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Homogenization of an elastic medium reinforced by anisotropic fibers

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Abstract

We present results in this Note concerning a vector version in the framework of linearized elasticity (see A. Sili, Homogenization of an elastic medium reinforced by anisotropic fibers, in press), of our previous work in which we have studied the homogenization of a scalar nonlinear monotone problem posed on a fibered medium (see A. Sili, Homogenization of a nonlinear monotone problem in an anisotropic medium, in press). Here, we assume that parallel elastic anisotropic fibers, periodically distributed with a period of size ε in a cube Ω, are surrounded by a soft elastic material, the elasticity coefficients of this material being in the ratio ε^2 with those of the fibers. We prove that the homogenized problem is nonlocal and involves variables linked together with the anisotropy of the fibers. ***To cite this article: A. Sili, C. R. Mecanique 331 (2003).***
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Résumé

Homogénéisation d'un milieu élastique renforcé par des fibres anisotropes. Nous présentons dans cette Note les résultats concernant la version vectorielle, dans le cadre de l'élasticité linéarisée (voir A. Sili, Homogenization of an elastic medium reinforced by anisotropic fibers, à paraître) de notre précédent travail sur l'homogénéisation d'un problème scalaire non linéaire monotone posé dans un milieu fibré (voir A. Sili, Homogenization of a nonlinear monotone problem in an anisotropic medium, à paraître). Ici, nous supposons que les fibres élastiques, parallèles et anisotropes, sont périodiquement réparties dans un cube Ω avec une période de taille ε, et entourées d'un matériau élastique mou, les coefficients de ce matériau étant dans un rapport de ε^2 avec ceux des fibres. Nous montrons que le problème homogénéisé est non local et fait apparaître des variables dûes à l'anisotropie. ***Pour citer cet article : A. Sili, C. R. Mecanique 331 (2003).***
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1. Introduction

We consider an heterogeneous elastic medium, the reference configuration of which is assumed to be the cube $\Omega = \omega \times (-\frac{1}{2}, \frac{1}{2}) = \omega \times I$. We assume that Ω is made from two different elastic materials: the first is that

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contained in a set F_ε of parallel fibers of radius $r\varepsilon$, $0<r<\frac{1}{2}$, obtained by an ε-periodic repetition of one of them. The second material which surrounds the fibers is assumed to be a soft material and corresponds to the region $M_\varepsilon=\Omega-F_\varepsilon$ of Ω. We shall consider a problem with mixed homogeneous boundary conditions in two cases: the first case corresponds to Dirichlet conditions on the union of the lower face $\Omega_{-1/2}=\omega\times\{-\frac{1}{2}\}$ and the upper face $\Omega_{1/2}=\omega\times\{\frac{1}{2}\}$ of Ω; this part of the boundary will be denoted by Γ_D. We denote by Γ_N the rest of the boundary of Ω which is subject to Neumann conditions. Hence,

$$\Gamma_D=\Omega_{-1/2}\cup\Omega_{-1/2};\qquad \Gamma_N=\partial\Omega-\Gamma_D \tag{1}$$

In the second case, we assume Dirichlet conditions only on the lower face $\Omega_{-1/2}$ (still denoted by Γ_D) and Neumann conditions elsewhere (Γ_N). The domain ω is a partition of squares C_ε^i of size $\varepsilon>0$, each of them containing the disk ω_ε^i of center $x_\varepsilon^i=\varepsilon(i_1,i_2)$ and of radius $r\varepsilon$ with some $0<r<\frac{1}{2}$. The cell Y_ε^i is then defined by $Y_\varepsilon^i=C_\varepsilon^i\times I$, so that:

$$\Omega=\bigcup_{i\in I_\varepsilon}Y_\varepsilon^i,\quad \text{where } I_\varepsilon=\left\{i=(i_1,i_2)\in\mathbb{Z}^2,\ C_\varepsilon^i\subset\omega\right\} \tag{2}$$

The fiber F_ε^i and the set F_ε of all the fibers are defined by:

$$F_\varepsilon^i=\omega_\varepsilon^i\times I,\qquad F_\varepsilon=\bigcup_{i\in I_\varepsilon}F_\varepsilon^i \tag{3}$$

The region occupied by the soft material will be denoted by M_ε:

$$M_\varepsilon=\Omega-F_\varepsilon \tag{4}$$

Throughout this Note, we denote by $x=(x',x_3)$ the generic point of $\mathbb{R}^3$. The Latin indices $i,j,k,l,\ldots$ run over the set $\{1,2,3\}$ and the Greek indices $\alpha,\beta,\ldots$ (except ε which takes its values in a decreasing sequence of integers) run over the set $\{1,2\}$. Unless otherwise stated, the Einstein summation convention with repeated indices will be used.

