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Symmetrization of Lagrangian gas dynamic in dimension two and multidimensional solvers

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Abstract

We propose a new formulation of multidimensional Euler equations in Lagrange coordinates as a system of conservation laws associated with free-divergent constraints. This formulation leads to a natural class of entropic Lagrangian schemes, based on a multidimensional node solver. For the sake of simplicity the study is done in 2D, but most of the ideas can be generalized in 3D. ***To cite this article: B. Després, C. Mazeran, C. R. Mecanique 331 (2003).***
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Résumé

Symétrisation de la dynamique des gaz en dimension deux et solveurs multidimensionnels. Nous proposons dans cette Note une nouvelle formulation de la dynamique des gaz multidimensionnelle en coordonnées lagrangienne sous la structure d'un système de lois de conservation associées à des contraintes de divergence nulle. Ce formalisme conduit à une classe de schémas lagrangiens entropiques basé sur un solveur aux noeuds multidimensionnel. Pour des raisons de simplicité d'exposé, l'étude est présentée en dimension deux d'espace. La plupart des idées peuvent se généraliser en dimension trois d'espace. ***Pour citer cet article : B. Després, C. Mazeran, C. R. Mecanique 331 (2003).***
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Introduction. Les résultats présentés dans cette Note sont extraits de [1] et peuvent être considérés comme l'extension multidimensionelle non triviale de [9]. Nous restreignons la présentation à la dimension deux d'espace, mais la plupart du matériel se généralise sans peine en dimension trois. En coordonnées eulériennes, le système de la dynamique des gaz (1) est mathématiquement bien posé et a été longuement étudié sur le plan théorique

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et numérique (voir par exemple [4] pour une référence générale). Toutefois au niveau numérique l'inconvénient inhérent à ce système de coordonnées est d'introduire une diffusion excessive, régularisant les discontinuités physiques. Au contraire le système de coordonnées lagrangiennes, dont le principe est de suivre la matière dans son déplacement, s'affranchit de ce désavantage et se révèle, en théorie, particulièrement bien adapté aux problèmes de frontières libres. Il est cependant connu que la dynamique des gaz en coordonnées lagrangiennes souffre encore du manque d'un formalisme intrinséquement multidimensionnel. Au-delà de la dimension un d'espace, le système d'équations a toujours été étudié sous une forme non conservative, avec opérateur différentiel dans le repère eulérien. Du point de vue numérique ceci se traduit par des méthodes essentiellement uni-dimensionnelles, construites à partir d'un « splitting » directionnel et de solveurs de Riemann un-D. Force est de constater qu'aucun solveur réellement multidimensionnel, justifié sur la plan théorique, n'a été validé jusqu'à présent.

Formalisme canonique et symétrisation. Notre idée de départ est d'exploiter la remarquable observation de [2,10], consistant à écrire le système de la dynamique des gaz (1) dans le repère lagrangien sous la forme conservative (2). Le système obtenu peut-être vu comme un système de lois de conservation *physiques* et *géométriques*, ces dernières traduisant la correspondance entre repère eulérien et lagrangien. Le résultat frappant est que (2) a été prouvé n'être que faiblement hyperbolique, alors que le système des équations d'Euler est hyperbolique en cordonnées eulériennes. Nous en déduisons que l'entropie physique n'est pas strictement concave en fonction des nouvelles inconnues de (2). Nous savons par contre qu'elle est strictement concave en fonction de ses arguments physiques usuels [4]. En remarquant que S est une entropie pour (2) associée à un flux nul, notre idée est d'étendre à la dimension 2 la forme canonique des systèmes de lois de conservation avec flux d'entropie nul proposée dans [3]. Cette généralisation est en effet possible (Théorème 2.1) et donne au système de la dynamique des gaz lagrangien une structure multidimensionnelle canonique (4)–(6). Ce qui distingue notre approche de [2] est de mettre en évidence et de considérer comme partie intégrante du formalisme les contraintes de divergence nulle (6). Le point essentiel est que ce sont précisément ces contraintes différentielles qui permettent de découpler les deux parties du système (4), et de prouver que (4) est symétrisable.

