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A two scale method for modulated vibration modes of large, nearly repetitive, structures

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Abstract

Nearly repetitive structures can present at least two kinds of vibration modes: localized modes and modulated ones. In this Note, the multiple scale method is applied to characterize a packet of modulated modes. In this respect, only small size problems are to be solved: periodic problems posed on a few basic cells and amplitude equations, which define a sort of homogenized model for this type of modes. It is established that the influence of the non-repetitive part of the structure is accounted by a boundary condition. ***To cite this article: E.M. Daya et al., C. R. Mecanique 331 (2003).***
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Résumé

Une méthode asymptotique à deux échelles pour les modes de vibrations modulés des longues structures presque répétitives. Les structures à forme presque répétitive peuvent avoir au moins deux types de modes, localisés ou modulés. Dans cette Note, la méthode des échelles multiples est appliquée pour caractériser un paquet de modes modulés. Pour cela, on n'a à résoudre que des problèmes de petite taille : des problèmes périodiques posés sur quelques cellules de base et des équations d'amplitudes, qui définissent une sorte de modèle homogénéisé pour ce type de modes. On montre que l'influence de la partie non répétitive de la structure est prise en compte par une condition aux limites. ***Pour citer cet article : E.M. Daya et al., C. R. Mecanique 331 (2003).***
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Mots-clés : Mécanique des solides numérique ; Solides et structures

1. Introduction

Large structures exhibiting a nearly periodic form are used in many domains, as aerospace industry. Such structures, such as the one depicted in Fig. 1, present localized modes and modulated modes. The existence of

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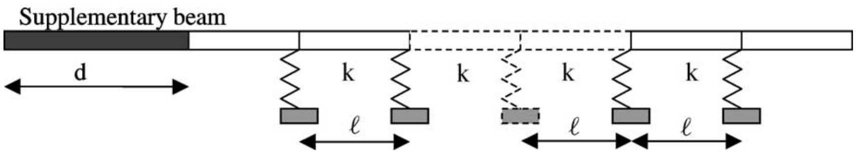

Fig. 1. The considered nearly repetitive structure.

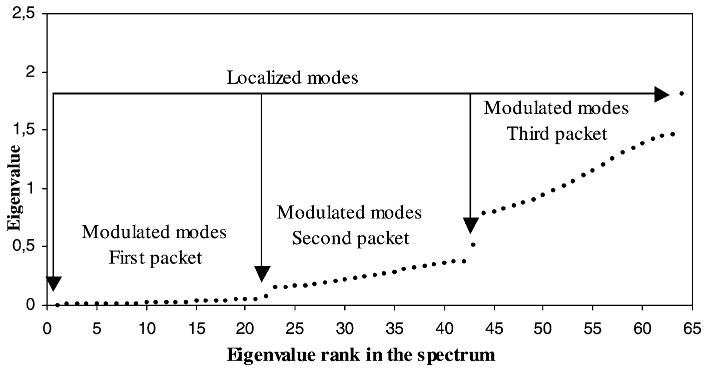

Fig. 2. The 65 first eigenvalues obtained by direct simulation for $L = 200$, $d = \ell = 10$, $K = 100\,000$, clamped beam.

the localized modes is well known as the vibration localization phenomenon [1,2]. Two main techniques have been considered to study vibrations of nearly periodic systems: the wave propagation method and the transfer matrix technique. Using the Lyapunov exponents of the transfer matrix, one measures the degree of wave localization in multi-coupled nearly periodic systems [2]. Concerning nearly periodic structures, previous work was devoted to understanding the vibration localization phenomenon. A literature review can be found in [1].

In this Note, one considers the modulated modes. Generally, these modes are closely located in well separated bands, see Fig. 2. They appear as slow modulations of periodic ones, see Fig. 3. Because of the latter property, the multiple scale asymptotic method [3] can be applied to describe this class of modes as established recently in the case of large periodic structures [4,5]. In these works, an equivalent continuum model for the modulated modes has been obtained. This model involves differential equations, whose coefficients can be obtained by solving some problems posed on a few basic cells. It is not a simple matter to deduce the boundary conditions to be associated with these differential equations. Here this question is revisited in the case of a repetitive structure coupled with a non-repetitive one.