The characteristic function of a set B will be denoted by χ_B and its Lebesgue measure by $|B|$. We introduce the square Y, the disk D of radius $r<\frac{1}{2}$ centered in Y:

$$Y=\left(-\frac{1}{2},\frac{1}{2}\right)^2,\qquad D=D(0,r) \tag{5}$$

We denote by $C_\#(Y)$ the space of continuous functions in $\mathbb{R}^2$ that are Y-periodic.

Let A be a fourth order tensor satisfying the following hypotheses: for almost all $(x,y)\in\Omega\times Y$, for all $i,j,k,l\in\{1,2,3\}$ and for all symmetric 2×2 tensor $e=(e_{ij})$,

$$A_{ijkl}=A_{jikl}=A_{klij} \tag{6}$$

$$y\to A_{ijkl}(x,y)\text{ is periodic with period }Y \tag{7}$$

$$A_{ijkl}\in L^\infty\big(\overline{\Omega};C_\#(Y)\big) \tag{8}$$

$$\exists m>0,\quad A_{ijkl}(x,y)e_{kl}e_{ij}\geqslant m e_{ij}e_{ij} \tag{9}$$

Let

$$f\in\big(L^2(\Omega)\big)^3 \tag{10}$$

We consider the variational formulation of the equilibrium equation of a body Ω which is only subject to volume forces f^ε, its lower and upper faces being fixed. We assume that the forces in the direction x_3 are more important than the one in the transversal directions; more precisely, $f^\varepsilon=(\varepsilon f_\alpha,f_3)$. Setting

$$\Phi=\left\{\phi\in\big(H^1(\Omega)\big)^3\colon\ \phi=0\text{ on }\Gamma_D\right\} \tag{11}$$

we consider the problem:

$$\begin{cases} u_\varepsilon \in \Phi, \\ \displaystyle\int_\Omega \left(\chi_{F_\varepsilon} A\left(x', x_3, \frac{x'}{\varepsilon}\right) + \varepsilon^2 \chi_{M_\varepsilon} A\left(x', x_3, \frac{x'}{\varepsilon}\right)\right) e(u_\varepsilon) e(\phi)(x)\,\mathrm{d}x \\ \quad = \displaystyle\int_\Omega \big(\varepsilon f_\alpha(x)\phi_\alpha(x) + f_3(x)\phi_3(x)\big)\,\mathrm{d}x, \quad \forall \phi \in \Phi \end{cases} \tag{12}$$

Under hypotheses (6)–(10), the use of the Korn's inequality and the Lax–Milgram theorem gives existence and uniqueness of the solution u_ε for each ε. The homogenization procedure consists in the study of the behaviour of u_ε as ε tends to zero.

The main difficulty comes from the fact that the radius of each fiber has the same order than the size ε of the period of the structure and the order of magnitude of the coefficients in the fibers are more important than the one of the coefficients in the matrix, the ratio being ε^2. An other choice of the scalings for the loading terms and for the ratio between the elasticity coefficients inside the fibers and outside them was considered in [1].

We perform the asymptotic analysis without any assumption on the isotropy of the material in the fibers, nor on the one in the matrix, so that we expect to obtain a coupling homogenized problem involving in particular the anisotropy of the fibers as it was shown in the scalar case, see [14]. For a single rod with a constant radius, the one-dimensional model is derived in [3]; a different approach is presented in [4] for partially heterogeneous straight rods and in [2] for curved rods.

In the case of rigid isotropic fibers with very thin radius r_ε, i.e., under the assumption $r_\varepsilon/\varepsilon \to 0$, $\lambda_\varepsilon \to \infty$, $\mu_\varepsilon \to \infty$, where λ_ε and μ_ε are the Lamé coefficients of the fibers, it was proved in [13] that the homogenized material is a second gradient material. This situation was addressed in a more general context in [5]. The commutativity of the limits first with respect to the thickness of the structure and then with respect to the size of the period or conversely was considered in [6–9] and [10].