Construction d'un schéma multidimensionnel. Sur cette assise théorique, nous cherchons à construire une classe de schémas numériques de type volume-finis avec solveurs multidimensionnels. Trois principes naturels guident cette construction : conserver la masse de chaque cellule, satisfaire la condition locale de production d'entropie et assurer au schéma d'être conservatif. La première question qui se pose est de savoir où placer les degrés de liberté des inconnues. Nous choisissons de discrétiser les inconnues *géométriques* sur les *arêtes* des cellules, car elles satisfont à des conditions de divergence. Les variables *physiques* étant quant à elles les inconnues du système de lois de conservation classique (4), elles sont approchées par leur *moyenne sur la cellule*. Nous proposons une discrétisation de la deuxième partie de (2) qui assure pour tout temps la condition de divergence nulle (6) dès lors qu'elle est vérifiée à l'instant initial (Proposition 3.1). La discrétisation de (4) est fondée sur une condition suffisante de conservation de la masse (Proposition 3.2) et conduit à la formulation aux nœuds (12). Pour achever la définition du schéma nous construisons les flux aux nœuds sous les critères d'entropie et de caractère conservatif (Proposition 3.3). Un fait remarquable est que la condition d'entropie, inspirée de [3], permet de reconsidérer les solveurs de Riemann aux faces utilisés classiquement et d'envisager toute une classe de solveurs aux nœuds qui sont adaptés au cas considéré. Afin d'illustrer cette nouvelle approche nous présentons deux exemples numériques.

1. Introduction

The results presented in this Note come from [1]. Gas dynamic in Lagrange coordinates still suffers from the lack of an intrinsic multidimensional formulation, from both the theorical and numerical point of view. For a long time it has been usually written as a system of non-conservative PDEs with eulerian derivatives, associated with geometrical equations for displacement of the moving frame. At the numerical level, the traditional approach

consists in solving 1D-Riemann problems on each edges of a triangulation, as if the problem was a pure local one-d problem. A drawback of such a method is that the Riemann solver provides only face-centered normal component velocities, whereas velocities are needed at nodes to move the mesh. Moreover this displacement remains a problem, at least because usual techniques are not mass preserving. This can be seen as an incompatibility between node velocities and Riemann edges velocity.

2. Canonical formalism and symmetrization

The starting idea (see [2]) is that the Euler system for compressible gas dynamics

$$\begin{cases} \partial_t(\rho) + \partial_x(\rho u) + \partial_y(\rho v) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) + \partial_y(\rho uv) = 0 \\ \partial_t(\rho v) + \partial_x(\rho uv) + \partial_y(\rho v^2 + p) = 0 \\ \partial_t(\rho e) + \partial_x(\rho ue + pu) + \partial_y(\rho ve + pv) = 0 \end{cases} \tag{1}$$

can be rewritten as a system of conservation laws in the initial Lagrangian frame. Let us assume for the sake of simplicity that the pressure law is given by the perfect gas approximation $p = (\gamma - 1)\rho(e - \frac{1}{2}u^2 - \frac{1}{2}v^2)$ so that (1) is closed, but other choices are possible provided there are thermodynamically consistant. The system of gas dynamics (1) can be rewritten:

$$\begin{cases} \partial_t(\rho J) = 0 \\ \partial_t(\rho Ju) + \partial_X(pM) + \partial_Y(-pB) = 0 \\ \partial_t(\rho Jv) + \partial_X(-pL) + \partial_Y(pA) = 0 \\ \partial_t(\rho Je) + \partial_X(puM - pvL) + \partial_Y(pAv - pBu) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \partial_t A - \partial_X u = 0 \\ \partial_t B - \partial_X v = 0 \\ \partial_t L - \partial_Y u = 0 \\ \partial_t M - \partial_Y v = 0 \end{cases} \tag{2}$$

where the set (A, B, L, M) gives the entries of the Jacobian matrix $\frac{\partial(x,y)}{\partial(X,Y)}$

$$A = \partial_X x, \quad B = \partial_X y, \quad L = \partial_Y x, \quad M = \partial_Y y \tag{3}$$

and J denotes its determinant. A drawback of (2) is that the system proves to be only weakly hyperbolic (cf. [2,10]), which may appear as striking since it is well known that the Euler equations are hyperbolic. Let us define the physical entropy with the fundamental law of thermodynamics $T\,\mathrm{d}S = \mathrm{d}\varepsilon + p\,\mathrm{d}\tau$, where $\tau = \rho^{-1}$ and $e = \varepsilon + (u^2 + v^2)/2$. We deduce from the lack of hyperbolicity that the physical entropy S cannot be strictly concave in function of all unknowns $(\rho J, \rho Ju, \rho Jv, \rho Je, A, B, L, M)$.

