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Incremental energy minimization in dissipative solids

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Abstract

The incremental energy minimization is examined as a method of determining solution paths for time-independent dissipative solids. Isothermal quasi-static deformations are considered, and the deformation work is locally decomposed into the increments in free energy and intrinsic dissipation. General conditions necessary for the applicability of the minimization procedure are derived and discussed. ***To cite this article: H. Petryk, C. R. Mecanique 331 (2003).***

Résumé

Minimisation d'énergie excédentaire dans solides dissipatifs. Nous étudions la minimisation d'énergie excédentaire comme méthode possible de détermination des solutions sous forme de chemin de déformation pour des solides dissipatifs indépendant de temps. On considère les déformations quasi-statiques, et le travail de déformation est décomposé localement en deux parties, l'excédent d'énergie libre, et la dissipation intrinsèque. Nous discutons les conditions dérivées ici, nécessaires afin de rendre la procédure de minimisation applicable au problème posé. ***Pour citer cet article : H. Petryk, C. R. Mecanique 331 (2003).***

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Mots-clés : Solides et structures ; Matériaux dissipatifs ; Plasticité ; Énergie ; Stabilité des chemins

1. Introduction

This paper is concerned with a variational method of determining the response of an inelastic body to slowly varying loading. It is based on the concept of the minimization of *incremental* energy supplied to the examined *system*, apparently first developed by Petryk [1–3] for a class of rate-independent problems at finite deformation. Essentially the same idea can also be found in more recent papers by other authors, e.g., [4–6]. An advantage of the incremental energy minimization method is that it provides a criterion of choice if non-unique solutions exist. Justification of that approach follows from its consistency with Hill's bifurcation theory [7,8], as shown in [2], and with the concept of instability of plastic deformation processes or paths [3]. The non-convex minimization of an

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incremental energy supply enables crossing multiple bifurcation points with automatic selection of the post-critical deformation path in finite element simulations [9,10]. A general condition of applicability is that the global tangent stiffness matrix must be symmetric.

The deformation work, which is the principal part of the minimized energy functional, is usually expressed as an integral of the stress power. According to classical thermodynamics, the work supplied in isothermal quasi-static deformation can be decomposed into the increments in free energy and intrinsic dissipation. With this decomposition adopted, the aim of this paper is to examine the applicability of the incremental energy minimization to time-independent dissipative solids when the increments of internal state variables are treated as independent unknowns. As opposed to the closely related earlier works dealing with path stability, see [11,4] for comparison, the main task here is to characterize the minimization procedure itself along with its prerequisities. In particular, a symmetry restriction on the dissipation function is derived as a condition necessary for the intrinsic consistency between the first- and second-order minimization. This symmetry restriction has not been mentioned in some recent works dealing with numerical applications of the minimization procedure.

2. Minimization of incremental energy

Following the approach originated in [1,2], consider the minimization problem

$$\Delta E \to \min \quad \text{subject to kinematical constrains} \tag{1}$$

where ΔE is the increment in energy to be supplied from external sources to the mechanical *system*, consisting of the deformed body and the loading device, in order to produce quasi-statically a deformation increment, generally with the help of additional perturbing forces. The prefix Δ denotes a virtual increment from a given state, corresponding to an increment $\Delta\lambda$ of a loading parameter λ used to define changes in external constrains. For inelastic solids with path-dependent energy consumption, (1) is fundamentally different from the usual energy minimization at *fixed* λ.

Suppose that the minimum in (1) exists, and denote it by ΔE^0. Then (1) can be rewritten as

$$\Delta E^0 \leqslant \Delta E \quad \text{subject to kinematical constrains} \tag{2}$$

Under the restriction to isothermal processes, ΔE is defined as follows

$$\Delta E = \Delta W + \Delta\Omega \tag{3}$$

where ΔW is the total work increment supplied to the deforming body, and $\Delta\Omega$ is a potential energy increment of the loading device, assuming that the incremental loading is *conservative* in an overall sense. A further insight into the meaning of ΔE is gained if the mechanical system is imagined to be placed in a heat reservoir. Then ΔE can be identified with a change in the internal energy of the resulting compound thermodynamic system {deforming body + loading device + heat reservoir} [11].