In the case of a single rod, it was proved in [3] that the reduction problem 3d–1d leads to a limit problem which consists of a system in which arise a Bernoulli–Navier type displacement u, another displacement v due to the anisotropy of the cylinder and a two-dimensional displacement w. The case of a cylinder having heterogeneities through its axis was studied in [11]. In the homogenization process we consider here, it is natural to expect to keep at the limit some sides of the limit problem obtained in [3], since we are in presence of several thin cylinders (the fibers). Actually, we prove that the homogenized problem involves a variable u linked together with the displacements in the fibers. This variable can be seen as an 'homogenized form' of the Bernoulli–Navier displacement obtained in [3]. In particular, the transversal displacements u_α depend on the variable x' and they are strongly linked with another variable $z(x, y)$ which describes the displacements outside the fibers, see below for the definition of the corresponding spaces. Similarly, the anisotropic variable v and the two-dimensional displacement w of [3] take here a homogenized form.

Globally, our approach is similar to that developped in [3]: in the first step, the limit problem is written as a system involving different variables (Theorem 2.1); then, after writing some of these variables in terms of the solutions of elementary problems which depend only on the coefficients of the tensor A, we give the final form of the homogenized problem (Theorem 2.2). Note the nonlocal character of the equation given in Theorem 2.2: the macroscopic displacements u in the fibers are connected with the displacements z outside them, see the precise definition of u and z below. In other words, these two variables are not independent and the homogenized equation given in Theorem 2.2 must be seen as a single equation involving the pair (u, z) rather than a system of a two equations on u and z.

In addition to the space Φ defined by (11), we will use the following spaces and notations.

We denote by y^R the point $y^R = (-y_2, y_1)$ obtained from y by a rotation. As usual, the strain tensor $e(\phi)$ is defined componentwise by: $e_{ij}(\phi)(x) = \frac{1}{2}(\frac{\partial\phi_i}{\partial x_j} + \frac{\partial\phi_j}{\partial x_i})(x)$ while the notation $e^y_{\alpha\beta}(\phi)(x, y) = \frac{1}{2}(\frac{\partial\phi_\alpha}{\partial y_\beta} + \frac{\partial\phi_\beta}{\partial y_\alpha})(x, y)$

will be used for the components of the strain tensor in the basic cell Y. We also use the abbreviated notation $e(\phi) = \begin{pmatrix} e_{\alpha\beta}(\phi) & e_{\alpha 3}(\phi) \\ e_{\alpha 3}(\phi) & e_{33}(\phi) \end{pmatrix}$. If B denotes a fourth order tensor, then the scalar product of the matrices $Be(\phi)$ and $e(\phi)$ will be denoted as: $Be(\phi)\ e(\phi) = \sum_{i,j} (Be(\phi))_{i,j} (e(\phi))_{i,j}$. We denote by $H^1_\#(Y)$ the space of functions in $H^1_{\mathrm{Loc}}(\mathbb{R}^2)$ which are Y-periodic and by $C_\#(Y)$ (resp. $C^\infty_\#(Y)$), the space of continuous (resp. infinitely differentiable) functions in $\mathbb{R}^2$ which are Y-periodic.

We denote by $H^1_0(I)$ (resp. $H^1_b(I)$) the space of functions u in the one-dimensional Sobolev space $H^1(I)$ whith zero trace on the boundary of I (resp. such that $u(-\frac{1}{2}) = 0$); the space $H^2_0(I)$ (resp. $H^2_b(I)$) denotes the subspace of $H^1_0(I)$ (resp. $H^1_b(I)$) of functions with derivative in $H^1_0(I)$ (resp. $H^1_b(I)$). We denote by $H^1_m(D)$) the subspace of functions in $H^1(D)$) with zero average over D.

We set:

$$\mathcal{U} = \big\{u = (u_1, u_2, u_3),\ u_\alpha \in H^1(\Omega) \cap L^2\big(\omega; H^2_0(I)\big),\ u_3 \in L^2\big(\omega; H^1_0(I)\big)\big\} \tag{13}$$

$$\mathcal{V} = \big\{v = (v_1, v_2, v_3)\colon\ \exists c \in L^2\big(\omega; H^1_0(I)\big),\ v_\alpha(x,y) = c(x) y^R_\alpha \text{ in } \Omega \times D,\ v_3 \in L^2\big(\Omega; H^1_m(D)\big)\big\} \tag{14}$$

$$\mathcal{W} = \Big\{w = (w_1, w_2, 0)\colon\ w_\alpha \in L^2\big(\Omega; H^1_m(D)\big),\ \int_D (y_1 w_2 - y_2 w_1)\,\mathrm{d}y = 0,\ \text{a.e. in } \Omega\Big\} \tag{15}$$

