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Flow in wavy tube structure: asymptotic analysis and numerical simulation

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Abstract

This paper deals with the study of the stationary, incompressible, 2D flow of a fluid in a thin wavy tube. In this work, we consider a domain which is the union of two wavy tubes depending on a small parameter. The asymptotic expansion is constructed. The method of partial asymptotic decomposition is applied. The numerical implementation of this method for the extrusion process is developed. The new physical effects are discussed. ***To cite this article: A. Ainser et al., C. R. Mecanique 331 (2003).***

Résumé

Écoulement dans une structure tubulaire ondulée : analyse asymptotique et résolution numérique. Nous considérons ici le mouvement bi-dimensionnel et stationnaire d'un fluide incompressible à l'intérieur d'un domaine constitué de tubes ondulés. La méthode de décomposition asymptotique partielle du domaine est mise en place et des résultats numériques, obtenus pour la modélisation de procédés d'extrusion seront présentés afin de justifier l'application de cette méthode. ***Pour citer cet article : A. Ainser et al., C. R. Mecanique 331 (2003).***

Keywords: Computational fluid mechanics; Asymptotic expansion; Stokes equation; Partial asymptotic domain decomposition; Extrusion; Numerical solution

Mots-clés : Mécanique des fluides numérique ; Développement asymptotique ; Équation de Stokes ; Décomposition asymptotique partielle du domaine ; Extrusion ; Solution numérique

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Cette analyse consiste à étudier l'écoulement d'un fluide à l'intérieur d'un domaine constitué de deux tubes ondulés dépendant d'un petit paramètre $\varepsilon > 0$.

Le problème de Stokes dans une structure tube constituée de cylindres fins a été considéré dans [3] en utilisant la méthode de décomposition asymptotique partielle du domaine (MAPDD). Cette méthode consiste à séparer le problème initial en petits sous-problèmes afin de réduire le coût total de la solution numérique.

Tout d'abord, il nous faut construire une solution asymptotique pour le problème afin de décrire et de justifier l'application de la MAPDD. Cette analyse confirme la localisation des effets de couche limite au voisinage des zones de transition ainsi que la convergence de la solution asymptotique vers une solution périodique à l'intérieur des tubes. La justification numérique proposée ici, est l'application de cette méthode pour simuler un procédé d'extrusion de polymère.

1. Introduction

This paper deals with the study of the stationary, incompressible, 2D flow of a fluid in a thin wavy tube. In this work, we consider a domain G^ε which is the union of two wavy tubes depending on a small parameter $\varepsilon > 0$.

Flows in thin layers have been considered in [1,2]. The Stokes problem in tube structure constituted by thin cylinders has been studied in [3]. The same method is applied in the present study. The proposed partial asymptotic domain decomposition method (MAPDD), described in [4], splits the initial Stokes problem in some smaller sub-problems and essentially reduces the total numerical cost of the total numerical solution.

In the first section, we build and justify an asymptotic solution for the Stokes problem. In Section 2, the MAPDD is described and justified. In Section 3, the MAPDD method is validated numerically by analyzing the flow in a wavy tube. This analysis confirmed the localized character of the transition zone between the periodic flows as well as the rapid convergence of the asymptotic solution toward the periodic one. However, the main advantages of the proposed method are clearly visible for big 3D problems, where the full solution is often difficult to obtain. As a final application, we use the MAPDD method for analyzing the polymer extrusion process.

The detailed description of the flow is presented in terms of velocity and pressure distribution, particle tracking and residence time analysis. The calculations predict the existence of a backward flow which has to be confirmed experimentally.

2. Stokes equations in two wavy tubes

2.1. Problem

Consider three sets $Y_i \subset [0,1] \times (0,1/2]$ $(0 \leqslant i \leqslant 2)$ defined by

$$Y_i := \{(y_1, y_2) \in \mathbb{R}^2\colon\ 0 < y_1 < 1 \text{ and } -h_i^+(y_1) < y_2 < h_i^+(y_1)\} \quad \text{for } 0 \leqslant i \leqslant 2$$

where $h_i^\pm : \mathbb{R} \to (0, 1/2)$ are functions of class $\mathcal{C}^2(\mathbb{R})$ and such that there exists m: $(\forall) t \in \mathbb{R},\ h_i^\pm(t) \geqslant m > 0$ (for $0 \leqslant i \leqslant 2$). We assume that the functions $h_1^\pm$ and $h_2^\pm$ are 1-periodic.