The mathematical expression for the basic physical quantity ΔW may depend on the problem. In particular, for a continuous body that in a fixed reference configuration occupies a domain B of volume V, the deformation work increment ΔW can be split at constant temperature as follows

$$\Delta W = \int_B \Delta w \,\mathrm{d}V = \int_B (\Delta\phi + \Delta\mathcal{D})\,\mathrm{d}V = \int \dot{W}\,\mathrm{d}t, \quad \dot{W} = \int_B \dot{w}\,\mathrm{d}V, \quad \dot{w} = \dot{\phi} + D \tag{4}$$

where a superimposed dot denotes the forward rate (one-sided material time derivative) with respect to time t, ϕ is the Helmholtz free energy density, and $\mathcal{D}$ is the intrinsic dissipation density, being a path-dependent integral of a local dissipation function D. The constitutive law used to determine ΔW is here time-independent and local. The analysis can be extended to global internal variables [4].

Let a local material state $\mathcal{G}$ be characterized in terms of the deformation gradient $\mathbf{F}$ and internal state variables considered in the reference configuration, denoted collectively by $\boldsymbol{\alpha}$, so that $\mathcal{G} := (\mathbf{F}, \boldsymbol{\alpha})$ at a fixed temperature. The two functions ϕ and D needed to determine ΔW from (4) are assumed to be given. For a smooth free energy density function ϕ, denoted for simplicity by the same symbol as its value, we have the usual formulae

$$\phi = \phi(\mathbf{F}, \boldsymbol{\alpha}), \qquad \phi_{,\mathbf{F}} = \mathbf{S}, \qquad -\phi_{,\boldsymbol{\alpha}} = \boldsymbol{A} \tag{5}$$

where the comma as a subscript denotes the partial Fréchet derivative, $\mathbf{S}$ is the transposed nominal stress, and $\boldsymbol{A}$ stands for the set of the thermodynamic driving forces conjugate to $\boldsymbol{\alpha}$. A continuous dissipation function D for a time-independent material is assumed to satisfy

$$D = D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) \geqslant 0, \quad D(r\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) = rD(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) \quad \text{for all } r > 0 \tag{6}$$

The condition of thermodynamic compatibility

$$D = \boldsymbol{A} \cdot \dot{\boldsymbol{\alpha}} \tag{7}$$

equivalent to $\dot{w} = \mathbf{S} \cdot \dot{\mathbf{F}}$, is necessary for $\dot{\boldsymbol{\alpha}}$ to be compatible with the current thermodynamic forces $\boldsymbol{A}(\mathcal{G})$, while (7) fails for other *virtual* $\dot{\boldsymbol{\alpha}}$. A central dot denotes full contraction (or a scalar product). It will be convenient to rewrite the condition (7) in a more mathematical form as follows

$$\dot{\boldsymbol{\alpha}} \in \mathcal{L}(\mathcal{G}), \quad \mathcal{L}(\mathcal{G}) = \left\{\dot{\boldsymbol{\alpha}}\colon\ D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) = \boldsymbol{A}(\mathcal{G}) \cdot \dot{\boldsymbol{\alpha}}\right\} \tag{8}$$

$\mathcal{L}(\mathcal{G})$ being thus defined as a cone in $\dot{\boldsymbol{\alpha}}$-space within which the dissipation function D yields the actual value of the dissipation rate. Note that D is necessarily linear in $\dot{\boldsymbol{\alpha}}$ within that cone.

The question arises under which circumstances the minimizer in (2) is a solution to the incremental boundary-value problem, satisfying locally the constitutive law. Whatever might be an approximate sense of a solution in small but finite increments, the minimization procedure (2) should yield an exact solution in the first-order rates in the limit as the magnitude of all increments tends to zero. To examine this, applicability of the minimization rule (1) or (2) must be checked for ΔE developed up to the *second*-order terms.