$$\left\{\begin{aligned} \mathcal{Z} = \Big\{& z = (z_1, z_2, z_3) \in \big(L^2\big(\Omega; H^1_\#(Y)\big)\big)^3,\ \int_D z_i\,\mathrm{d}y = 0,\ \forall i = 1,2,3,\ \exists u = (u_1, u_2, 0) \in \mathcal{U}, \\ & \exists c \in L^2\big(\omega; H^1_0(I)\big)\colon\ z_\alpha(x,y) = -y_\beta \frac{\partial u_\alpha}{\partial x_\beta}(x) + c(x) y^R_\alpha \text{ in } \Omega \times D,\ \forall \alpha = 1,2, \\ & z_3(x,y) = -y_\alpha \frac{\partial u_\alpha}{\partial x_3}(x) \text{ in } \Omega \times D\Big\} \end{aligned}\right. \tag{16}$$

For the sake of brevity, we put:

$$\mathcal{T} = \mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \mathcal{Z} \tag{17}$$

Before giving the results, let us recall the definition of the two-scale convergence (see [12]) for which we use the symbol $\rightharpoonup\!\!\rightharpoonup$.

Definition 1.1. A sequence z_ε in $L^2(\Omega)$ two-scale converges to a function $z \in L^2(\Omega \times Y)$ if:

$$\forall \phi \in \mathcal{D}\big(\Omega; C^\infty_\#(Y)\big), \quad \int_\Omega z_\varepsilon(x) \phi\left(x, \frac{x'}{\varepsilon}\right) \mathrm{d}x \longrightarrow \iint_{\Omega\, Y} z(x,y) \phi(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{18}$$

In order to describe the homogenized problem corresponding to the situation where Dirichlet condition is only prescribed on the lower face $\Omega_{-1/2}$ an analogous space still denoted by $\mathcal{T}$ is defined by replacing in (13), (14) and (17) the spaces $H^1_0(I)$ and $H^2_0(I)$ respectively by $H^1_b(I)$ and $H^2_b(I)$. Taking into account this remark, Theorem 2.1 below is concerned with the case $\Gamma_D = \Omega_{-1/2} \cup \Omega_{1/2}$ as well as the case $\Gamma_D = \Omega_{-1/2}$.

Our main results are the following.

2. Statement of the results

Theorem 2.1. *Assume* (6)–(10). *Let u_ε be the sequence of solutions of* (12).

There exist

$$(u, v, w, z) \in \mathcal{T} \tag{19}$$

such that for all $\alpha,\beta=1,2$, *the following convergences hold true*:

$$e_{\alpha\beta}(u_\varepsilon)\chi_{F_\varepsilon} \rightharpoonup\rightharpoonup e^y_{\alpha\beta}(w)\chi_D(y) \tag{20}$$

$$e_{\alpha 3}(u_\varepsilon)\chi_{F_\varepsilon} \rightharpoonup\rightharpoonup \frac{1}{2}\left(\frac{\partial v_3}{\partial y_\alpha}+\frac{\partial c}{\partial x_3}y^R_\alpha\right)\chi_D(y) \tag{21}$$

$$e_{33}(u_\varepsilon)\chi_{F_\varepsilon} \rightharpoonup\rightharpoonup \left(\frac{\partial u_3}{\partial x_3}-y_\alpha\frac{\partial^2 u_\alpha}{\partial x_3^2}\right)\chi_D(y) \tag{22}$$

$$e_{\alpha\beta}(u_\varepsilon)\chi_{M_\varepsilon} \rightharpoonup\rightharpoonup \left(e_{\alpha\beta}(u)+e^y_{\alpha\beta}(z)\right)\chi_{(Y-D)}(y) \tag{23}$$

$$e_{\alpha 3}(u_\varepsilon)\chi_{M_\varepsilon} \rightharpoonup\rightharpoonup \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_3}+\frac{\partial z_3}{\partial y_\alpha}\right)\chi_{(Y-D)}(y) \tag{24}$$

$$e_{33}(u_\varepsilon)\chi_{M_\varepsilon} \rightharpoonup\rightharpoonup 0 \tag{25}$$

