Available online at www.sciencedirect.com

SCIENCE DIRECT

C. R. Mecanique 331 (2003) 531–536

Comptes Rendus Mecanique

# Asymptotics for eigenvalues of the Laplacian with a Neumann boundary condition on a thin cut-out tube

Marina Yu. Planida

*The Bashkir State Pedagogical University, October Revolution st., 3a, 450000 Ufa, Russia*

Received 26 May 2003; accepted 6 June 2003

Presented by Évariste Sanchez-Palencia

**Abstract**

We consider the eigenvalue problem for the Laplace operator in a bounded three-dimensional domain where a thin tube is cut out. Imposing a Neumann boundary condition on the boundary of this tube, we construct asymptotics for eigenvalues on the small parameter that is a diameter of the tube. ***To cite this article: M.Yu. Planida, C. R. Mecanique 331 (2003).***


**Résumé**

**Comportement asymptotique des valeurs propres du laplacien avec conditions de Neumann sur un tube extrait.** On considère l'opérateur de Laplace dans un domaine tridimensionnel borné dont on a extrait un tube fin, avec la condition aux limites de Neumann. Nous construisons le développement asymptotique des valeurs propres pour des valeurs petites du diamètre du tube. ***Pour citer cet article : M.Yu. Planida, C. R. Mecanique 331 (2003).***


*Keywords:* Vibrations; Asymptotique; Valeur propre

*Mots-clés :* Vibrations ; Asymptotics ; Eigenvalue

## 1. Introduction

We consider the eigenvalue problem for the Laplace operator in a bounded domain and impose a Neumann boundary condition on a boundary of a thin cut-out tube. The asymptotics for an eigenvalue of this singular perturbed problem is constructed under the assumption the tube shrinks to a closed curve.

Let $x = (x_1, x_2, x_3)$, $\Omega \subset \mathbb{R}^3$ be a bounded simply-connected domain having an infinitely differentiable boundary $\Gamma$, $\gamma \in C^\infty$ be a closed curve with no self-intersection lying in the plane $x_3 = 0$, $\gamma \subset \Omega$. In a vicinity of $\gamma$ we introduce coordinates $(y, s)$, $y = (y_1, y_2)$, where $s$ is a natural parameter (the arc) of the curve $\gamma$, $y_1 = x_3$, $y_2$ is a distance to $\gamma$ in the plane $x_3 = 0$ measured along the inward normal to the curve $\gamma$ considered as a boundary of two-dimensional domain in the plane $x_3 = 0$. By $\omega$ we denote a simply-connected domain in $\mathbb{R}^2$ having

*E-mail address:* planida@bspu.ru (M.Yu. Planida).


doi:10.1016/S1631-0721(03)00125-6

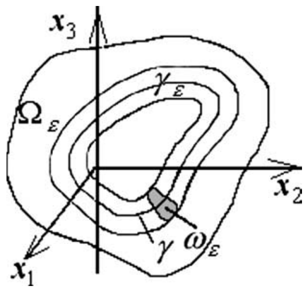


Fig. 1.

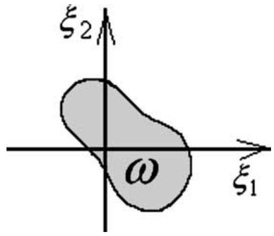


Fig. 2.

smooth boundary and containing the origin. We set $\omega_\varepsilon = \{y \in \mathbb{R}^2\colon\ \varepsilon^{-1} y \in \omega\}$, $\gamma_\varepsilon = \{x \in \mathbb{R}^3\colon\ s \in \gamma,\ y \in \omega_\varepsilon\}$, $\Omega_\varepsilon = \Omega \setminus \overline{\gamma_\varepsilon}$, $0 < \varepsilon \ll 1$ (cf. Figs. 1, 2). In [1] it was proved that if $\lambda_0$ is a simple eigenvalue of the boundary value problem

$$-\Delta\psi_0 = \lambda_0\psi_0 \quad \text{as } x \in \Omega, \qquad \sigma_1 \frac{\partial \psi_0}{\partial \tau} + \sigma_2 \psi_0 = 0 \quad \text{as } x \in \Gamma \tag{1}$$