For the boundary, we introduce the notation $\Gamma_i^\pm := \{(y_1, \pm h_i^\pm(y_1))\colon\ 0 < y_1 < 1\}$ for $0 \leqslant i \leqslant 2$ and we assume that the curves given by $(\Gamma_1^\pm - \mathbf{e}_1) \cup \Gamma_0^\pm \cup (\Gamma_2^\pm + \mathbf{e}_1)$ are of class $\mathcal{C}^2$.

Let $\varepsilon > 0$ be a small parameter with $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}^\star$.

We define the domain G_0^ε, obtained from Y_0 by a homothetic contraction in $1/\varepsilon$ times (with respect to 0) and the domains $G_1^\varepsilon := (\bigcup_{l=1}^{n} \overline{\varepsilon\{Y_1 - l\mathbf{e}_1\}})'$ and $G_2^\varepsilon := (\bigcup_{l=1}^{n-1} \overline{\varepsilon\{Y_2 + l\mathbf{e}_1\}})'$, where for any set A, $\bar{A}$ is the closure and A' is the set of the interior points. In this study, we consider the thin domain $G^\varepsilon := G_1^\varepsilon \cup G_0^\varepsilon \cup G_2^\varepsilon$. Let $\Sigma_i^\varepsilon := \overline{G}^\varepsilon \cap \{x_1 = i\}$ $(i = -1, 1)$ be the lateral sides of the domain and $\Gamma_\varepsilon^\pm$ be the upper and lower parts of $\partial G^\varepsilon \setminus \{\Sigma_{-1}^\varepsilon \cup \Sigma_1^\varepsilon\}$.

The stationary flow of a viscous incompressible fluid through the channel G^ε is governed by the following Stokes system:

$$\begin{cases} -\nu\Delta\mathbf{v}^\varepsilon + \nabla p^\varepsilon = \mathbf{f}, \quad \operatorname{div}\mathbf{v}^\varepsilon = 0 \quad \text{in } G^\varepsilon \\ \mathbf{v}^\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon^\pm, \quad \mathbf{v}^\varepsilon = \varepsilon^2\boldsymbol{\varphi}_\varepsilon \quad \text{on } \Sigma_i^\varepsilon, \quad i = -1, 1 \end{cases} \tag{1}$$

Let $H^1_{\mathrm{div}}(G^\varepsilon)$ be the space of vector valued functions from $[H^1(G^\varepsilon)]^2$ with vanishing divergence and equal to zero on the boundaries $\Gamma_\varepsilon^\pm$ and $V(G^\varepsilon)$ the subspace of $H^1_{\mathrm{div}}(G^\varepsilon)$ of functions vanishing on the whole boundary. We denote by $\mathcal{W}_i$ the space of divergence free functions $\mathbf{u} \in \mathcal{C}^\infty_{\mathrm{per}}((\bigcup_{l\in\mathbb{Z}}(\overline{Y_i + l\mathbf{e}_1}))')$, where $\mathcal{C}^\infty_{\mathrm{per}}$ is the space of $\mathcal{C}^\infty$ functions which are 1-periodic in y_1; the closure of $\mathcal{W}_i$ with respect to the usual norm of $[H^1(Y_i)]^2$ is denoted by $W(Y_i)$.

Let the function $\boldsymbol{\varphi}_\varepsilon$ be such that $\boldsymbol{\varphi}_\varepsilon = \boldsymbol{\varphi}_i(x/\varepsilon)$ on G_i^ε with $\boldsymbol{\varphi}_i \in W(Y_i)$ (for $i = 1, 2$). Moreover, we assume that

$$\int_{-h_1^-(0)}^{h_1^+(0)} (\boldsymbol{\varphi}_1)_1(0, y_2)\,\mathrm{d}y_2 = \int_{-h_2^-(0)}^{h_2^+(0)} (\boldsymbol{\varphi}_2)_1(0, y_2)\,\mathrm{d}y_2 := \kappa \tag{2}$$

where $(\cdot)_1$ stands the first component.