2. Two-scale analysis

2.1. Basic expansions

Consider the bending motions of the elastic beam of Fig. 1, as a representative of nearly repetitive structures. It is the assembly of a periodic beam of length L and a supplementary one, of length d. The periodic part has N identical cells, whose length is $\ell = L/N$. The equations for the vibration modes can be split in three parts, one for the supplementary beam, one for the repetitive part and the matching between these two parts:

$$\frac{\mathrm{d}^4 V_s}{\mathrm{d}x^4} - \lambda V_s = 0, \quad x \in [-d, 0] \tag{1}$$

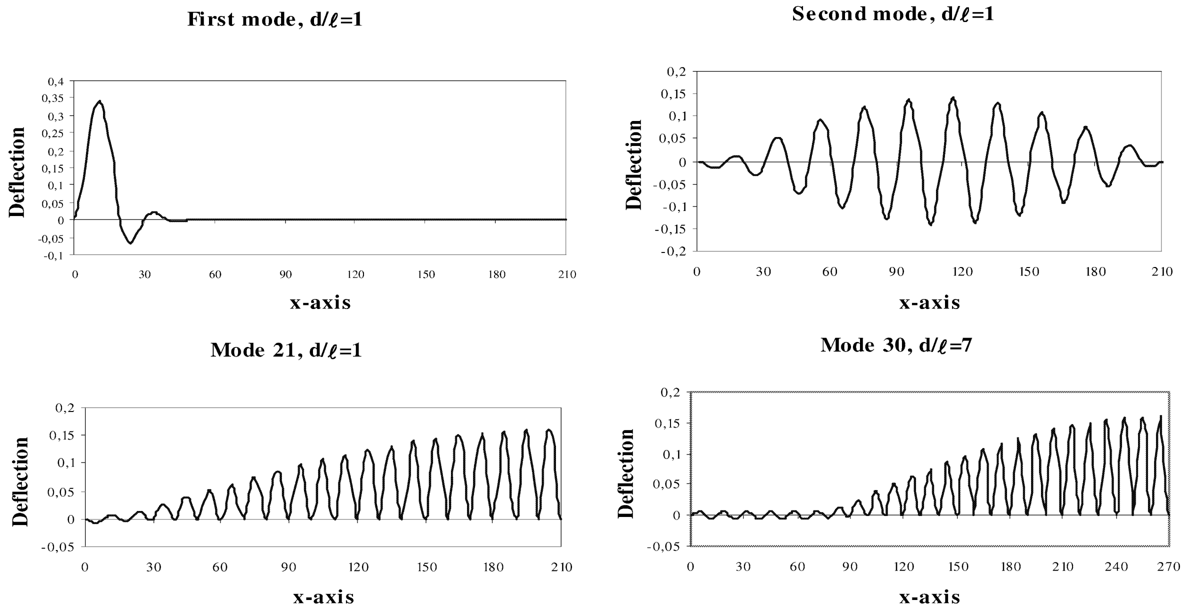

Fig. 3. Some eigenmodes.

$$\frac{\mathrm{d}^4 V_r}{\mathrm{d}x^4} - \lambda V_r = 0, \quad x \in [0, L], \qquad \left[\left[\frac{\mathrm{d}^3 V_r}{\mathrm{d}x^3}(i\ell)\right]\right] = -K V_r(i\ell), \quad i = 1, \ldots, N-1 \tag{2}$$

$$V_s(0) = V_r(0), \quad V_s'(0) = V_r'(0), \quad V_s''(0) = V_r''(0), \quad V_s'''(0) = V_r'''(0) \tag{3}$$

where $\lambda = \rho S\omega^2/EI$, $K = \frac{k}{EI}$, k is the spring stiffness and ω^2 is the square of the natural frequency. $V_s(x)$ (respectively $V_r(x)$) is the deflection in the supplementary beam (respectively periodic beam). Obviously, Eqs. (1)–(3) have to be completed by boundary conditions.