Denote by $\mathcal{G}_0 = (\mathbf{F}_0, \boldsymbol{\alpha}_0)$ a fixed initial state from which any increment is initiated. Assuming that the function ϕ is twice Fréchet differentiable, we obtain

$$\Delta\phi = \Delta_1\phi + \Delta_2\phi + \mathrm{o}\big(\rho^2\big), \quad \rho = a\|\Delta\mathbf{F}\| + \|\Delta\boldsymbol{\alpha}\|, \quad 0 < a = \text{const} \tag{9}$$

where $\mathrm{o}(\rho^2)/\rho^2 \to 0$ as $\rho \to 0$, $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$ denotes a norm in an appropriate (Hilbert) space, and

$$\Delta_1\phi = \mathbf{S}(\mathcal{G}_0) \cdot \Delta\mathbf{F} - \boldsymbol{A}(\mathcal{G}_0) \cdot \Delta\boldsymbol{\alpha}, \qquad \Delta_2\phi = \tfrac{1}{2}\Delta\mathbf{S} \cdot \Delta\mathbf{F} - \tfrac{1}{2}\Delta\boldsymbol{A} \cdot \Delta\boldsymbol{\alpha}, \tag{10}$$

Under the assumption that the function $D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha})$ possesses the Fréchet derivative $D_{,\boldsymbol{\alpha}}$ with respect to $\boldsymbol{\alpha}$, we have

$$D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}_0 + \Delta\boldsymbol{\alpha}) = D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}_0) + D_1(\dot{\boldsymbol{\alpha}}, \Delta\boldsymbol{\alpha}) + \|\dot{\boldsymbol{\alpha}}\|\mathrm{o}\big(\|\Delta\boldsymbol{\alpha}\|\big), \qquad D_1(\dot{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}}) \equiv \hat{\boldsymbol{\alpha}} \cdot D_{,\boldsymbol{\alpha}}(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}_0) \tag{11}$$

where $D_1(r\dot{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}}) = rD_1(\dot{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})$ for every $r > 0$ and $D_1(\dot{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})$ is linear with respect to $\hat{\boldsymbol{\alpha}}$. It follows that

$$\Delta\mathcal{D} = \Delta_1\mathcal{D} + \Delta_2\mathcal{D} + \mathrm{o}\big(\|\Delta\boldsymbol{\alpha}\|^2\big) \tag{12}$$

on a radial path of constant direction of the tangent ($\dot{\boldsymbol{\alpha}}$), where

$$\Delta_1\mathcal{D} = D(\Delta\boldsymbol{\alpha}, \boldsymbol{\alpha}_0), \qquad \Delta_2\mathcal{D} = \tfrac{1}{2}D_1(\Delta\boldsymbol{\alpha}, \Delta\boldsymbol{\alpha}) \tag{13}$$

In contrast to (10), the formulae (13) do not apply to arbitrary circuitous paths in $\boldsymbol{\alpha}$-space, cf. [12]. In this paper we restrict attention to smooth paths of bounded curvature that start from $\mathcal{G}_0$ and are kept fixed as $\|\Delta\boldsymbol{\alpha}\| \to 0$. Under the simplifying assumption that D is Gateaux differentiable with respect to $\dot{\boldsymbol{\alpha}}$ at $\dot{\boldsymbol{\alpha}} \neq \mathbf{0}$ (but *not* at $\dot{\boldsymbol{\alpha}} = \mathbf{0}$), and using $D = D_{,\dot{\boldsymbol{\alpha}}} \cdot \dot{\boldsymbol{\alpha}}$ as the well-known consequence of $(6)_2$, validity of (13) is extended to such 'direct' paths. The case when D is not differentiable with respect to $\dot{\boldsymbol{\alpha}}$ can be examined separately.

