's type, is derived by rewriting the computations in [5] with minor modifications. The weak formulation of (7) reads: to find a number $\lambda \in \mathbb{R}$ and a nontrivial function $U^0 \in \mathcal{H}_0$ such that

$$\big(\nabla U^0, \nabla V\big)_{\Omega^-} = \lambda \left\{ \big(U^0, V\big)_{\Omega^-} + \frac{(\operatorname{meas}_n \Omega^+)}{(\operatorname{meas}_{n-1}\Upsilon)^2} \int_{\Upsilon} U^0(x)\,\mathrm{d}s \int_{\Upsilon} V(x)\,\mathrm{d}s \right\} \quad \forall V \in \mathcal{H}_0 \tag{8}$$

where $\mathcal{H}_0 = \{V \in H^1(\Omega^-) \mid V \text{ is a constant on } \Upsilon\}$.

At the same time, the approximations developed in [2] uphold the following facts. First, for each $\lambda > 0$ there exists a sequence of eigenvalues $\Lambda_{N(h)}(h)$ in (4) converging towards λ as $h \to 0$. Second, for any eigenvalue $\beta_k > 0$ of the Neumann spectral problem

$$-\Delta W(x) = \beta W(x), \quad x \in \Omega^+; \qquad \partial_n W(x) = 0, \quad x \in \partial\Omega^+ \tag{9}$$

there exists an eigenvalue $\Lambda_{N(h)}(h)$ of problem (2) such that

$$\big|\Lambda_{N(h)}(h) - \beta_k\big| \leqslant C_k h^{1/4} \tag{10}$$

We note that, in view of convergence (5), in both cases, the eigenvalue number $N(h)$ grows indefinitely as $h \to +0$.

The principal goal of our paper is to detect the convergence rate in (5) and to clarify the dependence on k of bounds in the asymptotic accuracy estimates of type (10). We use the so-called direct and inverse reductions, as outlined in [8,9], to obtain these estimates. We state the general results in Section 2. In the one-dimensional case, a detailed analysis of the results is performed in Section 3. In this case, i.e., $n = 1$, the eigenvalues λ_k and β_k of the resulting problems (5) and (10) are given by (25) and the estimates obtained are even more precise than in the case where $n > 1$. As a matter of fact, when $n = 1$, among the results we have that, under the restriction $hk^3 = \mathrm{O}(1)$, the discrepancy in (5) reads $|h^{-1}\Lambda_k(h) - \lambda_k| = \mathrm{O}(hk^4)$, while the restriction $hk^4 = \mathrm{O}(1)$ provides the bound $C_k h^{1/4} = \mathrm{O}(k^3 h^{1/4})$ in (10).

We emphasize that in what follows all constants $\mathbf{h}_p$ and $\mathbf{c}_p$ do not depend on both either the parameter h or the eigenvalue number $k = 1, 2, \ldots.$

2. Estimates for convergence rates

The following assertion results from the *inverse reduction procedure* when an approximation to a solution of the original problem (2) is constructed from a solution of the resulting problem (7) and the classical lemma on "almost eigenvalues" is applied. In the sequel $\varkappa_k$ denotes the multiplicity of the eigenvalue λ_k in (6),

$$\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+\varkappa_k-1} < \lambda_{k+\varkappa_k} \tag{11}$$

Proposition 1. *There exist constants* $\mathbf{h}_1 > 0$ *and* $\mathbf{c}_1 > 0$ *such that, for any eigenvalue* λ_k *of problem* (8) *with multiplicity* $\varkappa_k$ *and for any integer* $l \in (0, \varkappa_k]$, *the condition*

$$h \leqslant \mathbf{h}_1 l^{-1}(1+\lambda_k)^{-1} \tag{12}$$

provides at least l *eigenvalues* $\Lambda_j(h), \ldots, \Lambda_{j+l-1}(h)$ *of problem* (3) *satisfying the estimate*

$$\left|\Lambda_p(h) - h\lambda_k\right| \leqslant \mathbf{c}_1 l h^2 (1+\lambda_k)^2 \tag{13}$$

To compensate for obvious lacks of information on spectrum (4) given by Proposition 1, we employ the *direct reduction procedure* which, in contrast to the inverse reduction, provides an approximation solution to the resulting problem (7) based on a solution to the original problem (2). These two reductions lead to completed results on relations between spectra (4) and (6) (see [8,9] and [5] for a detailed discussion).

