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On two-dimensional water waves in a canal

Vladimir Kozlov [a], Nikolay Kuznetsov [b]

[a] *Department of Mathematics, Linköping University, 581 83 Linköping, Sweden*
[b] *Laboratory for Mathematical Modelling of Wave Phenomena, Institute for Problems in Mechanical Engineering, Russian Academy of Sciences, V.O., Bol'shoy pr. 61, St. Petersburg 199178, Russia*

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Abstract

This Note deals with an eigenvalue problem that contains a spectral parameter in a boundary condition. The problem for the two-dimensional Laplace equation describes free, time-harmonic water waves in a canal having uniform cross-section and bounded from above by a horizontal free surface. It is shown that there exists a domain for which at least one of eigenfunctions has a nodal line with both ends on the free surface. Since Kuttler essentially used the non-existence of such nodal lines in his proof of simplicity of the fundamental sloshing eigenvalue in the two-dimensional case, we propose a new variational principle for demonstrating this latter fact. ***To cite this article: V. Kozlov, N. Kuznetsov, C. R. Mecanique 331 (2003).***

Résumé

Sur les ondes de surface bidimensionnelles dans un canal. Cette Note porte sur un problème aux valeurs propres avec le paramètre spectral dans la condition aux limites. Le problème pour l'équation de Laplace bidimensionnelle décrit les ondes de surface libres, harmoniques dans le temps, dans un canal limité supérieurement par une surface libre horizontale. On démontre l'existence d'un domaine tel que au moins une des fonctions propres a une ligne nodale avec les deux extrémités sur la surface libre. Kuttler avait utilisé essentiellement la non - existence de telles lignes nodales dans sa démonstration du caractère simple de la valeur propre de ballottement dans le cas bidimensionnel. Nous proposons donc un nouveau principe variationnel pour prouver ce fait. ***Pour citer cet article : V. Kozlov, N. Kuznetsov, C. R. Mecanique 331 (2003).***
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Mots-clés : Mécanique des fluides ; Oscillations libres d'une fluide ; Courbe nodale ; Valeur propre fondamenale ; Principe variationnelle

1. Introduction: statement of the problem

Let an inviscid, incompressible, heavy fluid (water) occupy an infinitely long canal bounded above by a free surface of finite width. Let the surface tension be negligible and let the water motion be small-amplitude, irrotational, and two-dimensional in the planes normal to the generators of the canal bottom B. Under these assumptions the following spectral problem describes the free time-harmonic oscillations of water:

E-mail addresses: vlkoz@mai.liu.se (V. Kozlov), nikuz@wave.ipme.ru (N. Kuznetsov).

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$$u_{xx} + u_{yy} = 0 \quad \text{in } W, \qquad u_y = \nu u \quad \text{on } F, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } B \tag{1}$$

Here $u(x, y)$ is the velocity potential of the flow and rectangular Cartesian coordinates (x, y) are taken in the plane of the motion, with the origin and the x-axis in the mean free surface, whereas the y-axis is directed upwards. The canal cross-section W is a bounded, simply connected domain whose piecewise smooth boundary ∂W has no cusps. One of the open arcs forming ∂W is an interval F of the x-axis (the free surface of water), and the bottom $B = \partial W \setminus \overline{F}$ is the union of open arcs, lying in the half-plane $y < 0$, complemented by the corner points (if there are any) connecting these arcs. We complement (1) by the following orthogonality condition

$$\int_F u \, \mathrm{d}x = 0 \tag{2}$$

thus excluding the zero eigenvalue. The spectral parameter ν in (1) is equal to ω^2/g, where ω is the radian frequency of water oscillations and g is the acceleration due to gravity.

This problem – it is usually referred to as the sloshing problem – has been the subject of a great number of studies over more than two centuries (a historical review is given by Fox and Kuttler [1]). It is well known since the 1950s that problem (1), (2) has a discrete spectrum; that is, there exists a sequence of eigenvalues $0 < \nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_n \leqslant \cdots$, each having a finite multiplicity equal to the number of repetitions; moreover, $\nu_n \to \infty$ as $n \to \infty$, and the corresponding eigenfunctions $\{u_n\}_1^\infty \subset H^1(W)$ form a complete system in an appropriate Hilbert space.