where $\tau$ denotes inward normal to $\partial\Omega_\varepsilon$, $(\sigma_1, \sigma_2) = (1, 0)$ or $(\sigma_1, \sigma_2) = (0, 1)$, then there exists a unique eigenvalue $\lambda_\varepsilon$ of the problem

$$-\Delta\psi_\varepsilon = \lambda_\varepsilon\psi_\varepsilon \quad \text{in } \Omega_\varepsilon, \qquad \frac{\partial \psi_\varepsilon}{\partial \tau} = 0 \quad \text{on } \partial\gamma_\varepsilon, \qquad \sigma_1 \frac{\partial \psi_\varepsilon}{\partial \tau} + \sigma_2 \psi_\varepsilon = 0 \quad \text{on } \Gamma \tag{2}$$

converging to $\lambda_0$ as $\varepsilon \to 0$, and this perturbed eigenvalue is simple.

In the present paper using the method of matched asymptotics expansions [2–4], the asymptotics for this $\lambda_\varepsilon$ are constructed as $\varepsilon \to 0$. We notice that for particular case when the curve is a circle the asymptotics for the solution to the Poisson equation was constructed in [5].

## 2. Construction of asymptotics

Since the function $\psi_0$ does not meet the needed boundary condition on $\gamma_\varepsilon$, we employ the method of matched asymptotics expansions in order to construct the asymptotics of $\psi_\varepsilon$ in a vicinity of $\gamma_\varepsilon$. These asymptotics are constructed in terms of 'inner' variables $\xi = y\varepsilon^{-1}$.

For small $y$ the Laplace operator rewritten to $(y, s)$ becomes

$$\Delta = \Delta_y - \left( t(s) + \sum_{i=1}^{\infty} d_i(s) y_2^i \right) \frac{\partial}{\partial y_2} + \left( 1 + \sum_{i=2}^{\infty} h_{i-1}(s) y_2^{i-1} \right) \frac{\partial^2}{\partial s^2} - \sum_{i=1}^{\infty} p_i(s) y_2^i \frac{\partial}{\partial s} \tag{3}$$

where $\Delta_y$ is the two-dimensional Laplace operator with respect to variables $y, t$, and $b_i, h_q, p_m \in C^\infty(\gamma)$; $|t|$ is the curvature of the curve $\gamma$, while eigenfunction $\psi_0$ is expanded into a series

$$\psi_0\big(x(y; s)\big) = P_0(s) + P_1(y; s) + P_2(y; s) + \mathrm{O}\big(r^3\big), \quad r \to 0 \tag{4}$$

$$P_0 = c_{00}, \quad P_1 = c_{10} y_1 + c_{01} y_2, \quad P_2 = \frac{c_{20} y_1^2 + 2c_{11} y_1 y_2 + c_{02} y_2^2}{2}, \quad c_{ij}(s) = \left. \frac{\partial^{i+j} \psi_0}{\partial y_1^i \partial y_2^j} \right|_{y=0} \tag{5}$$

where $r = |y|$. Moreover, by (1) and (3),

$$c_{20} + c_{02} = t c_{01} - c_{00}'' - \lambda_0 c_{00} \tag{6}$$

In accordance with the method of matched asymptotics expansions [2], we rewrite (4) in the variables $\xi$ and see that asymptotics for the eigenfunction $\psi_\varepsilon$ in a vicinity of $\gamma_\varepsilon$ should be sought as

$$\psi_\varepsilon = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots \tag{7}$$

$$v_i(\xi; s) \sim P_i(\xi; s), \quad \rho \to \infty \tag{8}$$

where $\rho = |\xi|$. Substituting (3), (7) into (2), and substituting in the equality obtained for the variables $\xi$, we get the boundary value problem for $v_0$:

$$\Delta_\xi v_0 = 0, \quad \xi \in \mathbb{R}^2 \setminus \overline{\omega}, \qquad \frac{\partial v_0}{\partial \tau} = 0, \quad \xi \in \partial\omega \tag{9}$$