3. Conditions for applicability of the incremental energy minimization

3.1. First-order minimization

We have $\Delta E = \dot{E}\Delta t$ to first order, where from (3), (4) and (5),

$$\dot{E} = \int_B \big(D(\dot{\boldsymbol\alpha}, \boldsymbol\alpha_0) - \boldsymbol{A}(\mathcal{G}_0)\cdot\dot{\boldsymbol\alpha}\big)\,\mathrm{d}V + \int_B \mathbf{S}(\mathcal{G}_0)\cdot\dot{\mathbf{F}}\,\mathrm{d}V + \dot{\Omega} \tag{14}$$

By the principle of virtual power, and excluding unilateral constraints for simplicity, the current global state is in mechanical equilibrium if and only if the sum of the last two terms in (14) has a constant value for all virtual (kinematically admissible) velocity fields. This is required for applicability of the minimization (1) to first order, along with another necessary condition

$$D(\dot{\boldsymbol\alpha}, \boldsymbol\alpha_0) - \boldsymbol{A}(\mathcal{G}_0)\cdot\dot{\boldsymbol\alpha} \geqslant 0 \quad \text{for all virtual } \dot{\boldsymbol\alpha} \tag{15}$$

which must hold almost everywhere in B, since otherwise $\dot{E}$ would be unbounded from below.

If the above two conditions imposed on the initial *state* of incremental energy minimization are satisfied then, from (8), $\dot{E} \geqslant \dot{E}^0$ holds true if and only if the minimizing rate $\dot{\boldsymbol\alpha}^0 \in \mathcal{L}(\mathcal{G}_0)$ almost everywhere in B. If, moreover, also $\dot{\boldsymbol\alpha} \in \mathcal{L}(\mathcal{G}_0)$ in B then $\dot{E} = \dot{E}^0$. Clearly, to select $\dot{\boldsymbol\alpha}^0$, the energy minimization (2) must be extended to the second-order terms, which is examined in Subsection 3.2.

Remark 1. In general, convexity of the dissipation function D with respect to $\dot{\boldsymbol\alpha}$ need not be assumed in advance when applying the incremental energy minimization. However, fulfillment of (15) is related to *convexification* (i.e. finding a convex envelope) of D; this problem is not pursued further here.

Denote by superscript 0 the quantities evaluated along a path determined by applying the minimization procedure (1) successively. The condition (15) must hold along that path at the beginning of a next time step, since otherwise no minimum of $\dot{E}$ could be found at that instant and the minimization procedure would break down. Therefore, by treating the left-hand expression in (15) as a function of time at variable $\mathcal{G}_0$ but fixed $\dot{\boldsymbol\alpha}$, its forward increment must be non-negative if the expression vanishes initially, which is the case if $\dot{\boldsymbol\alpha} \in \mathcal{L}(\mathcal{G}_0)$. For a time step arbitrarily small, this yields

$$D_1\big(\dot{\boldsymbol\alpha}, \dot{\boldsymbol\alpha}^0\big) - \dot{\boldsymbol{A}}^0\cdot\dot{\boldsymbol\alpha} \geqslant 0 \quad \text{for all } \dot{\boldsymbol\alpha} \in \mathcal{L}(\mathcal{G}_0) \tag{16}$$

The above argument and result (16) are, in essence, the same as those established first by Nguyen (cf. [13]); however, his starting point was the assumed maximum dissipation principle, and not the minimization (1) as here.

3.2. Second-order local minimization with respect to $\dot{\boldsymbol\alpha}$

We begin with the remark that in energy calculations with accuracy to the second-order terms, the leading terms like in $\Delta_1\mathcal{D}$ or $\Delta_1\phi$, and hence $\Delta\boldsymbol\alpha$ or $\Delta\mathbf{F}$ therein, must be also correct to second order, e.g., on substituting $\Delta\boldsymbol\alpha = \dot{\boldsymbol\alpha}\,\Delta t + \frac{1}{2}\ddot{\boldsymbol\alpha}(\Delta t)^2$. In turn, in the second-order term $\Delta_2 w$ it suffices to substitute $\Delta\boldsymbol\alpha = \dot{\boldsymbol\alpha}\Delta t$. Here and below, all derivatives are evaluated in $\mathcal{G}_0$.