(u,v,w,z) *is the unique solution of the problem*:

$$\begin{cases}
(u,v,w,z)\in\mathcal{T},\\
\displaystyle\int\limits_{\Omega\times D} A\begin{pmatrix} e^y_{\alpha\beta}(w) & \frac{1}{2}\left(\frac{\partial v_3}{\partial y_\alpha}+\frac{\partial c}{\partial x_3}y^R_\alpha\right)\\ \frac{1}{2}\left(\frac{\partial v_3}{\partial y_\alpha}+\frac{\partial c}{\partial x_3}y^R_\alpha\right) & \frac{\partial u_3}{\partial x_3}-y_\alpha\frac{\partial^2 u_\alpha}{\partial x_3^2}\end{pmatrix}\begin{pmatrix} e^y_{\alpha\beta}(\bar w) & \frac{1}{2}\left(\frac{\partial \bar v_3}{\partial y_\alpha}+\frac{\partial \bar c}{\partial x_3}y^R_\alpha\right)\\ \frac{1}{2}\left(\frac{\partial \bar v_3}{\partial y_\alpha}+\frac{\partial \bar c}{\partial x_3}y^R_\alpha\right) & \frac{\partial \bar u_3}{\partial x_3}-y_\alpha\frac{\partial^2 \bar u_\alpha}{\partial x_3^2}\end{pmatrix}\mathrm{d}x\,\mathrm{d}y\\
\quad\displaystyle+\int\limits_{\Omega\times(Y-D)} A\begin{pmatrix} e_{\alpha\beta}(u)(x)+e^y_{\alpha\beta}(z) & \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_3}+\frac{\partial z_3}{\partial y_\alpha}\right)\\ \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_3}+\frac{\partial z_3}{\partial y_\alpha}\right) & 0\end{pmatrix}\\
\quad\quad\begin{pmatrix} e_{\alpha\beta}(\bar u)(x)+e^y_{\alpha\beta}(\bar z) & \frac{1}{2}\left(\frac{\partial \bar u_\alpha}{\partial x_3}+\frac{\partial \bar z_3}{\partial y_\alpha}\right)\\ \frac{1}{2}\left(\frac{\partial \bar u_\alpha}{\partial x_3}+\frac{\partial \bar z_3}{\partial y_\alpha}\right) & 0\end{pmatrix}\mathrm{d}x\,\mathrm{d}y\\
\quad\displaystyle=\int\limits_{\Omega}\int\limits_{Y}\left(f_\alpha(x)\bar u_\alpha(x)+f_3(x)\left(\bar u_3(x)+\bar z_3(x,y)\right)\right)\,\mathrm{d}x\,\mathrm{d}y\\
\forall(\bar u,\bar v,\bar w,\bar z)\in\mathcal{T}
\end{cases} \tag{26}$$

Remark 2.1. Problem (26) is a well posed problem on the space $\mathcal{U}\times\mathcal{V}\times\mathcal{W}\times\mathcal{Z}$ which is a Hilbert space for the norm

$$\begin{cases}
\left(\left\|\frac{\partial u_3}{\partial x_3}\right\|^2_{L^2(\Omega)}+\sum\limits_\alpha\left\|\frac{\partial^2 u_\alpha}{\partial x_3^2}\right\|^2_{L^2(\Omega)}+\sum\limits_{\alpha,\beta}\left(\left\|e_{\alpha\beta}(u)\right\|^2_{L^2(\Omega)}+\left\|\frac{\partial v_3}{\partial y_\alpha}+\frac{\partial c}{\partial x_3}y^R_\alpha\right\|^2_{L^2(\Omega\times D)}\right.\right.\\
\quad\left.\left.+\left\|e^y_{\alpha\beta}(w)\right\|^2_{L^2(\Omega\times D)}+\left\|e^y_{\alpha\beta}(z)\right\|^2_{L^2(\Omega\times Y)}\right)+\sum\limits_\alpha\left\|\frac{\partial z_3}{\partial y_\alpha}\right\|^2_{L^2(\Omega\times Y)}\right)^{1/2}
\end{cases} \tag{27}$$

Convergences (21)–(25) suggest that the solution u_ε of (12) behaves as:

$$\left(\frac{u_\alpha(x)}{\varepsilon}+z_\alpha\left(x,\frac{x'}{\varepsilon}\right)+\varepsilon\, w_\alpha\left(x,\frac{x'}{\varepsilon}\right),\ z_3\left(x,\frac{x'}{\varepsilon}\right)+u_3(x)+\varepsilon\, v_3\left(x,\frac{x'}{\varepsilon}\right)\right).$$