For $\mathbf{f} \in [H^{-1}(G^\varepsilon)]^2$ and $\boldsymbol{\varphi}_\varepsilon$ satisfying (2), problem (1) has a unique solution $(\mathbf{v}^\varepsilon, p^\varepsilon) \in H^1_{\mathrm{div}}(G^\varepsilon) \times L^2(G^\varepsilon)/\mathbb{R}$ (see [5]).

2.2. Asymptotic expansion

In the following, we suppose that $\mathbf{f}(x) = (f_1(x_1), 0)^t$ with f_1 a known function of $L^2(-1, 1)$. We look for an asymptotic solution in the form

$$\mathbf{v}_a^\varepsilon(x) = \varepsilon^2 \begin{cases} \mathbf{v}_1\left(\frac{x}{\varepsilon}\right) + \mathbf{v}_{bl-1}\left(\frac{x}{\varepsilon} + \frac{1}{\varepsilon}\mathbf{e}_1\right) + \mathbf{v}_{bl0}\left(\frac{x}{\varepsilon}\right) \quad \text{if } x \in G_1^\varepsilon \\ \mathbf{v}_{bl0}\left(\frac{x}{\varepsilon}\right) \quad \text{if } x \in G_0^\varepsilon \\ \mathbf{v}_2\left(\frac{x}{\varepsilon}\right) + \mathbf{v}_{bl0}\left(\frac{x}{\varepsilon}\right) + \mathbf{v}_{bl1}\left(\frac{x}{\varepsilon} - \frac{1}{\varepsilon}\mathbf{e}_1\right) \quad \text{if } x \in G_2^\varepsilon \end{cases} \tag{3}$$

For $i = 1, 2$, $\mathbf{v}_i(y) = \mathbf{w}_i^\alpha(y)$, the unique solution to the following problem: find $(\mathbf{w}_i^\alpha, \pi_i^\alpha) \in W(Y_i) \times L^2(Y_i)/\mathbb{R}$ such that $-\nu\Delta\mathbf{w}_i^\alpha + \nabla\pi_i^\alpha = \alpha\mathbf{e}_1$ in $(\bigcup_{l\in\mathbb{Z}}\overline{(Y_i + l\mathbf{e}_1)})'$, where α is such that $\int_{-h_i^-(0)}^{h_i^+(0)}(\mathbf{w}_i^\alpha)_1(0, y_2)\,\mathrm{d}y_2 = \kappa$.

The boundary layers terms $\mathbf{v}_{bl-1}$ and $\mathbf{v}_{bl1}$ are the exponentially decaying solutions of Stokes systems in semi-infinite channel $\Omega_1 := (\bigcup_{l=1}^\infty(\overline{Y_1 + l\mathbf{e}_1}))'$ and $\Omega_2 := (\bigcup_{l=1}^\infty(\overline{Y_2 - l\mathbf{e}_1}))'$ respectively, with boundary condition $\mathbf{v}_{bl-1} + \mathbf{v}_1 = \boldsymbol{\varphi}_1$ on $\overline{\Omega}_1 \cap \{y_1 = 0\}$ and $\mathbf{v}_{bl1} + \mathbf{v}_2 = \boldsymbol{\varphi}_2$ on $\overline{\Omega}_2 \cap \{y_1 = 0\}$ (see [6]).

Denote by $\Omega = ([\bigcup_{l=1}^\infty(\overline{Y_1 - l\mathbf{e}_1})] \cup \overline{Y_0} \cup [\bigcup_{l=1}^\infty(\overline{Y_2 + l\mathbf{e}_1})])'$. The boundary layer $\mathbf{v}_{bl0} \in H^1_{\mathrm{div}}(\Omega)$ is an exponentially decaying function which satisfies the Stokes equations with a right-hand side equal to $f_1\mathbf{e}_1$ on Y_0.