In his note [2], Kuttler tried to investigate the behaviour of nodal lines of the eigensolution u_n. (In the spectral theory of boundary value problems, studies of the patterns of nodal lines ascends to the classical works of Courant [3].) The approach in [2] is based on the following key lemma:

Lemma. *Nodal lines of an eigenfunction of problem* (1), (2) *have one end on the free surface and the other one on the bottom.*

Examining the proof of this lemma shows that there is a gap in Kuttler's reasoning. Our attempt to fill in the gap resulted in constructing an example of sloshing eigenfunction that has a nodal line with both ends on the free surface (see Proposition 2.6). The construction involves one of the velocity potentials defined in $\mathbb{R}^2_-$ and having singularities on $\partial\mathbb{R}^2_-$; earlier these potentials were used for demonstrating the existence of point eigenvalues embedded in the continuous spectrum of the water-wave problem (see [4], Chapter 4).

Since all results formulated in [2] are proved by using the above fallacious lemma, it is necessary to check whether they are true. It occurs that one of the main results in [2], the simplicity of the fundamental eigenvalue, is valid (see Proposition 3.1), but the proof of this fact is far from being obvious. It involves a new variational principle for an equivalent spectral problem in which stream function appears instead of the velocity potential.

2. Nodal lines and domains of the velocity potential

In this section, we construct an example of the sloshing problem, possessing an eigenfunction that has only one nodal line whose both ends are on F (Section 2.1), and consider some simple properties of nodal domains in Section 2.2.

2.1. Example

Our example involves a particular pair velocity potential/stream function introduced in [4], Section 4.1.1. The simplest example of this kind was proposed by McIver [5], but for our purpose we need another one that has more nodal lines. Here we investigate nodal lines of u and v simultaneously in order to obtain the required example, whereas in [4] properties of the level lines were studied only for v.

For $\nu = 3/2$ we consider the following two functions:

$$u(x, y) = \int_0^\infty \frac{\cos k(x-\pi) + \cos k(x+\pi)}{k-\nu} \mathrm{e}^{ky}\,\mathrm{d}k \tag{3}$$

$$v(x, y) = \int_0^\infty \frac{\sin k(x-\pi) + \sin k(x+\pi)}{\nu-k} \mathrm{e}^{ky}\,\mathrm{d}k \tag{4}$$

where both numerators vanish at $k = \nu = 3/2$, and so the integrals are usual infinite integrals. It is easy to verify that u and v are conjugate harmonic functions in $\mathbb{R}^2_-$ such that $u(-x, y) = u(x, y)$ and $v(-x, y) = -v(x, y)$. Moreover, u and v are infinitely smooth up to $\partial\mathbb{R}^2_- \setminus \{x = \pm\pi,\ y = 0\}$ and well-known facts from the theory of distributions imply that $[u_y - \nu u]_{y=0}$ is equal to a linear combination of Dirac's measures at $x = \pi$ and $x = -\pi$. Therefore,

$$u_y = \nu u \quad \text{on } \partial\mathbb{R}^2_- \setminus \{x = \pm\pi, y = 0\} \tag{5}$$

The calculated nodal lines of u and v are shown in Fig. 1 (bottom) and we proceed with formulations of several assertions proving that the location of the lines is as plotted. It is clear that the negative y-axis is a nodal line of v and another nodal line is described by

Proposition 2.1. *Apart from $\{x = 0,\ y < 0\}$, there is only one nodal line of $v(x, y)$ in $\mathbb{R}^2_-$, which is smooth, symmetric about the y-axis, and has both ends on the x-axis so that the right one, say $(x_0, 0)$, lies between the origin and the point $(\pi, 0)$.*

The latter nodal line serves as the bottom B in our example; the right half of this line is shown by the dashed line in Fig. 1 (bottom), where the bullet marks the position of $(\pi, 0)$ and the solid lines are nodal lines of u. Since (5) holds for u and the Cauchy–Riemann equations yield that the Neumann condition in (1) is fulfilled on the so defined bottom B, we see that u satisfies the sloshing problem in the domain W between this B and the x-axis.