For the eigenvalue λ_k in (11), we introduce the relative distance d_k from $(1+\lambda_k)^{-1}$ to the nearest point $(1+\lambda_j)^{-1} \neq (1+\lambda_k)^{-1}$, that is

$$d_k = \min\left\{\frac{1+\lambda_k}{1+\lambda_{k-1}} - 1, 1 - \frac{1+\lambda_k}{1+\lambda_{k+\varkappa_k}}\right\} \tag{14}$$

Theorem 1. *There exist constants* $\mathbf{h}_2 > 0$ *and* $\mathbf{c}_2 > 0$ *such that, for any* $\varkappa_k$*-multiple eigenvalue* λ_k (*cf.* (11)) *of the resulting problem* (8), *the condition*

$$h \leqslant \mathbf{h}_2 \varkappa_k^{-1}\left(1+d_k^{-1}\right)^{-1}(1+\lambda_k)^{-1} \tag{15}$$

ensures that the inclusion

$$\left(1+h^{-1}\Lambda_p(h)\right)^{-1} \in \left[\frac{1}{2}\left(\frac{1}{1+\lambda_k} + \frac{1}{1+\lambda_{k+\varkappa_k}}\right), \frac{1}{2}\left(\frac{1}{1+\lambda_k} + \frac{1}{1+\lambda_{k-1}}\right)\right] \tag{16}$$

occurs only for the eigenvalues $\Lambda_k(h), \ldots, \Lambda_{k+\varkappa_k-1}(h)$ *of the original problem* (3). *These eigenvalues satisfy the estimate*

$$\left|\Lambda_p(h) - h\lambda_k\right| \leqslant \mathbf{c}_2 h^2 (1+\lambda_k)^2 \tag{17}$$

Moreover, under the condition

$$h \leqslant \mathbf{h}_2 \varkappa_k^{-1}\left(1+d_k^{-1}\right)^{-1}\left\{1+\frac{1}{2}(\lambda_k + \lambda_{k+\varkappa_k})\right\}^{-1} \tag{18}$$

the segment

$$\left[\frac{h}{2}(\lambda_k+\lambda_{k-1}), \frac{h}{2}(\lambda_k + \lambda_{k+\varkappa_k})\right] \tag{19}$$

contains only the above-mentioned eigenvalues $\Lambda_k(h), \ldots, \Lambda_{k+\varkappa_k-1}(h)$ *as well.*

Note that Theorem 1 reveals convergence (5) with a fixed k and, in addition, formula (17) delineates the convergence rate. In particular, any segment

$$[\mu_-, \mu_+] \subset (\lambda_{k-1}, \lambda_k) \tag{20}$$

becomes free of the points $h^{-1}\Lambda_p(h)$ for a sufficiently small $h > 0$ and the next assertion, originating in the direct reduction as well, provides a bound for such h.

Proposition 2. *The segment* $[h\mu_-, h\mu_+]$ *related to* (20) *does not contain an eigenvalue of problem* (3) *as long as*

$$h \leqslant \mathbf{h}_3\big(1+\rho^{-1}\big)^{-1}(1+\lambda_k)^{-2} \tag{21}$$

where $\rho = \min\{\lambda_k - \mu_+, \mu_- - \lambda_{k-1}\}$ *and* $\mathbf{h}_3$ *is a certain constant which does not depend either on the endpoints* $\mu_\pm$ *and the eigenvalue* λ_k*, nor on the parameter* h*.*

For the medium frequencies, only the inverse reduction can be applied, which leads to the following assertion.