Function $\mathbf{v}_a^\varepsilon$ belongs to $[H^1_{\mathrm{div}}(G^\varepsilon)]^2$ and satisfies the boundary condition $\mathbf{v}_a^\varepsilon = 0$ on $\Gamma_\varepsilon^\pm$ and $\mathbf{v}_a^\varepsilon(\pm 1, x_2) = \varepsilon^2[\boldsymbol{\varphi}_\varepsilon + \mathbf{v}_{bl0}(\pm 1/\varepsilon, x_2/\varepsilon)]$ on $\Sigma_{\pm 1}^\varepsilon$. However, it can be modified by adding a function $\boldsymbol{\mathcal{D}}^\varepsilon \in [H^1_{\mathrm{div}}(G^\varepsilon)]^2$ such that $\boldsymbol{\mathcal{D}}^\varepsilon + \mathbf{v}_a^\varepsilon = \varepsilon^2\boldsymbol{\varphi}_\varepsilon$ on ∂G^ε, $\boldsymbol{\mathcal{D}}^\varepsilon = 0$ if $-1 + \varepsilon < x_1 < 1 - \varepsilon$ and $\|\boldsymbol{\mathcal{D}}^\varepsilon\|_{H^1(G^\varepsilon)} \leqslant \varepsilon^2 c \exp(-\lambda/\varepsilon)$, where c and λ are positive constants which do not depend on ε.

Denote $\mathbf{V}_a^\varepsilon = \mathbf{v}_a^\varepsilon + \boldsymbol{\mathcal{D}}^\varepsilon$. Function $\mathbf{V}_a^\varepsilon$ is a solution to the following problem:

$$\begin{cases} \mathbf{V}_a^\varepsilon \in [H^1_{\mathrm{div}}(G^\varepsilon)]^2 \text{ such that } \mathbf{V}_a^\varepsilon - \varepsilon^2\boldsymbol{\varphi}_\varepsilon \in V(G^\varepsilon), \text{ and} \\ \nu\int_{G^\varepsilon}\nabla\mathbf{V}_a^\varepsilon \cdot \nabla\mathbf{w} = \int_{G^\varepsilon}\mathbf{f}\cdot\mathbf{w} + \nu\int_{G^\varepsilon}\nabla\boldsymbol{\mathcal{D}}^\varepsilon \cdot \nabla\mathbf{w} \quad (\forall)\mathbf{w} \in V(G^\varepsilon) \end{cases} \tag{4}$$

The following estimate holds

$$\left\|\mathbf{v}^\varepsilon - \mathbf{V}_a^\varepsilon\right\|_{[H^1(G^\varepsilon)]^2} \leqslant c\varepsilon^2 \exp(-\lambda/\varepsilon) \tag{5}$$

where c and λ are two positive constants independent of ε.

2.3. Method of asymptotic partial decomposition of domain

Introduce another parameter $\delta = K\varepsilon[|\ln\varepsilon|]$, with some finite $K \in \mathbb{N}^*$.

Let $V_{\mathrm{dec}}^{\delta,\varepsilon}(G^\varepsilon)$ be the space

$$V_{\mathrm{dec}}^{\delta,\varepsilon}\big(G^\varepsilon\big) := \bigg\{\mathbf{u} \in V\big(G^\varepsilon\big)\colon\ \mathbf{u}(x) = \varepsilon^2\left[\mathbf{v}_1\left(\frac{x}{\varepsilon}\right) - \boldsymbol{\varphi}_\varepsilon(x)\right] \text{ if } -1+\delta < x_1 < -\delta,$$
$$\mathbf{u}(x) = \varepsilon^2\left[\mathbf{v}_2\left(\frac{x}{\varepsilon}\right) - \boldsymbol{\varphi}_\varepsilon(x)\right] \text{ if } \varepsilon+\delta < x_1 < 1-\delta\bigg\} \tag{6}$$