It is clear that the function

$$v_0(\xi; s) \equiv P_0(s) \tag{10}$$

satisfies (9) and (8). Substituting (3), (7) in (2) and writing out the problems for $v_1,\ v_2$, we see that

$$\Delta_\xi v_1 = 0, \quad \xi \in \mathbb{R}^2 \setminus \overline{\omega}, \qquad \frac{\partial v_1}{\partial \tau} = 0, \quad \xi \in \partial\omega \tag{11}$$

$$\Delta_\xi v_2 = t\frac{\partial v_1}{\partial \xi_2} - \frac{\partial^2 v_0}{\partial s^2} - \lambda_0 v_0, \quad \xi \in \mathbb{R}^2 \setminus \overline{\omega}, \qquad \frac{\partial v_2}{\partial \tau} = 0, \quad \xi \in \partial\omega \tag{12}$$

Here we also bear in mind that $v_0$ is independent of $\xi$.

Now we are going to study the solvability of the problems (11), (12) and find the asymptotics of $v_1,\ v_2$ as $\rho \to \infty$. We start from the following obvious statement.

**Lemma 2.1.** *Let $Z_n(\xi; s)$ be an arbitrary harmonic polynomial of $n$-th degree with respect to the variables $\xi$ whose coefficients depend on $s$. Then there exists a harmonic function $V(\xi; Z_n(\xi; s))$ defined in $\mathbb{R}^2 \setminus \overline{\omega}$ satisfying the homogeneous Neumann condition on $\partial\omega$ and the asymptotics*

$$V\big(\xi; Z_n(\xi; s)\big) = Z_n(\xi; s) + \big(a(s)\sin\varphi + b(s)\cos\varphi\big)\rho^{-1} + \mathrm{O}\big(\rho^{-2}\big) \tag{13}$$

*as $\rho \to \infty$. Here $\varphi$ is polar angle.*

**Corollary 2.2.** (a) *There exists, harmonic in $\mathbb{R}^2 \setminus \overline{\omega}$, functions $V(\xi; \xi_1),\ V(\xi; \xi_2)$ satisfying the homogeneous Neumann condition on $\partial\omega$ and asymptotics*

$$V(\xi; \xi_i) = \xi_i + \frac{1}{2\pi}\sum_{j=1}^{2} m_{ij}\frac{\partial \ln\rho}{\partial \xi_j} + \mathrm{O}\big(\rho^{-2}\big), \quad \rho \to \infty \tag{14}$$

(b) *The matrix $M(\omega) = (m_{ij})_{i,j=1,2}$ is positive defined, symmetric and determined by the domain $\omega$.*

Item (a) is implied by Lemma 2.2 while the validity of item (b) can be established by analogy with [3,7].

**Remark 1.** We note that when $\omega$ is a unit disk, we have $V(\xi; \xi_1) = (\rho + \rho^{-1})\sin\varphi,\ V(\xi; \xi_2) = (\rho + \rho^{-1})\cos\varphi$ and, therefore, the assertion (14) yields that $M = 2\pi E$, where $E$ is a unit matrix.

Due to Corollary 2.2 the function

$$v_1(\xi; s) = c_{10}(s)V(\xi; \xi_1) + c_{01}(s)V(\xi; \xi_2) \tag{15}$$

is a solution of the boundary value problem (11) and has the asymptotics

$$v_1(\xi; s) = P_1(\xi; s) + \big(A(s)\sin\varphi + B(s)\cos\varphi\big)\rho^{-1} + \mathrm{O}\big(\rho^{-2}\big) \tag{16}$$

as $\rho \to \infty$. Thus, it satisfies (8), moreover, by (15), (14),

$$A = \frac{1}{2\pi}(c_{10}m_{12} + c_{01}m_{22}), \qquad B = \frac{1}{2\pi}(c_{10}m_{11} + c_{01}m_{21}) \tag{17}$$

Let us proceed to the problem (12). We seek $v_2$ in the form:

$$v_2=\frac{t}{2}\xi_2 v_1-\frac{\rho^2}{4}(c''_{00}+\lambda_0 c_{00})+\tilde v_2 \tag{18}$$