Suppose first that (2) holds to second order for $\dot{\boldsymbol\alpha} = \dot{\boldsymbol\alpha}^0 \in \mathcal{L}(\mathcal{G}_0)$ and the only difference between both sides of (2) is due to $\ddot{\boldsymbol\alpha} \neq \ddot{\boldsymbol\alpha}^0$. This can be shown to imply locally

$$D\big(\Delta\boldsymbol\alpha^0, \boldsymbol\alpha_0\big) - \boldsymbol{A}(\mathcal{G}_0)\cdot\Delta\boldsymbol\alpha^0 = \mathrm{o}\big(\|\Delta\boldsymbol\alpha^0\|^2\big) \tag{17}$$

Next, on using (12) and (9), taking $\ddot{\boldsymbol\alpha} = \mathbf{0}$ and substituting (17), and rearranging with the help of (13), (10), and (8), it is shown that the fulfillment of (2) to second order implies [11]

$$\lim_{\Delta t\to 0}\big(\Delta_2 w - \Delta_2 w^0\big)/(\Delta t)^2 \geqslant 0 \text{ at } \dot{\mathbf{F}} = \dot{\mathbf{F}}^0 \quad \Longleftrightarrow \quad J(\dot{\boldsymbol\alpha}) \geqslant J\big(\dot{\boldsymbol\alpha}^0\big) \text{ for all } \dot{\boldsymbol\alpha} \in \mathcal{L}(\mathcal{G}_0) \tag{18}$$

almost everywhere in B, where $\dot{\mathbf{F}}^0$ is fixed and

$$J(\dot{\boldsymbol{\alpha}}) = \tfrac{1}{2}\dot{\boldsymbol{\alpha}} \cdot \phi_{,\boldsymbol{\alpha}\boldsymbol{\alpha}} \cdot \dot{\boldsymbol{\alpha}} + \tfrac{1}{2}D_1(\dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}}) + \dot{\mathbf{F}}^0 \cdot \phi_{,\mathbf{F}\boldsymbol{\alpha}} \cdot \dot{\boldsymbol{\alpha}} \tag{19}$$

We note that fulfillment of (18) requires $\mathrm{d}J(r\dot{\boldsymbol{\alpha}}^0)/\mathrm{d}r = 0$ at $r = 1$, which yields

$$D_1\big(\dot{\boldsymbol{\alpha}}^0, \dot{\boldsymbol{\alpha}}^0\big) - \dot{\boldsymbol{A}}^0 \cdot \dot{\boldsymbol{\alpha}}^0 = 0 \tag{20}$$

as a consequence of (2), where $\dot{\boldsymbol{A}}^0$ is related to $(\dot{\mathbf{F}}^0, \dot{\boldsymbol{\alpha}}^0)$ through (5). Hence, $J(\dot{\boldsymbol{\alpha}}^0) = \frac{1}{2}\dot{\mathbf{F}}^0 \cdot \phi_{,\mathbf{F}\boldsymbol{\alpha}} \cdot \dot{\boldsymbol{\alpha}}^0$ and $\Delta_2 w^0 = \frac{1}{2}\Delta\mathbf{S}^0 \cdot \Delta\mathbf{F}^0 = \frac{1}{2}\dot{\mathbf{S}}^0 \cdot \dot{\mathbf{F}}^0(\Delta t)^2$, to second order.

For *existence* of a minimum in (18) it is necessary that $J(\dot{\boldsymbol{\alpha}})$ be bounded from below, which requires

$$\dot{\boldsymbol{\alpha}} \cdot \phi_{,\boldsymbol{\alpha}\boldsymbol{\alpha}} \cdot \dot{\boldsymbol{\alpha}} + D_1(\dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}}) \geqslant 0 \quad \text{for all } \dot{\boldsymbol{\alpha}} \in \mathcal{L}(\mathcal{G}_0) \tag{21}$$

Condition (21) is imposed on the current *state* and related to directional stability of equilibrium, cf. [13,11].