Theorem 2. *There exist constants* $\mathbf{h}_4 > 0$ *and* $\mathbf{c}_4 > 0$ *such that, for any eigenvalue* β_k *of problem* (9) *of multiplicity* κ_k *and for any integer* $l \in (0, \kappa_k]$*, the condition*

$$h \leqslant \mathbf{h}_4 l^{-4}(1+\beta_k)^{-2} \tag{22}$$

provides at least l *eigenvalues* $\Lambda_j(h), \dots, \Lambda_{j+l-1}(h)$ *of problem* (3) *verifying estimate* (10) *with the constant*

$$C_k = \mathbf{c}_4 l(1+\beta_k)^{3/2} \tag{23}$$

3. Analysis in the one-dimensional case

In order to comment on and clarify our results for many-dimensional domains, we consider the simplest one-dimensional problem (2), where

$$\Omega^- = (-T,0), \quad \Omega^+ = (0,T), \quad \Sigma^\pm = \{\pm T\}, \quad \Upsilon = \{0\}$$
$$\Delta u = u'', \quad u' := \partial_x u, \quad \partial_{n^\pm} u = \pm u'$$

In this case, we have the explicit solutions of the resulting spectral problems (7) and (9),

$$\begin{aligned} &-U''(x) = \lambda U(x), \quad x \in (-T,0); \quad U'(-T) = 0, \quad \lambda U(0) = T^{-1}U'(0) \\ &-W''(x) = \beta W(x), \quad x \in (0,T); \quad U'(-T) = 0, \quad U'(0) = 0 \end{aligned} \tag{24}$$

Specifically, the spectra of problems $(24)_1$ and $(24)_2$ are respectively composed of the simple eigenvalues

$$\lambda_k = T^{-2} z_k^2 \quad \text{and} \quad \beta_k = T^{-2}\pi^2 k^2 \tag{25}$$

where $k = 1, 2, \dots$ and z_k are nonnegative roots of the transcendental equation $z = -\operatorname{tg} z$. It is not difficult to find out that $z_1 = 0$ and, for $k = 2, 3, \dots$ as $k \to \infty$,

$$z_k = \pi\left(k - \frac{3}{2}\right) + \left[\pi\left(k - \frac{3}{2}\right)\right]^{-1} - \frac{4}{3}\left[\pi\left(k - \frac{3}{2}\right)\right]^{-3} + \mathrm{O}\left(\left[\pi\left(k - \frac{3}{2}\right)\right]^{-5}\right) \tag{26}$$

These computations allow us to redefine the assertions and asymptotic formulae presented in the previous section while adapting them to ordinary differential equations. Indeed, by modifying constants $\mathbf{h}_p$ and $\mathbf{c}_p$ in the bounds from restrictions (12), (15), (18) and (22) and estimates (13), (17) and (10) together with (23), on account of (25) and (26), we can replace $1+\lambda_k$, $1+d_k^{-1}$, and $1+\beta_k$ by k^2, k, and k^2, respectively (note that $\Lambda_1(h) = \lambda_1 = \beta_1 = 0$ and, therefore, we do not need to discuss these eigenvalues). Moreover, $l = \varkappa_k = \kappa_k = 1$ because the eigenvalues are simple.

The upper bounds in (13) and (17) now become equal to $\mathbf{c}_{5,6}h^2k^4$. At the same time, restriction (15) with the bound $\mathbf{h}_6 k^{-3}$ is much harder than restriction (12) with the bound $\mathbf{h}_5 k^{-2}$. To explain this disagreement, we compare results presented in Proposition 1 and Theorem 1. By (26), we have

$$0 < \mathbf{c}_0 hk \leqslant |h\lambda_k - h\lambda_{k\pm1}| \leqslant \mathbf{C_0} hk \tag{27}$$