We define the partially decomposed problem as a variational problem (4) stated on the restricted space $V_{\mathrm{dec}}^{\delta,\varepsilon}(G^\varepsilon)$

$$\begin{cases} \text{Find } \mathbf{v}_d^\varepsilon \in H^1_{\mathrm{div}}\big(G^\varepsilon\big) \text{ such that } \mathbf{v}_d^\varepsilon - \varepsilon^2\boldsymbol{\varphi}_\varepsilon \in V_{\mathrm{dec}}^{\delta,\varepsilon}(G^\varepsilon) \text{ and} \\ \nu\int_{G^\varepsilon} \nabla\mathbf{v}_d^\varepsilon\cdot\nabla\mathbf{w} = \int_{G^\varepsilon} f_1\mathbf{e}_1\cdot\mathbf{w} \quad (\forall)\mathbf{w}\in V_{\mathrm{dec}}^{\delta,\varepsilon}(G^\varepsilon) \end{cases} \tag{7}$$

As in [4], we prove that for all $J \in \mathbb{N}^\star$ there exist $K \in \mathbb{N}^\star$ and $c \in \mathbb{R}_+$ such that

$$\left\|\mathbf{v}^\varepsilon - \mathbf{v}_d^\varepsilon\right\|_{[H^1(G^\varepsilon)]^2} \leqslant c\varepsilon^J \tag{8}$$

where c is a constant which does not depend on ε.

This estimate shows that the partially decomposed solution $\mathbf{v}_d^\varepsilon$ is exponentially close to the solution $\mathbf{v}^\varepsilon$ of the initial problem (1).

The same result holds true in the three-dimensional case, when Y_i are the domains $\{(y_1, y_2, y_3) \in \mathbb{R}^3\colon\ 0 < y_1 < 1,\ -h^-(\varphi, y_1) < r < h^+(\varphi, y_1)\}$, where r and φ are the polar coordinates in y_2y_3-plane and $h^\pm$ are 2π-periodic in φ and positive functions of $\mathcal{C}^2(\mathbb{R}\times[0,1])$. The below numerical experiment is developed for an extrusion process in such three-dimensional wavy tube. Another but similar approach was used in [7] to reduce the linear Stokes problem in an infinite domain with some outlets to the Stokes problem set in a truncated bounded domain.

Estimate (8) is generalized to the case of a wavy tube structure, analogous to the tube structure of [3] but with G_1^ε type domains instead of the thin cylinders of [3].

3. Application to the polymer extrusion

The polymer extrusion is an important part of industrial polymer processing. It is mainly used in polymer forming, blending and mixing. To perform this operation, two main engineering solutions exist: single and twin screw extruder. We will consider a model problem with the geometry of the domain that is a part of the real twin screw geometry (Fig. 1), where the main advantages are: the possibility to process polymers with no enough adhesion at the barrel wall; improved mixing and residence time distribution; development of new material based on the reactive extrusion. The twin screw extruder contains two intermeshing (or not intermeshing) co-rotating (or counter-rotating) screws; this geometry is time dependent and therefore the numerical analysis is very difficult.

To simplify the problem, we neglect the intermeshing area and therefore only the flow inside one screw is considered as a model problem. An additional advantage of this configuration is that we can change the frame of reference by considering the rotating cylindrical barrel (instead of the screw) with opposite angular velocity which makes the geometry stationary. In this new frame of reference, the centrifugal and Corriolis forces should be taken in account, but for high viscosity polymers these inertial forces are not important and generally could be neglected

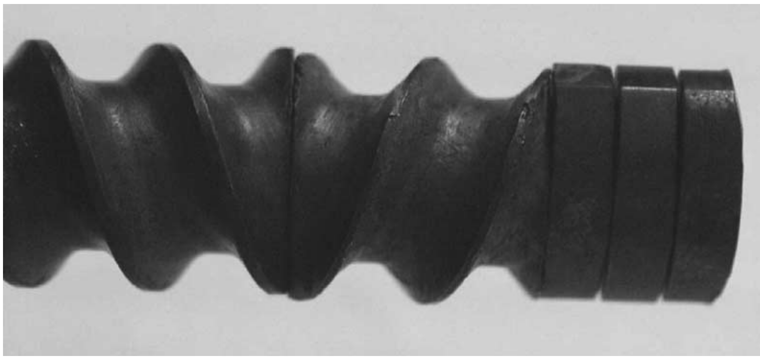

Fig. 1. Twin screw geometry.