We guess that (2) could probably be considered as an extension of the class of one-d system of conservation laws with a zero entropy flux studied in [3]. We just recall that one-d system of conservation laws with zero entropy flux, Galilean invariance and reversibility for smooth solutions find a canonical formalism, see [3]. Since we desire the physical entropy to be strictly concave in function of the unknowns of the system, it leaves little choice but to choose $U = (\tau, u, v, e)^{\mathrm{T}}$ as the principal unknown. Then it is well known that $S(U)$ is strictly concave with respect to U [4]. Following [3] we define $\Psi = (p, -u, -v)^{\mathrm{T}}$ and find the canonical structure of the system:

Theorem 2.1. *System* (2) *can be rewritten*:

$$\rho_0\partial_t U + \partial_X \begin{pmatrix} \mathcal{B}^X\Psi \\ -\frac{1}{2}(\Psi, \mathcal{B}^X\Psi) \end{pmatrix} + \partial_Y \begin{pmatrix} \mathcal{B}^Y\Psi \\ -\frac{1}{2}(\Psi, \mathcal{B}^Y\Psi) \end{pmatrix} = 0, \quad \textit{given that } \rho_0 = \rho(t=0, X, Y), \quad \textit{with} \tag{4}$$

$$\mathcal{B}^X = \begin{pmatrix} 0 & M & -L \\ M & 0 & 0 \\ -L & 0 & 0 \end{pmatrix} \quad \textit{and} \quad \mathcal{B}^Y = \begin{pmatrix} 0 & -B & A \\ -B & 0 & 0 \\ A & 0 & 0 \end{pmatrix} \tag{5}$$

Both matrices are symmetric but nonconstant. However there are linked with the free divergent constraint

$$\partial_X\mathcal{B}^X + \partial_Y\mathcal{B}^Y = 0 \tag{6}$$

Proposition 2.2. *Assume that the free divergence constraint* (6) *holds at* $t = 0$. *Then the system of Eqs.* (4) *is well posed in the sense that* (4) *is symmetrizable and the system*

$$\begin{cases} \partial_t A - \partial_X u = 0 \\ \partial_t B - \partial_X v = 0 \\ \partial_t L - \partial_Y u = 0 \\ \partial_t M - \partial_Y v = 0 \end{cases} \tag{7}$$

for the evolution of (A, B, L, M) *is integrable and well posed.*

Actually (4)–(7) is decoupled, since U is the unknown of the symmetrizable system (4) and (A, B, L, M) is the unknown of the decoupled system (7) whose fluxes are linear function of U. To our knowledge, it is the first time that Euler equations written in Lagrange coordinate are considered as a hyperbolic system of conservation laws with differential constraints. Let us quote, however, that this class of system as for long been considered by Godunov [5] and Boillat [6]. Actually the canonical structure (4) is also shared by the system of ideal magnetohydrodynamics in dimension two and three [7]). We immediatly get:

Lemma 2.3. *Let* p *be a thermodynamically consistant pressure law. For all systems of the form* (4)–(6) *then the entropy flux is zero, that is smooth solutions of* (2)–(7) *satisfy* $\partial_t S = 0$ *whereas discontinuous solutions satisfy* $\partial_t S \geqslant 0$.

3. Construction of a 2D solver

This section is devoted to the presentation of a a finite-volume scheme based upon the mathematical structure of (4)–(6), (7). Let us consider a mesh in the (X, Y) plane, whose current cell is noted j (its current neigbour is k). The surface of the cell Ω_j is s_j. In most of practical calculations, cells are either triangular or quadrilateral, but this is not important in this presentation. An edge between two cells is referred to with its cell numbers jk: as an example the length of the edge between the cell Ω_j and the cell Ω_k is l_{jk}. Similarly the outgoing normal from Ω_j to Ω_k is $\vec{n}_{jk} = (n^X_{jk}, n^Y_{jk})$. We also need a notation for the nodes. In order to simplify the notation, we have chosen to use the letters r and s only for nodes: for example the edge jk connects necessarily two nodes, referred to for example as the node number r and the node number s. As A, B, L and M satisfy divergence equations, we locate their degree of freedom on the edges and note $A_{jk}, B_{jk}, \ldots$ their approximate averaged on the edge jk at time t. This is a very important difference compared with [2] where this quantities are discretized inside the cell. As for U, since it is a classical unknown of a classical system of conservation law, its degrees of freedom are choosen inside the cell; we note U_j the approximate averaged value of U in the cell j at time t.