By (18), (12), (11), (10), (8), (5) and (6), the boundary value problem for $\tilde v_2$ can be written as follows:

$$\begin{gathered}\Delta_\xi \tilde v_2=0,\quad \xi\in\mathbb{R}^2\setminus\overline{\omega},\qquad \frac{\partial\tilde v_2}{\partial\tau}=-\frac{t}{2}v_1\frac{\partial\xi_2}{\partial\tau}+\frac{1}{4}\frac{\partial\rho^2}{\partial\tau}(c''_{00}+\lambda_0c_{00}),\quad \xi\in\partial\omega\\ \tilde v_2(\xi;s)=Z_2(\xi;s)+\mathrm{o}\big(\rho^2\big),\quad \rho\to\infty\end{gathered} \tag{19}$$

In turn, $\tilde v_2$ is constructed as

$$\tilde v_2(\xi;s)=V\big(\xi;Z_2(\xi;s)\big)+\hat v_2(\xi;s),\quad \hat v_2(\xi;s)=\mathrm{o}(\rho),\quad \rho\to\infty \tag{20}$$

It follows from (20) and (19) that the boundary value problem for $\hat v_2$ has the form

$$\Delta_\xi \hat v_2=0,\quad \xi\in\mathbb{R}^2\setminus\overline{\omega},\qquad \frac{\partial\hat v_2}{\partial\tau}=-\frac{t}{2}v_1\frac{\partial\xi_2}{\partial\tau}+\frac{1}{4}\frac{\partial\rho^2}{\partial\tau}(c''_{00}+\lambda_0c_{00}),\quad \xi\in\partial\omega \tag{21}$$

It is known that there exists the solution to the problem (21) having the asymptotics

$$\hat v_2(\xi;s)=G(s)\ln\rho+\mathrm{O}(1),\quad G(s)=\frac{1}{2\pi}\int\limits_{\partial\omega}\frac{\partial\hat v_2}{\partial\tau}\,\mathrm{d}l_\xi \tag{22}$$

as $\rho\to\infty$. Calculating $G(s)$ by the boundary condition in (21), (15) and employing (14), we derive that

$$G(s)=-\frac{t(s)}{4\pi}(c_{10}m_{12}+c_{01}m_{22})+\frac{1}{2\pi}|\omega|(c''_{00}+\lambda_0c_{00}) \tag{23}$$

It follows from (18), (20), (22) that the boundary value problem (12) has a solution $v_2$ satisfying the asymptotics

$$v_2(\xi;s)=P_2(\xi;s)+G(s)\ln\rho+\mathrm{O}(1),\quad \rho\to\infty \tag{24}$$

and, therefore, this solution satisfies (8).

**Remark 2.** Observe that rewriting asymptotics of the function $\varepsilon^2 v_2$ (as $\rho\to\infty$) in the variables $x$ is the origin of the term $(-\varepsilon^2\ln\varepsilon G(s))$. To eliminate this term, in the inner expansion we introduce an additional term $\varepsilon^2\ln\varepsilon v_{2,1}$ obeying the asymptotics:

$$v_{2,1}(\xi;s)=G(s)+\mathrm{o}(1) \tag{25}$$

Thus, the leading terms of the asymptotics $\psi_\varepsilon$ should be constructed as

$$\psi_\varepsilon\approx v_0+\varepsilon v_1+\varepsilon^2\ln\varepsilon v_{2,1}+\varepsilon^2 v_2 \tag{26}$$

Now we substitute (26), (3) into (2), pass to the variables $\xi$ in the equality obtained and equate the coefficient of $\varepsilon^2\ln\varepsilon$. This procedure gives the following boundary value problem:

$$\Delta_\xi v_{2,1}=0,\quad \xi\in\mathbb{R}^2\setminus\overline{\omega},\qquad \frac{\partial v_{2,1}}{\partial\tau}=0,\quad \xi\in\partial\omega \tag{27}$$