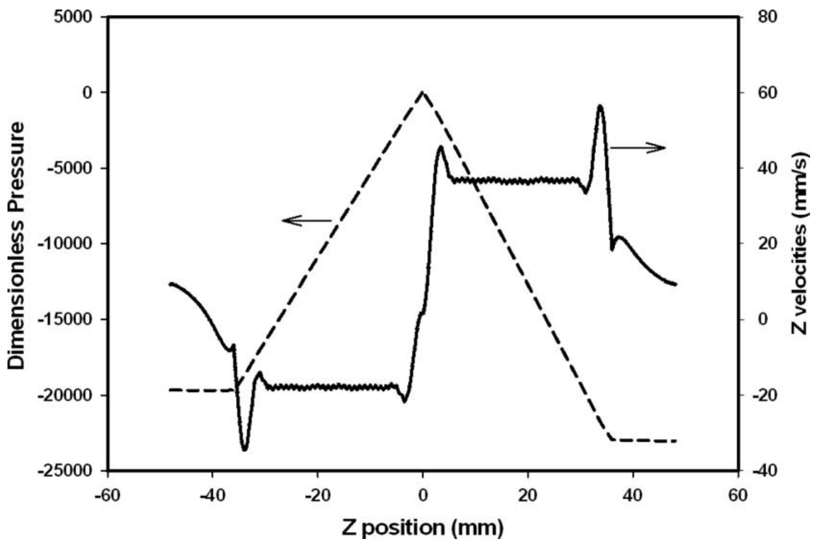

Fig. 2. Dimensionless pressure and velocity.

(as well as all the inertial effects). We should mention that the rheological behavior of polymers is described by complex differential or integral viscoelastic constitutive models but in this study we consider them as a Newtonian fluid with constant viscosity which is normalized to 1 in the calculations.

Nowadays, it is difficult to solve practical 3D problems in entire flow domain due to complexity of the geometry and excessive memory requirements. The above asymptotic domain decomposition technique proposes an alternative approach by splitting the flow domain and solving smaller problems joined by appropriate boundary conditions. For each of two domains, we suppose that the fully developed periodic flow is reached at the inflow and outflow sections. The velocity distribution of this periodic flow is calculated by solving a small 3D cell problem in order to impose velocity field at the inlet and outlet of each sub-domain. The dimensions of the problem are those of the twin screw extruder at the laboratory (Clextral BC21). The operating conditions are as follows: flow rate (700 mm^3/s) and screw angular velocity (31.5 rad/s).

The problem in each sub-domain is discretized in terms of classical Galerkin formulation. These approximations are relatively expensive for 3D problems but they provided improved accuracy for the pressure and the velocity fields and therefore they were implemented in our code. The resolution of the linearized system of equations is done