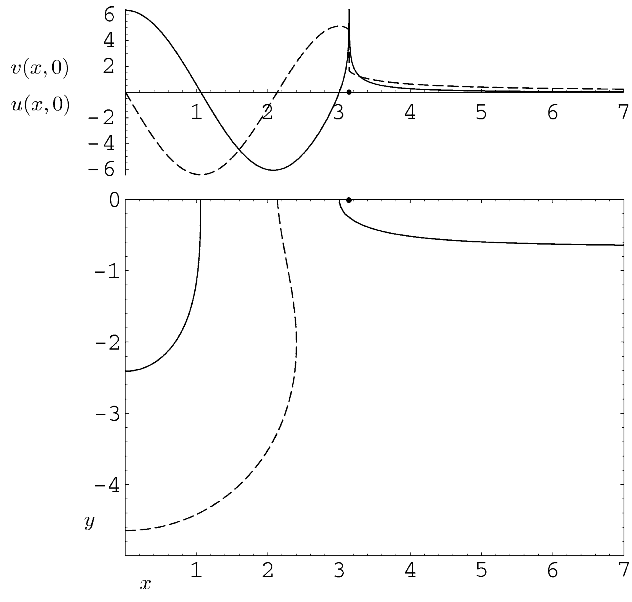

Fig. 1. Nodal lines of u (solid lines) and v (dashed line) given by (3) and (4), respectively, with $\nu = 3/2$.

Moreover, Fig. 1 (bottom) shows that there is only one nodal line of u in this water domain W and this property will be stated in Proposition 2.6 below.

Our proof of Proposition 2.1 is based on the next lemma illustrated in Fig. 1 (top), where $u(x,0)$ (solid line) and $v(x,0)$ (dashed line) are plotted.

Lemma 2.2. *Function $v(x,0)$ has the following properties on the half-axis $x \geqslant 0$:*

(i) *$v(x,0)$ is continuous on $[0,\pi]$ and on $[\pi,+\infty)$, but*

$$v(x,0) \to \mathrm{Si}(3\pi) + \frac{\pi}{2} \mp \frac{\pi}{2} \quad \text{as } x \to \pi \pm 0 \tag{6}$$

(ii) *there are exactly two zeroes of $v(x,0)$ on $[0,\pi)$, at $x=0$ and at some point $x_0 \in (2\pi/3,\pi)$;*

(iii) *$v(x,0) < 0$ for $x \in (0,x_0)$ and there is only one point $x_m \in (0,x_0)$, where $v(x,0)$ attains minimum;*

(iv) *$v(x,0) > 0$ for $x > x_0$;*

(v) *$v(x,0)$ is a monotonically decreasing convex function for $x > \pi$ and it tends to zero as $x \to +\infty$.*

The proof of this lemma is based on various representations for $v(x,0)$ following from formulae 3.722.5 and 3.354.1 in Gradshteyn and Ryzhik [6] (see [4], Section 4.1.1 for details).

The behaviour of $u(x,0)$ is more complicated then that of $v(x,0)$ and the corresponding integral representation following from 3.722.7 and 3.354.2 in [6] has the form: $u(x,0) = 2\pi H(\pi - x)\cos\nu x + \int_0^\infty [\mathrm{e}^{-|x-\pi|k\nu} + \mathrm{e}^{-(x+\pi)k\nu}]\frac{k\,\mathrm{d}k}{1+k^2}$. It implies that there is a logarithmic singularity at $x=\pi$ and that the function is positive, monotonically decreasing, and convex for $x>\pi$; moreover, it tends to zero as $x \to +\infty$. The existence of zeroes is given by

Lemma 2.3. *There are exactly two zeroes of $u(x,0)$ on $(0,\pi)$ and the function changes sign at these zeroes. The first zero is at $x = x_m$ (see (iii) in Lemma 2.2) and the second one at $x = x_M$, $x_m < x_M < \pi$.*

An immediate consequence of this lemma is

Corollary 2.4. *Two nodal lines of $u(x,y)$ emanate from $(x_m,0)$ and $(x_M,0)$.*