3.1. Discretization of (6), (7)

Let us define on each edge $C_{jk} = n^X_{jk} M_{jk} - n^Y_{jk} B_{jk}$ and $D_{jk} = n^X_{jk} L_{jk} - n^Y_{jk} A_{jk}$. The discrete free divergence constraint can be expressed:

$$\sum_k l_{jk} C_{jk} = \sum_k l_{jk} D_{jk} = 0 \tag{8}$$

Assuming that the velocity (u_r, v_r) at each node r is a degree of freedom of the method, the natural discrete equation for C and D are

$$\partial_t C_{jk} = \frac{v_s - v_r}{l_{jk}} \quad \text{and} \quad \partial_t D_{jk} = \frac{u_s - u_r}{l_{jk}} \tag{9}$$

where (r, s) are the positive-oriented vertices of edge jk.

Proposition 3.1. *Assume that* (8) *holds at initial time. Then, whatever the node velocities* (u_r, v_r), *it holds at all time for all solutions of* (9).

3.2. Discretization of (4)

A natural continuous in time finite volume discretization of (4) is

$$s_j \rho_j^0 \partial_t U_j + \sum_k l_{jk} \left[n_{jk}^X \begin{pmatrix} \mathcal{B}_{jk}^X \Psi_{jk} \\ -\frac{1}{2}(\Psi_{jk}, \mathcal{B}_{jk}^X \Psi_{jk}) \end{pmatrix} + n_{jk}^Y \begin{pmatrix} \mathcal{B}_{jk}^Y \Psi_{jk} \\ -\frac{1}{2}(\Psi_{jk}, \mathcal{B}_{jk}^Y \Psi_{jk}) \end{pmatrix} \right] = 0 \tag{10}$$

where Ψ_{jk} is still unknown. We have given in the introduction some reasons why we need a node localized formulation; we first decide to construct some 'fluxes' on nodes under the conservation of mass criteria.

Proposition 3.2. *Let us rely the choice of Ψ_{jk} to the choice of $\Psi_{jr,s}$ by $\Psi_{jk} = (\Psi_{jr} + \Psi_{js})/2$ and replace $(\Psi_{jk}, \mathcal{B}_{jk}^X \Psi_{jk})$ by $\frac{1}{2}(\Psi_{jr}, \mathcal{B}_{jk}^X \Psi_{jr}) + \frac{1}{2}(\Psi_{js}, \mathcal{B}_{jk}^X \Psi_{js})$. Then the finite-volume method* (10) *is mass preserving. Let us introduce l_{jr} (a length) and θ_{jr} (an angle) such that*

$$\mathcal{M}_{jk} = l_{jk} n_{jk}^X \mathcal{B}_{jk}^X + l_{jk} n_{jk}^Y \mathcal{B}_{jk}^Y, \quad \mathcal{N}_{jr} = \frac{1}{2}\mathcal{M}_{jk^+} + \frac{1}{2}\mathcal{M}_{jk^-} = l_{jr} \begin{pmatrix} 0 & \cos\theta_{jr} & \sin\theta_{jr} \\ \cos\theta_{jr} & 0 & 0 \\ \sin\theta_{jr} & 0 & 0 \end{pmatrix} \tag{11}$$

In (11) *k^+ and k^- refer to some edge numbers such that the edge-node-edge sequence jk^-, r, jk^+ is positively oriented. With these notations we get a fully node localized formulation*

$$s_j \rho_j^0 \partial_t U_j + \sum_r \begin{pmatrix} \mathcal{N}_{jr} \Psi_{jr} \\ -\frac{1}{2}(\Psi_{jr}, \mathcal{N}_{jr} \Psi_{jr}) \end{pmatrix} = 0 \tag{12}$$

Since the velocity (u_r, v_r) is needed to move the mesh and $\Psi = (p, -u, -v)^{\mathrm{T}}$, it is reasonable to impose $u_{jr} = u_r$ and $v_{jr} = v_r$ at each node r. We achieve to construct Ψ_{jr} by two criteria: stability via entropy inequality and conservativity.

Theorem 3.3. *There exist a local definition of flux Ψ_{jr} such that the scheme* (12) *is entropic and totally conservative* (*see* [1]): *$\partial_t S_j \geqslant 0$, and $\partial_t(\sum_j s_j \rho_j^0 U_j) = 0$: here $s_j \rho_j^0 = s_j(t)\rho_j(t)$ is constant.*

This kind of property is based on the splitting of each symmetric matrix $\mathcal{N}_{jr}$ into a symmetric non positive part and a symmetric non negative part [3]. Such a decomposition is, of course, non unique, which brings a class of entropic schemes. We do not give an explicit construction of Ψ_{jr} here, and refer the reader to [1] for a complete presentation. This in turn implies some positivity properties for both the internal energy and the density. The scheme is locally conservative for all variables.