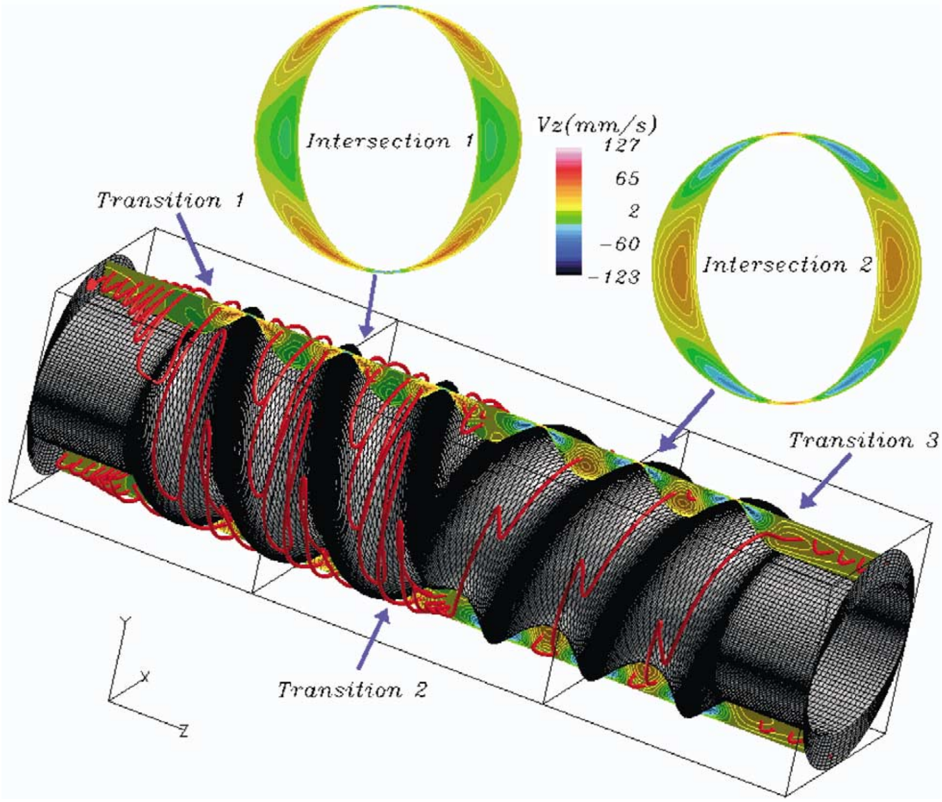

Fig. 3. Problem domains, finite element mesh, velocity distribution and typical streamline.

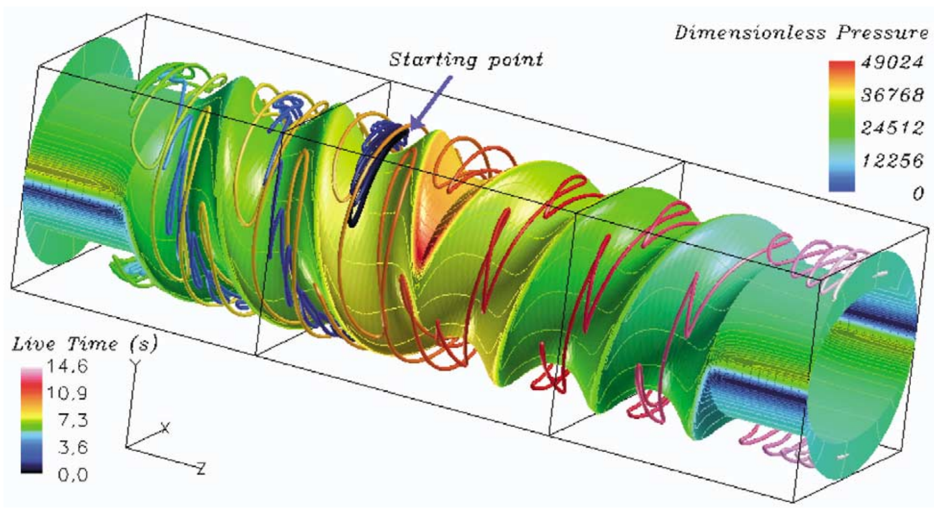

Fig. 4. Dimensionless pressure distribution and streamline from the back flow.

by using the iterative solver based on the bi-conjugate stabilized algorithm of van der Vorst [8] in combination of LU incomplete factorization based on the drop tolerance.

In Fig. 3, we present the geometry of the problem together with three sub-domains of decomposition, the finite element mesh, the extrusion velocity (V_z) distribution at two cross sections and one cutting plane and finally a typical streamline with lifetime of about 9 s. The calculated V_z profile proves the periodic character of the solution far from the three transition zones (Figs. 3 and 2). Due to the difference of the pressure drop in the right handed screw (positive) and in the left handed screw (negative), the pressure driven flow is negative in the channel of the right handed screw (Fig. 2 and Fig. 3 cross-section 1) and positive in the channel of the left handed screw (Fig. 2 and Fig. 3 cross-section 2). From our graphics, we can estimate the size of the transition zones which is for example about 6 mm from the center of the zone 2 (Fig. 3) and corresponds to 1/4 of screw pitch. The streamline shape is in agreement with the well known helical motion inside the channels of the screws. The dimensionless pressure distribution together with another (more exotic) streamline (44 s lifetime) are presented in Fig. 4. These computations have proved the existence of a back flow in the right handed screw which, according to our knowledge, has never been mentioned in the literature and is waiting to find its experimental confirmation.

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