3.3. *The symmetry condition*

The first variation of $J(\dot{\boldsymbol{\alpha}})$ at $\dot{\boldsymbol{\alpha}} = \dot{\boldsymbol{\alpha}}^0$ reads

$$\delta J(\dot{\boldsymbol{\alpha}}^0) = \tfrac{1}{2}D_1\big(\dot{\boldsymbol{\alpha}}^0 + \delta\dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}}^0\big) + \tfrac{1}{2}D_1\big(\dot{\boldsymbol{\alpha}}^0, \dot{\boldsymbol{\alpha}}^0 + \delta\dot{\boldsymbol{\alpha}}\big) - D_1\big(\dot{\boldsymbol{\alpha}}^0, \dot{\boldsymbol{\alpha}}^0\big) - \dot{\boldsymbol{A}}^0 \cdot \delta\dot{\boldsymbol{\alpha}} \tag{22}$$

for an infinitesimal $\delta\dot{\boldsymbol{\alpha}}$. More rigorously, the variation (22) represents the Gateaux differential of J at $\dot{\boldsymbol{\alpha}}^0$ in the direction of $\delta\dot{\boldsymbol{\alpha}}$. Clearly, if (18) holds, then the *one-sided* variation $\delta J(\dot{\boldsymbol{\alpha}}^0) \geqslant 0$ whenever $\dot{\boldsymbol{\alpha}}^0 + \delta\dot{\boldsymbol{\alpha}} \in \mathcal{L}(\mathcal{G}_0)$. If the minimizer $\dot{\boldsymbol{\alpha}}^0$ is an interior point of $\mathcal{L}(\mathcal{G}_0)$ with respect to the direction of $\delta\dot{\boldsymbol{\alpha}}$, in the sense that the equivalence:

$$\big(\,\dot{\boldsymbol{\alpha}}^0 + \delta\dot{\boldsymbol{\alpha}} \in \mathcal{L}(\mathcal{G}_0) \quad\Longleftrightarrow\quad \dot{\boldsymbol{\alpha}}^0 - \delta\dot{\boldsymbol{\alpha}} \in \mathcal{L}(\mathcal{G}_0)\big) \quad \text{holds for infinitesimal } \delta\dot{\boldsymbol{\alpha}} \tag{23}$$

then $\delta J(\dot{\boldsymbol{\alpha}}^0) = 0$ for $\delta\dot{\boldsymbol{\alpha}}$ satisfying (23). Moreover, a similar variation of the left-hand expression in (16) must also vanish at $\dot{\boldsymbol{\alpha}} = \dot{\boldsymbol{\alpha}}^0$, as a consequence of (16) and (20). Jointly, this requires

$$D_1\big(\dot{\boldsymbol{\alpha}}^0, \dot{\boldsymbol{\alpha}}^0 + \delta\dot{\boldsymbol{\alpha}}\big) = D_1\big(\dot{\boldsymbol{\alpha}}^0 + \delta\dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}}^0\big) \quad \text{subject to (23)} \tag{24}$$

This is the symmetry restriction imposed on the function D_1, that is, on the dissipation function D itself. In fact, the condition (24) should be satisfied *in advance*, being necessary for applicability of the incremental energy minimization (1) to the first- and second-order simultaneously. Other forms of (24) can be specified, e.g., if $D_1(\dot{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})$ linear with respect to $\dot{\boldsymbol{\alpha}}$ in a convex $\mathcal{L}(\mathcal{G}_0)$ then (24) holds for every $\dot{\boldsymbol{\alpha}}^0 \in \mathcal{L}(\mathcal{G}_0)$ if and only if

$$D_1(\dot{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}}) = D_1(\hat{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}}) \quad \text{for all } \dot{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}} \in \mathcal{L}(\mathcal{G}_0). \tag{25}$$

Remark 2. In the special case of classical (single-mode) plasticity, the cone $\mathcal{L}(\mathcal{G}_0)$ reduces to a single ray in $\dot{\boldsymbol{\alpha}}$-space, and the symmetry condition (24) is satisfied automatically. Another special case when (24) holds trivially is when $D(\dot{\boldsymbol{\alpha}})$ is state-independent. In general, (24) represents an additional restriction imposed on the constitutive law.