Since restriction (12) provides $\mathbf{C}_0 hk \leqslant \mathbf{C}_0\mathbf{h}_5 k^{-1}$, the interval $(h\lambda_k - \mathbf{c}_5 h^2k^4, h\lambda_k - \mathbf{c}_5 h^2 k^4)$ of width $2\mathbf{c}_5 h^2k^4 \leqslant 2\mathbf{c}_5\mathbf{h}_5^2$, indicated in (13), can contain the eigenvalues $h\lambda_{k\pm1}$ and $\Lambda_{k\pm1}(h)$ in addition to $h\lambda_k$ and $\Lambda_k(h)$. On the other hand, under the harder restriction (15), the width of the interval generated by (17) satisfies $2\mathbf{c}_6 h^2k^4 \leqslant 2\mathbf{c}_6\mathbf{h}_6^2 k^{-2}$ and therefore can be made smaller than the bound $\mathbf{C}_0 hk \leqslant \mathbf{C}_0 \mathbf{h}_6 k^{-2}$ in (27) by a proper choice of $\mathbf{h}_6$. In other words, Proposition 1 cannot specify how many eigenvalues of the problem (3) fulfill estimate (13) and hence it describes only a *collective asymptotics* of eigenvalues. In contrast, Theorem 1 detects exactly one eigenvalue $\Lambda_k(h)$ which verifies estimate (17) and inclusion (16) so that the *individual asymptotics* of the eigenvalue is met and, for fixed k, the convergence $h^{-1}\Lambda_k(h) \to \lambda_k$ with the rate $\mathrm{O}(hk^4)$ is thus confirmed. Furthermore, restriction (18), obtaining the similar bound $\mathbf{h}_8 k^{-3}$ as in (15), ensures absence of "extraneous" eigenvalues $\Lambda_j(h)$ with $j \neq k$ in segment (19).

Rewriting (12) under the form $h(1+\lambda_k) \leqslant \mathbf{h}_1$, we see that Proposition 1 covers the whole low-frequency range $[0, \mathbf{C}_1)$ of spectrum (4) but Theorem 1 only its part $[0, h^{1/3}\mathbf{C}_2)$ (recall that the necessary restriction (15) contains the value $1 + d_k^{-1}$ which, owing to the left inequality in (27), is of order $(1+\lambda_k)^{1/2}$ for the one-dimensional problem). Neither Propositions 1 and 2, nor Theorem 1 work inside the middle-frequency range $[\mathbf{C}_1, h^{-1}\mathbf{C}_3)$ of the spectrum. By virtue of (25), bound (23) in estimate (10) turns into $\mathbf{c}_7 h^{1/4} k^3$. Detecting collective asymptotics of eigenvalues (cf. [1,2,5]), Theorem 2 is only related to the part $[\mathbf{C}_1, h^{-1/2}\mathbf{C}_4)$ of the middle-frequency range of spectrum (4), since the bound in (21) takes the form $\mathbf{h}_7 k^{-4} = \mathbf{C}_4^2 \beta_k^{-2}$ with $\mathbf{C}_4 = T^{-2}\pi^2 \mathbf{h}_7^{1/2}$.

For the sake of brevity, we do not demonstrate here asymptotic forms for eigenvectors which can also be naturally divided into *individual* and *collective* (the latter are known as *quasimodes*). It is self-understood that individual asymptotics of eigenvectors are available provided the corresponding eigenvalues admit the individual asymptotics as well, i.e., only on a narrow part of the low-frequency range (cf. [5] for cases $n = 2, 3$). We also refer to [2] for more precise results on structures of eigenfunctions associated with the middle-frequencies of another scalar stiff problem.

Finally, let us note that the Weyl asymptotics for eigenvalues predicts that $\lambda_k = \mathrm{O}(k)$ for $\Omega \subset \mathbb{R}^2$ and $\lambda_k = \mathrm{O}(k^{2/3})$ for $\Omega \subset \mathbb{R}^3$. Such asymptotic behavior of eigenvalues reduces the growth in k of the bounds discussed above. However, we do not know proofs of such kind of results for nonlocal boundary conditions of Steklov's type. Moreover, conclusions derived from Theorem 1 are crucially based on the lower bound in (27) but, in general, estimates of this kind are not available in many-dimensional domains. It should be mentioned that if two different eigenvalues happen to be close each to the other, the value $(1 + d_k^{-1})^{-1}$ becomes small and condition (15) excessively restrictive. Hence, in such cases it is worth using the collective asymptotic forms too.

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