Proposition 3.4. *The global scheme* (9)–(12) *written in the Lagrangian reference frame (X, Y) has a complete interpretation as a scheme written on a moving mesh in the Eulerian reference frame (x, y).*

We then study the 2D plane Sod shock tube, on a regular mesh composed of three layers with rectangular cells. As expected on such a symetric mesh, the scheme preserves the unidimensional character of the flow and we observe that it exactly degenerates on a one-dimensional scheme. We provide in Fig. 1 *almost converged* solutions obtained with 3×1000 cells, to illustrate the stability of the method. The node flux is only *one order*. We then run the Saltzman test problem [8], which consists in solving a one-dimensional problem on a two-dimensional nonuniformly distorded mesh made with distorded quadrangular cells. It thus points out the purely geometrical errors generated by Lagrangian codes. The problem consists of a rectangular box whose left-hand side wall acts as a piston with a constant velocity. The one-dimensional symmetry is broken by an initial nonuniform mesh. Fig. 2 illustrates the Lagrangian mesh at time $t = 0.5$ for an usual finite-volume method and new one's. The tradional approach fails at time 0.6 due to an excessively distorded mesh (0.8 for other codes, see [8]), whereas

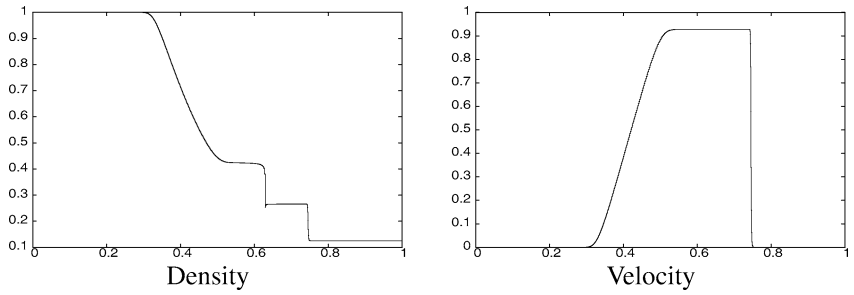

Fig. 1. 2D Sod shock tube at time $t = 0.14$; *Almost converged* solutions with the node solver approach.

Fig. 1. Le tube à choc de Sod à $t = 0{,}14$: les solutions « presque convergées » avec la nouvelles approche.

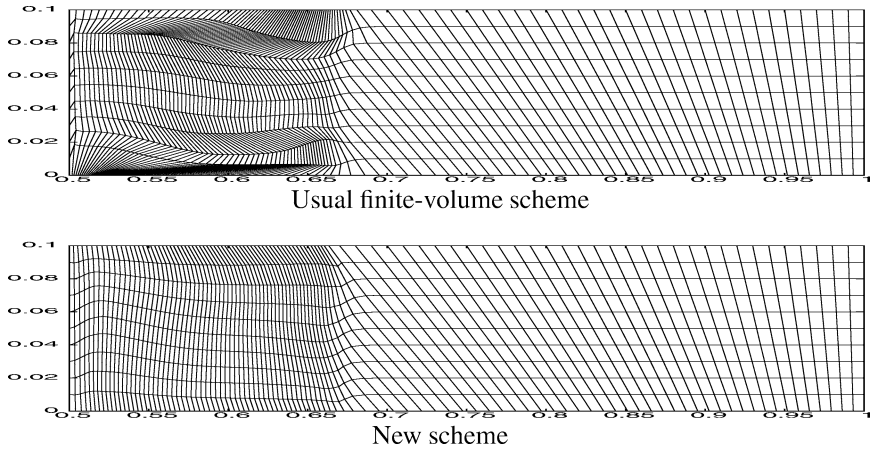

Fig. 2. Saltzman test problem; Lagrangian mesh at time $t = 0.5$.

Fig. 2. Le problème de Saltzman ; la maille de Lagrange à $t = 0{,}5$.

the new scheme almost preserves the unidimensional symmetry. It thus allows us to run the simulation after three reflections, until $t = 0.96$. Preleminary numerical results, realized with a particular choice of scheme, are very encouraging and will go on. Some of the ideas can be directly generalized in dimension three.

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