As explained in [4,5], the multiple scale analysis can be used to describe the modulated modes. The principle of this method can be described as follows. A small parameter η is introduced, for instance as the ratio between the length of the basic cell and the length of the whole structure. The starting point $\{\lambda_0, w_0(x)\}$ of the perturbation technique is solution of the eigenvalue problem (2) posed on few basic cells and with periodicity conditions. As it is classical [3], $V_r(x)$ and λ are sought as an integer-power series with respect to η:

$$V_r(x) = \sum_{i=0}^{\infty} \eta^i V_{ri}(x, X), \qquad \lambda = \sum_{i=0}^{\infty} \eta^i \lambda_i \tag{4}$$

where x is a local variable and $X = \eta x$ is a global variable that can describe the slow variation of the eigenmodes (Fig. 3). The mode V_r is assumed to be locally periodic, i.e., periodic with respect to the local variable x. Inserting the asymptotic expansions (4) into (2) and using the classical rules of the two-scale expansion method [3], one finds an asymptotic expansion in the form:

$$V_r(x, X) = A_0(X)w_0(x) + \eta\big(A_1(X)w_0(x) + A_0'(X)w_1(x)\big) + \theta\big(\eta^2\big) \tag{5}$$

where $w_1(x)$ is solution of a periodic problem. $A_0(X)$ and $A_1(X)$ are amplitude functions that can account for slow spatial modulations of the modes. These amplitude functions satisfy the following equations:

$$CA_0'' + \lambda_2 A_0 = 0 \tag{6}$$

$$CA_1'' + \lambda_2 A_1 = DA_0''' - \lambda_3 A_0 \tag{7}$$

where the constants C and D are determined from the periodic modes $w_0(x)$ and $w_1(x)$. The detailed definition of periodic problems satisfied by the modes $w_i(x)$ and the constants C, D can found in [1,4].

2.2. How to get boundary conditions for the amplitudes?

Because of the reduction of the 4th order equation (2) to the 2nd order one (6), it is not possible to satisfy all the boundary conditions and there exist boundary layers. Some local corrections have to be introduced in the expansions. As explained in [4], this correction w_{loc} can be defined by Floquet theory [6] and (4a) can be modified as:

$$V_r(x) = \sum_{i=0}^{\infty} \eta^i \big(V_{ri}(x, X) + \alpha_i w_{\text{loc}}(x)\big) \tag{8}$$

So, two asymptotic expansions have to be considered. The first one is valid near $x = 0$ where w_{loc} is the decreasing Floquet function and the second is valid near $x = L$ where w_{loc} is the increasing Floquet function. These functions correspond to the non periodic eigensolutions of the fundamental transfer matrix [1,6].

Let us consider, for example, a clamped beam at $x = L$. The deduced boundary conditions for the amplitudes A_0 and A_1 depend on the properties of the periodic mode w_0 that satisfies $w_0(L) = w_0'(L) = 0$ (case 2) or not (case 1) [4]:

$$A_0(L\eta) = 0 \quad \text{and} \quad A_1(L\eta) = C_2 A_0'(L\eta) \quad \text{in case 1} \tag{9}$$

$$A_0'(L\eta) = 0 \quad \text{and} \quad A_1'(L\eta) = C_3 A_0''(L\eta) \quad \text{in case 2} \tag{10}$$

The constants C_2 and C_3 are obtained from the periodic solutions w_i and from the Floquet function w_{loc}, in the same manner as the boundary conditions at $x = 0$, see (12)–(14) below.