Consider a minimizer $\dot{\boldsymbol{\alpha}}^0$ of $J(\dot{\boldsymbol{\alpha}})$ associated with some $\dot{\mathbf{F}}^0$ and another one $\dot{\boldsymbol{\alpha}}^0 + \delta\dot{\boldsymbol{\alpha}}^0$ associated with $\dot{\mathbf{F}}^0 + \delta\dot{\mathbf{F}}^0$ for an infinitesimal $\delta\dot{\mathbf{F}}^0$. Since each minimizer lies within $\mathcal{L}(\mathcal{G}_0)$ and satisfies (20), after some transformations with the use of (20), (22) and (5) it follows that

$$\delta\dot{\boldsymbol{\alpha}}^0 \cdot \dot{\boldsymbol{A}}^0 = \dot{\boldsymbol{\alpha}}^0 \cdot \delta\dot{\boldsymbol{A}}^0 \quad \text{and} \quad \delta\dot{\boldsymbol{\alpha}}^0 \cdot \phi_{,\boldsymbol{\alpha}\mathbf{F}} \cdot \dot{\mathbf{F}}^0 = \dot{\boldsymbol{\alpha}}^0 \cdot \phi_{,\boldsymbol{\alpha}\mathbf{F}} \cdot \delta\dot{\mathbf{F}}^0 \tag{26}$$

From now on, the superscript 0 is omitted for brevity. If a minimizer $\dot{\boldsymbol{\alpha}}$, related to $\dot{\mathbf{S}}$, is a single valued, continuous and piecewise differentiable (but never fully linear) function of $\dot{\mathbf{F}}$ then, from (26), (5) and by the time-independence of the material, we obtain

$$\delta\dot{\mathbf{S}} \cdot \dot{\mathbf{F}} = \dot{\mathbf{S}} \cdot \delta\dot{\mathbf{F}} \quad \text{and} \quad \dot{\mathbf{S}} = U_{,\dot{\mathbf{F}}}(\dot{\mathbf{F}}, \mathcal{G}_0), \quad U = \tfrac{1}{2}\dot{\mathbf{S}}\cdot\dot{\mathbf{F}} \tag{27}$$

This form of the constitutive rate relationship, postulated long ago by Hill [7], is shown here to follow from the minimization (18) applied to finding $\dot{\boldsymbol{\alpha}}$ as a function of $\dot{\mathbf{F}}$.

Remark 3. Note that the symmetry requirement (24) has *not* been used when deriving another symmetry condition (27). However, as shown above, the symmetry restriction (24) is required for the intrinsic consistency between the first- and second-order incremental energy minimization.

3.4. Second-order global minimization

Applicability of (2) up to second-order terms within the incremental constitutive framework with $\dot{\mathbf{S}}$ as a function of $\dot{\mathbf{F}}$ was thoroughly examined [2,3]. Here we only recall that the minimizer is shown to satisfy automatically the rate form of the principle of virtual power if an only if the potential form (27) holds, and then the second-order minimization (1) is shown to reduce to

$$\int_B U(\Delta\mathbf{F}, \mathcal{G}_0)\,\mathrm{d}V + \cdots = (\Delta t)^2 \int_B U(\dot{\mathbf{F}}, \mathcal{G}_0)\,\mathrm{d}V + \cdots \to \min \tag{28}$$

subject to kinematical constrains. In (28), only the leading term (corresponding to $\Delta_2 W$) is displayed, and the second-order terms added in case of configuration-dependent loading as well as all higher-order terms are dotted.

It should be noted that the minimizer determined from (28) under assumption (27) represents an exact solution in velocities, in the limit as $\Delta t \to 0$. Finite increments found from (28) are merely approximate unless further constitutive assumptions are introduced. Recent examples of the use of (28) can be found in several references listed in the Introduction.

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