It is obvious that $v_{2,1}(\xi;s)\equiv G(s)$ is a solution of the boundary value problem (27) and has the asymptotics (25). Replacing $v_i$ by their asymptotics at infinity (10), (16), (24), (25) in (26) and rewriting these asymptotics in the variable $y$, in accordance with method of matched asymptotics expansions, we see that the asymptotics of the eigenfunction outside a neighbourhood of $\gamma_\varepsilon$ and the asymptotics of the eigenvalue should be constructed as follows:

$$\psi_\varepsilon(x)=\psi_0(x)+\varepsilon^2\psi_2(x)+\cdots \tag{28}$$

$$\psi_2(x)=r^{-1}\big(A(s)\sin\varphi+B(s)\cos\varphi\big)+G(s)\ln r+\mathrm{O}(1),\quad r\to 0 \tag{29}$$

$$\lambda_\varepsilon=\lambda_0+\varepsilon^2\lambda_2+\cdots \tag{30}$$

Substituting (30) and (28) into (2), we arrive at the boundary value problem for $\psi_2$:

$$-\Delta\psi_2=\lambda_0\psi_2+\lambda_2\psi_0,\quad x\in\Omega\setminus\gamma,\qquad \sigma_1\frac{\partial\psi_2}{\partial\tau}+\sigma_2\psi_2=0,\quad x\in\Gamma \tag{31}$$

By analogy with [4] one can prove the following statement:

**Lemma 2.3.** *Let $a,b,g\in C^\infty(\gamma)$, $\|\psi_0\|_{L^2(\Omega)}=1$. Then there exist functions $\Psi_2^{(0)}(x;g)$, $\Psi_2^{(2)}(x;a,b)\in C^\infty(\overline{\Omega}\setminus\gamma)$, having the asymptotics*

$$\Psi_2^{(0)}\big(x;g(s)\big)(x)\sim g(s)\ln r,\qquad \Psi_2^{(2)}\big(x;a(s),b(s)\big)\sim r^{-1}\big(a(s)\sin\varphi+b(s)\cos\varphi\big)$$

*as $r\to 0$ and being the solutions of the boundary value problem* (31) *for $\lambda_2=\Lambda_2^{(0)}$ and $\lambda_2=\Lambda_2^{(2)}$, respectively, where*

$$\Lambda_2^{(0)}=2\pi\int_\gamma c_{00}g\,\mathrm{d}s,\qquad \Lambda_2^{(2)}=-2\pi\int_\gamma (c_{10}b+c_{01}a)\,\mathrm{d}s$$

It follows from Lemma 2.3 that the function $\psi_2=\Psi_2^{(2)}(x;A,B)+\Psi_2^{(0)}(x;G)$, where the quantities $A,\ B,\ G$ are defined by the equalities (17), (23), has the asymptotics (29) and is a solution to the boundary value problem (31) as

$$\lambda_2=|\omega|\int_\gamma\left(\lambda_0\psi_0^2-\left(\frac{\partial\psi_0}{\partial s}\right)^2\right)\mathrm{d}s-\int_\gamma\nabla_y\psi_0M(\omega)\left(\nabla_y\psi_0+\frac{1}{2}t(s)\psi_0\mathbf{e}_2\right)\mathrm{d}s \tag{32}$$

where $\mathbf{e}_2=(0,1)$.

Thus, the eigenvalue of the problem (2) has the asymptotics (30), (32). Rigorous justification of this asymptotics can be carried by analogy with [4].

## 3. Conclusing remarks

We note that if we impose Dirichlet boundary conditions on the tube's boundary, the variational properties of the eigenvalues say that $\lambda_\varepsilon-\lambda_0>0$. At the same time, for the problem considered in this paper, the quantity $\lambda_\varepsilon-\lambda_0$ has no definite sign. Indeed, assume that $S_{R_1}$ is a disk of radius $R_1>1$ with center at the origin, $\Omega=S_{R_1}\times(-\pi/2,\pi/2)$, $\gamma\subset\Omega\cap\{x_3=0\}$ is a unit circle with center at the origin, $(R,\Theta,x_3)$ are cylindrical coordinates associated with $x$. It is known that a simple eigenvalue of the problem (1) for $(\sigma_1,\sigma_2)=(0,1)$ (i.e., of the Dirichlet problem) is given by $\lambda_0=\varkappa^2+(\nu_j/R_1)^2$, and the associated eigenfunction normalized in $L^2(\Omega)$ is determined by the equality