Such test functions imply that the quantities $e^x_{\alpha\beta}(z)\chi_D(y)\frac{\partial u_3}{\partial x_\alpha}(x)$, and $(\partial z_\alpha/\partial x_3+\partial z_3/\partial x_\alpha)\chi_D(y)$ for $\alpha,\beta=1,2$ do not appear at the limit. This is indeed the case since the two-scale limits $\sigma_{\alpha\beta}(x,y)$ (resp. $\sigma_{\alpha 3}(x,y)$) of $A_{\alpha\beta kl}e_{kl}(u_\varepsilon)\chi_{F_\varepsilon}$ (resp. $A_{\alpha 3kl}e_{kl}(u_\varepsilon)\chi_{F_\varepsilon}$) are such that $\int_D y_\alpha\sigma_{\gamma\beta}\,\mathrm{d}y=\int_D y_\alpha\sigma_{\alpha 3}\,\mathrm{d}y=\int_D\sigma_{\alpha 3}\,\mathrm{d}y=0$, for each $\alpha,\beta,\gamma=1,2$ (there is no summation) and $\int_D(y_1\sigma_{23}+y_2\sigma_{13})\,\mathrm{d}y=0$.

In order to get another formulation of the homogenized problem (26), we first write the variables v and w in terms of u via elementary solutions $(\hat v_3^{(p)},\widehat{w}^{(p)})$, $p=1,2,3$, and $(\hat v_3,\widehat{w})$ which depend only on the coefficients

A_{ijkl} of the tensor A. In the case $\Gamma_D = \Omega_{-1/2}$, problem (26) is then written in the following form; the case $\Gamma_D = \Omega_{-1/2} \cup \Omega_{1/2}$ also leads to a problem on (u, z) but with supplementary terms due to the condition $c(x', -\frac{1}{2}) = c(x', \frac{1}{2}) = 0$, see [15].

Theorem 2.2. *In the case $\Gamma_D = \Omega_{-1/2}$, the homogenized problem* (26) *admits the following formulation*: (u, z) *is the unique solution of*

$$\begin{cases} (u,z) \in \mathcal{U} \times \mathcal{Z}, \\ \displaystyle\int_\Omega A^0(x) \begin{pmatrix} \frac{\partial^2 u_1}{\partial x_3^2} \\ \frac{\partial^2 u_2}{\partial x_3^2} \\ \frac{\partial u_3}{\partial x_3} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \bar{u}_1}{\partial x_3^2} \\ \frac{\partial^2 \bar{u}_2}{\partial x_3^2} \\ \frac{\partial \bar{u}_3}{\partial x_3} \end{pmatrix} \mathrm{d}x + \int_{\Omega \times (Y-D)} A \begin{pmatrix} e_{\alpha\beta}(u)(x) + e^y_{\alpha\beta}(z) & \frac{1}{2}\big(\frac{\partial u_\alpha}{\partial x_3} + \frac{\partial z_3}{\partial y_\alpha}\big) \\ \frac{1}{2}\big(\frac{\partial u_\alpha}{\partial x_3} + \frac{\partial z_3}{\partial y_\alpha}\big) & 0 \end{pmatrix} \\ \qquad \times \begin{pmatrix} e_{\alpha\beta}(\bar{u})(x) + e^y_{\alpha\beta}(\bar{z}) & \frac{1}{2}\big(\frac{\partial \bar{u}_\alpha}{\partial x_3} + \frac{\partial \bar{z}_3}{\partial y_\alpha}\big) \\ \frac{1}{2}\big(\frac{\partial \bar{u}_\alpha}{\partial x_3} + \frac{\partial \bar{z}_3}{\partial y_\alpha}\big) & 0 \end{pmatrix} \mathrm{d}x\,\mathrm{d}y \\ \quad = \displaystyle\iint_{\Omega\, Y} \big(f_\alpha(x)\bar{u}_\alpha(x) + f_3(x)\big(\bar{u}_3(x) + \bar{z}_3(x,y)\big)\big)\,\mathrm{d}x\,\mathrm{d}y \\ \forall (\bar{u}, \bar{z}) \in \mathcal{U} \times \mathcal{Z} \end{cases} \tag{28}$$

The coefficients A^0_{ij} of the homogenized 3×3 matrix A^0 are given in terms of the elementary solutions $(\hat{v}_3^{(p)}, \widehat{w}^{(p)})$, $p = 1, 2, 3$, and $(\hat{v}_3, \widehat{w})$.

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