Now, we discuss the similar treatment for the continuity conditions at $x = 0$ (3). These relations have to be satisfied at each order of the previous asymptotic development. By solving Eq. (2) and boundary conditions at $x = -d$, $V_s(x)$ is as follows:

$$V_s(x) = a\phi_1(\lambda, x) + b\phi_2(\lambda, x) \tag{11}$$

where $\phi_1(\lambda, x)$, $\phi_2(\lambda, x)$ are known functions, depending on λ and d, and a, b are arbitrary constants determined from the continuity conditions (3). The latter constants a and b are also expanded into series of η. Considering (8), (11) and using the derivative rules of the multiple scale method, one establishes that the continuity conditions (3) leads to:

$$\begin{bmatrix} w_0(0) & w_{\text{loc}}(0) & -\phi_1(\lambda,0) & -\phi_2(\lambda,0) \\ w_0'(0) & w_{\text{loc}}'(0) & -\phi_1'(\lambda,0) & -\phi_2'(\lambda,0) \\ w_0''(0) & w_{\text{loc}}''(0) & -\phi_1''(\lambda,0) & -\phi_2''(\lambda,0) \\ w_0'''(0) & w_{\text{loc}}'''(0) & -\phi_1'''(\lambda,0) & -\phi_2'''(\lambda,0) \end{bmatrix} \begin{bmatrix} A_0(0) \\ \alpha_0 \\ a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{at the first order} \tag{12}$$

$$\begin{bmatrix} w_0(0) & w_{\text{loc}}(0) & -\phi_1(\lambda,0) & -\phi_2(\lambda,0) \\ w_0'(0) & w_{\text{loc}}'(0) & -\phi_1'(\lambda,0) & -\phi_2'(\lambda,0) \\ w_0''(0) & w_{\text{loc}}''(0) & -\phi_1''(\lambda,0) & -\phi_2''(\lambda,0) \\ w_0'''(0) & w_{\text{loc}}'''(0) & -\phi_1'''(\lambda,0) & -\phi_2'''(\lambda,0) \end{bmatrix} \begin{bmatrix} A_1(0) \\ \alpha_1 \\ a_1 \\ b_1 \end{bmatrix} = -A_0'(0) \begin{bmatrix} w_1(0) \\ w_1'(0) \\ w_1''(0) \\ w_1'''(0) \end{bmatrix} \quad \text{at the second order} \tag{13}$$

One assumes that this matrix is invertible. By condensation of α_i, a_i, b_i, one deduces the boundary conditions for the amplitudes at $x = 0$:

$$A_0(0) = 0, \qquad A_1(0) = C_1 A_0'(0) \tag{14}$$

where C_1 is a constant defined from the known functions w_0, w_1, w_{loc}, ϕ_1, ϕ_2.

2.3. Second and third order estimates of the spectrum

The amplitude equation (6) can be solved analytically on account of the boundary conditions (9), (10) and (14a). This leads to an approximation of the spectrum close to λ_0:

$$\begin{cases} A_0(X) = \sin(n\pi X/L\eta) \\ \lambda(n) = \lambda_0 + Cn^2\pi^2/L^2 + \theta(\eta^3), \quad n = 1, 2, 3 \ldots, \end{cases} \quad \text{for case 1} \tag{15}$$

$$\begin{cases} A_0(X) = \sin(n\pi X/2L\eta) \\ \lambda(n) = \lambda_0 + Cn^2\pi^2/4L^2 + \theta(\eta^3), \quad n = 1, 3, 5 \ldots \end{cases} \quad \text{for case 2} \tag{16}$$

At this stage in the asymptotic two-scale analysis, one can note that the method permits to generate an infinite number of eigenvalues of the initial structure from a periodic mode. Formula (15) is identical to that obtained in reference [4] for a clamped periodic beam. To get the effect of the supplementary beam, one has to consider the analysis at the next order. The analysis is detailed for case 1 in what follows.

The operator in the left hand side of the amplitude equation (7) is singular, because of equation (6). Thus there is a solvability condition:

$$\int_0^{L\eta} (CA_1'' + \lambda_2 A_1) A_0 \, \mathrm{d}X = \int_0^{L\eta} (DA_0''' - \lambda_3 A_0) A_0 \, \mathrm{d}X \tag{17}$$

Considering the amplitude equation (6) and boundary conditions (9), (10), (15), one finds that λ_3 is in the form:

$$\lambda_3 = C(C_1 - C_2)\frac{n^2\pi^2}{L^3\eta^3}, \quad n = 1, 2, \ldots \tag{18}$$