More detailed behaviour of the nodal lines of u (it is obvious that they are symmetric about the y-axis) is described by

Lemma 2.5. *There are two nodal lines of u in $\{x>0,\ y<0\}$; one of them emanates from $(x_m,0)$ and crosses the negative y-axis and the other one emanates from $(x_M,0)$ and goes to infinity.*

This lemma together with Proposition 2.1 allows us to prove

Proposition 2.6. *Inside the domain bounded from below by the nodal line of v that has endpoints at $(\pm x_0,0)$ the sloshing eigenfunction u given by (3) has a single nodal line with endpoints $(\pm x_m,0)$.*

2.2. Properties of nodal domains

Let $N(u) = \{(x,y) \in \overline{W}\colon\ u(x,y) = 0\}$ be the set of nodal lines of a sloshing eigenfunction u. A connected component of $W \setminus N$ will be called a nodal domain. On account of (1), one concludes that each nodal domain has a piecewise smooth boundary without cusps. The following simple assertions from [2] is given here for the sake of completeness.

Lemma 2.7. *If R is a nodal domain of u, then $\overline{R} \cap F$ contains an interval of the x-axis.*

Lemma 2.8. *The number of nodal domains corresponding to u_n is less or equal to $n+1$.*

Kuttler's reasoning in [2], which is a version of the Courant's original proof [3], turns out to be the proof of the latter lemma when the unnecessary reference to the fallacious lemma is omitted.

An immediate consequence of Lemmas 2.7 and 2.8 is the following

Corollary 2.9. *The sloshing eigenfunction u_n cannot change sign more than $2n$ times on F.*

It should be noted that the number of nodal domains corresponding to u_n is less then $n+1$ in some cases. For instance, the eigenfunction constructed as the example in Section 2.1 has two nodal domains. However, the corresponding eigenvalue $\nu = 3/2$ is not the fundamental one. (This follows from Proposition 3.1(ii), which says that the fundamental eigenfunction has only one nodal line connecting F and $\overline{B}$.) Therefore, the number of nodal domains in the example which is equal to two is less than the maximal number permitted by Lemma 2.8 which is at least three. On the other hand, the eigenfunctions in a rectangle have the maximal number of nodal domains.

3. The fundamental eigenvalue is simple

Proposition 3.1. (i) *The fundamental eigenvalue of problem* (1), (2) *is simple.*

(ii) *The corresponding eigenfunction has only one nodal line connecting F and $\overline{B}$.*

For proving this assertion we use a variational principle for a boundary value problem that is equivalent to (1), (2) and involves the conjugate of u harmonic function v (stream function). The latter satisfies

$$v_{xx} + v_{yy} = 0 \quad \text{in } W, \qquad -v_{xx} = \nu v_y \quad \text{on } F, \qquad v = 0 \quad \text{on } B \tag{7}$$

Here the second condition is derived from the second condition in (1) by differentiation and application of the Cauchy–Riemann equations; the last condition in (7) is obtained from the last condition in (1) by an appropriate choice of the additive constant in v. It is clear that the multiplicity of ν as an eigenvalue of (1), (2) is the same as its multiplicity as an eigenvalue of (7).

Without loss of generality we assume that $F = \{-1 < x < 1,\ y = 0\}$. For formulating a variational principle for problem (7), we rewrite the boundary condition on F in the form:

$$v = \nu \mathcal{K} v_y \quad \text{on } F \tag{8}$$

Here $(\mathcal{K} f)(x) = \int_{-1}^{1} K(x,\xi) f(\xi)\,\mathrm{d}\xi$ and the symmetric kernel $K(x,\xi) = K(\xi,x)$ is equal to $K(x,\xi) = (1-x)(\xi+1)/2$ for $\xi < x$. It is clear that $\mathcal{K}$ is a symmetric, positive operator in $L_2(F)$. Finally, by $\mathcal{D}_N$ we denote the so-called Dirichlet–Neumann operator that maps ϕ given on F into $\mathcal{D}_N \phi = \Phi_y\big|_F$, where Φ must be found from the following Dirichlet problem: $\nabla^2 \Phi = 0$ in W, $\Phi = \phi$ on F, $\Phi = 0$ on B. It is well known that $\mathcal{D}_N$ is a positive, self-adjoint operator in $L_2(F)$. It follows from (7) and (8) that for finding the fundamental eigenvalue ν_1 one can use the following variational principle:

$$\nu_1 = \min_{w \in H_B^1(W)} \frac{\int_W |\nabla w|^2\,\mathrm{d}x\,\mathrm{d}y}{\int_F \mathcal{D}_N w\,(\mathcal{K}\mathcal{D}_N)\,w\,\mathrm{d}x} \tag{9}$$

where $H_B^1(W)$ is the subspace of $H^1(W)$ that consists of functions with vanishing traces on B. Since the operator defined by the quadratic form in the denominator is compact in $H_B^1(W)$, there exists a nontrivial function w^* for which the quotient (9) attains the minimum. Moreover, it is easy to verify that $\nabla^2 w^* = 0$ in W. Therefore, $\mathcal{D}_N w^* = w_y^*$, and so w^* is an eigenfunction of (7).

The first statement in Proposition 3.1 is an immediate consequence of the following

Lemma 3.2. *The fundamental eigenvalue of problem* (7) *is simple and the corresponding eigenfunction may be chosen to be positive in $W \cup F$.*

Our proof of the second statement in Proposition 3.1 is based on examining the critical points of $v(x,0)$.

Let us consider water domains satisfying the extra condition that W is contained within the semistrip bounded by $\overline{F}$ and two vertical rays going downwards from the endpoints of $\overline{F}$. This condition was first introduced in the work [7] by John (now it is usually referred to as John's condition), where the so-called water-wave problem was considered (see also [4], Chapters 3 and 4). It occurs, that if W satisfies John's condition, then the second statement of Proposition 3.1 may be improved.

Proposition 3.3. *Let v be the fundamental eigenfunction of problem* (7), *then $v \in C^1(\overline{F})$. Moreover, if W satisfies John's condition, then $\mp v_x(\pm 1, 0) > 0$. We recall that without loss of generality F is assumed to coincide with* $\{-1 < x < 1,\, y = 0\}$.

4. Discussion

Let us discuss a couple of open questions concerning the eigensolutions to problem (1), (2). The first of them is related to the number of sign changes on F of the eigenfunction u_n. Our Corollary 2.9 gives only a rough upper bound $2n$ for this number and the question is whether one can replace $2n$ by n as stated in [2], where the proof is based on the fallocious lemma. Of course, it follows from the explicit expression that the nth sloshing eigenfunction has n changes of sign when W is a rectangle and F is its top side.

Another open question is whether *all* eigenvalues of problem (1), (2) are simple. There is a number of particular geometries for which all eigenvalues are proved to be simple. Of course, this is obvious for rectangular domains whose top side is the free surface (by separation of variables one obtains the explicit expressions for both eigenvalues and eigenfunctions in this case). A less trivial result is given implicitly in §258 of Lamb's book [8], where Kirchhoff's solution is presented for the case when B is formed by two segments at $\pi/4$ to the vertical (we recall that it is assumed that $F = \{-1 < x < 1,\, y = 0\}$). For this triangle the eigenvalues are $\nu_n = \mu_n (\tanh \mu_n)^{(-1)^n}$, where $\mu_n,\ n = 1, 2, \ldots,$ are positive roots of $\cos 2\mu \cosh 2\mu = 1$. Since the roots of the last transcendental equation are simple, the sloshing eigenvalues are also simple. Recently, Kuznetsov and Motygin [9] established that all eigenvalues are simple for $W = \mathbb{R}^2_-$ when F consists either of one gap or of two equal gaps in the rigid dock covering W. Finally, for the domains which intersect the x-axis at right angles all eigenvalues with sufficiently large numbers are simple. This follows from the asymptotic formula $\nu = \pi n/2 - (\kappa_+ + \kappa_-)/4\pi + \mathrm{o}(n^{-1})$ as $n \to \infty$ that was proved by Davis [10]. Here κ_+ (κ_-) is the curvature of B at the right (left) intersection with the x-axis.

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