$$\psi_0(x)=\frac{2}{\pi R_1}\frac{I_0(\nu_jR/R_1)}{I_0'(\nu_j)}X(x_3) \tag{33}$$

where $X(x_3)=\sin(\varkappa x_3)$ for even $\varkappa$; $X(x_3)=\cos(\varkappa x_3)$ for odd $\varkappa$; $\nu_j$ are zeroes for Bessel function $I_0(\nu)$ of zero order. Since $\gamma$ is a circle, it follows from (33) that $\partial\psi_0/\partial s=0$ on $\gamma$ (and, therefore, the second term in the first integral in (32) disappears). Assume, in addition, that $\omega$ is a unit circle. Therefore, $M=2\pi E$ (see Remark 1).

Let us consider the case $\varkappa = 2$, and $\nu_j/R_1$ is not a zero of the Bessel function $I_0(\nu)$. Clearly, there exists such an $R_1$. Then $\psi_0 = \partial\psi_0/\partial s = 0$ on $\gamma$ and it follows from the formula (32) that:

$$\lambda_2 = -2\pi \|\nabla_y \psi_0\|^2_{L^2(\gamma)} < 0 \tag{34}$$

Now assume that $\varkappa = 1$ and $\nu_s/R_1$ is a zero of first derivative of the Bessel function $I_0(\nu)$ (clearly, for $s$ large enough such an $R_1$ does exist). Then $\nabla_y \psi_0 = 0$ on $\gamma$ and it follows from (32) that:

$$\lambda_2 = 2\pi \lambda_0 \|\psi_0\|^2_{L^2(\gamma)} > 0 \tag{35}$$

Therefore, the assertions (30), (34) and (35) imply that in the first case $\lambda_\varepsilon - \lambda_0 < 0$, while in the second $\lambda_\varepsilon - \lambda_0 > 0$.

We note that similar phenomena appears in two- and three-dimensional boundary value problems for the Laplace operator in domains with a small cavity, when the Neumann condition is imposed on the boundary of the cavity while the cavity shrinks to a point [6,7].

In conclusion we also note that in the case of the Robin condition on $\Gamma$, the convergence of the eigenvalues and the estimates of the inverse operator needed for a rigorous justification of the asymptotics can be proved exactly by analogy with [1] (in this paper they considered Dirichlet and Neumann boundary conditions on $\Gamma$). The construction of the asymptotics $\lambda_\varepsilon$ does not differ from that above.

## Acknowledgements

The author thanks R.R. Gadyl'shin and D.I. Borisov for attention to this work. The author is supported by RFBR (02-01-00693), by the program 'Leading Scientific School' (NSh-1446.2003.1) and by the program 'Universities of Russia' (UR.04.01.010).

## References

[1] M.Yu. Planida, On the convergence of solutions of singularly perturbed boundary-value problems for the Laplace operator, Math. Notes 71 (2002) 867–877.
[2] A.M. Il'in, Matching of Asymptotic Expansions of Solutions of Boundary Value Problems, American Mathematical Society, Providence, RI, 1992.
[3] R.R. Gadyl'shin, Ramification of a multiple eigenvalue of the Dirichlet problem for the Laplacian under singular perturbation of the boundary condition, Math. Notes 52 (1992) 1020–1029.
[4] M.Yu. Planida, Asymptotics of eigenvalues for a cylinder insulated on a narrow strip, Comput. Math. Math. Phys. 43 (2003) 403–413.
[5] S.A. Nazarov, M.V. Paukshto, Discrete models and homogenization in problems of elastisity theory, Lenigrad University, Leningrad, 1984 (in Russian).
[6] S. Ozawa, Spectra of domains spherical Neumann boundary, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 30 (1983) 259–277.
[7] V.G. Maz'ya, S.A. Nazarov, B.A. Plamenevskij, Asymptotic expansions of the eigenvalues of boundary value problems for the Laplace operator in domains with small holes, Math. USSR-Izv. 24 (1985) 321–345.