Thus, the asymptotic expansion of the eigenvalues λ at third order is:

$$\lambda = \lambda_0 + C^* n^2\pi^2/L^2, \quad n = 1, 2, \ldots, \text{ where } C^* = C\big(1 + (C_1 - C_2)/L\big) \tag{19}$$

The same analysis can be made for case 2 and one finds the approximation of the spectrum:

$$\lambda = \lambda_0 + C^* n^2\pi^2/4L^2, \quad n = 1, 3, 5, \ldots, \text{ where } C^* = C\Big(1 + \big(2(C_3 - C_1) - D\big)/L\Big) \tag{20}$$

3. Numerical results

Consider the structure depicted in Fig. 1. The material data are $E = 2.1 \times 10^{11}$, $\nu = 0.3$, $\rho = 7800$. The ends of the whole beam are clamped. The whole structure and the basic cell have been discretised by cubic beam elements. The whole structure has been split into 210 elements, which corresponds to 422 d.o.f. For the basic cell, only 22 d.o.f. are needed. The obtained eigenvalues λ are reported in Fig. 2. For this example, the first and 22th modes are localized modes. The modulated modes are closely located in well separated packets. Some modes for various aspect ratios d/ℓ are plotted on Fig. 3. The modulated modes appear as a slow modulation of a periodic one.

The following tables present some eigenvalues obtained from the present method at third order and those of the direct computation. In Tables 1–3, we report the first eight eigenvalues corresponding to the modulated modes

Table 1
Smallest eigenvalues. Clamped beam. $\ell = 1$, $k = 100\,000$, $N = 21$ cells, $d/\ell = 1$. Obtained values of the constants: $\lambda_0 = 97.4$, $C = 59.19$, $C^* = 52.72$

Mode number	2	3	4	5	6	7	8	9
Formula (19)	98.57	102.1	108.0	116.3	126.9	139.9	154.3	172.9
Direct computation	98.69	102.0	107.8	116.0	126.4	139.2	153.3	171.8

Table 2
Smallest eigenvalues. Same data as in Table 1, except $N = 50$ cells. Modified constant: $C^* = 57.25$

Mode number	2	3	4	5	6	7	8	9
Formula (19)	97.6	98.3	99.4	101.0	103.0	105.5	108.4	111.8
Direct computation	97.6	98.3	99.4	101.0	103.1	105.5	108.5	111.9

Table 3
Smallest eigenvalues. Same data as in Table 1, except $d/\ell = 3$. Modified constant: $C^* = 54.51$

Mode number	3	4	5	6	7	8	9	10
Formula (19)	98.6	102.2	108.3	116.8	127.7	141.1	156.8	175.0
Direct computation	98.6	102.5	108.6	116.5	125.4	135.6	148.9	165.3

Table 4
Five eigenvalues at the end of the first packet structure. Same data as in Table 1. $\lambda_0 = 500.3$, $C = -209.94$, $C^* = -196.96$

Mode number	17	18	19	20	21
Formula (20)	411.0	446.3	472.7	490.3	499.3
Direct computation	417.5	446.4	471.2	489.5	499.3

of the first packet for different values of the aspect ratios d/ℓ, $N = L/\ell$. From these results, one can note that the asymptotic two scale method at third order describes quite perfectly the eigenvalues near λ_0. Similar results have also been obtained (see Table 4) for the eigenvalues at the end of the first modulated modes packet using the formula (20) [1]. By comparison of Tables 1 and 2, it appears that the accuracy of the estimate of the spectrum is better when the number of cells is larger. The efficiency of the procedure does not seem to depend on the additional structure: see Table 3. Moreover, the boundary condition (14) remains valid for any d, see Fig. 3(d).

4. Conclusions

In this Note, the existence of an equivalent continuous model for the modulated modes of a typical nearly repetitive structure has been established. Especially, boundary conditions for the amplitude equations have been deduced. This continuous model accounts for the beginning or the end of a mode packet. It is possible to extend the model to describe a whole packet [1], but it is not easy to define the associated boundary conditions in any case.

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