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On the Lagrange problem about the optimal form for circular hollow columns

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Abstract

A new approach to the problem of finding the form of the strongest clamped circular column with thin walls of fixed volume and height is proposed. The same model describes also the form of the horizontal beam with rectangular vertical sections of a fixed height and variable widths. The existence and the uniqueness of the optimal column are proved here for the first time and the optimal form is found. ***To cite this article: Yu.V. Egorov, C. R. Mecanique 331 (2003).***

Résumé

Sur le problème de Lagrange de la forme optimale d'une colonne circulaire aux parois minces. On propose une nouvelle approche au problème de la forme de la colonne encastrée la plus solide aux parois minces aux volume et hauteur fixés. Le même modèle décrit aussi le problème de la forme optimale d'une poutre horisontale aux sections verticales rectangulaires dont l'hauteur est fixée et la largeur est variable. On montre l'existence et l'unicité d'une telle colonne par la première fois ici et on trouve la forme optimale. ***Pour citer cet article : Yu.V. Egorov, C. R. Mecanique 331 (2003).***

Keywords: Computational solid mechanics; Lagrange problem; Optimal column

Mots-clés : Mécanique des solides numérique ; Problème de Lagrange ; Colonne optimale

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On considére le problème d'optimisation de la forme d'une colonne encastrée la plus solide aux parois minces à volume V et hauteur h fixés. On suppose que les sections de la colonne sont les anneaux $R < r < R + \rho(x)$, centré à l'axe x avec $0 < x < h$, où R est fixé et $\rho(x) \leqslant \varepsilon R,\ \varepsilon \ll 1$. En négligeant les puissances de ρ superieures à 1,

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on voit que l'aire de la section est $2\pi R\rho(x)$ et le deuxième moment de l'aire est $I(x) = 2\pi R^3\rho(x)$. Selon Euler, la colonne a l'énergie potentielle

$$T = \int_0^h EI(x)z''(x)^2\,\mathrm{d}x - \lambda\int_0^h z'(x)^2\,\mathrm{d}x, \quad \int_0^h \rho(x)\,\mathrm{d}x = const$$

où E est le modulus de Young, λ est la charge axiale, $z(x)$ est la deflexion laterale.

La charge critique λ_{cr} est la valeur maximale de λ pour laquelle $T[z]$ est positive. On cherche la forme optimale de la colonne, i.e. la fonction $\rho(x)$ telle que la valeur λ_{cr} est maximale.

On a le même modèle pour le problème de la forme optimale d'une poutre horisontale aux sections rectangulaires dont l'hauteur est fixée et la largeur est variable. (Voir [1–3].)

Dans nos notes [4,5] le problème de Lagrange a été consideré pour les cas quand la condition $\int_0^h I(x)\,\mathrm{d}x = const$ est remplacée par la condition $\int_0^h I(x)^\alpha\,\mathrm{d}x = const$ avec $\alpha \in]0,1[$. Dans [4] nous avons consideré le cas $\alpha = 1/2$ correspondant à une colonne solide aux sections similaires (sans le théorème d'unicité), et dans [5] les autres valeurs de $\alpha \in]0,1[$. L'unicité a été demontrée dans [5]. Nous avons utilisé la condition necessaire d'optimalité sous la forme $I(x)^{\alpha-1} = z''(x)^2$ qui n'est pas applicable dans le cas $\alpha = 1$, considéré ici. Cette différence au fond nous oblige de chercher une autre méthode.

Nous montrons ici par la première fois l'existence et l'unicité de la colonne optimale et trouvons cette colonne.

1. Introduction

Consider the problem on the form of the strongest clamped circular column with thin walls of fixed volume and height. The same model describes also the form of the horizontal beam with rectangular vertical sections of a fixed height and variable widths (see [1–3], where this problem was discussed). This problem is essentially different from the similar problems considered in our previous notes [4,5]. We prove the existence and the uniqueness theorems and find this optimal form numerically.

Consider a vertical column which horizontal sections are rings centered at the points of the x-axis for $0 \leqslant x \leqslant h$ and bounded by two circles of radii R and $R(x) + \rho(x)$, where R is fixed and $\rho(x) \leqslant \varepsilon R,\ \varepsilon \ll 1$. Neglecting powers of ρ greater than 1, we find that the cross-sectional area is $2\pi R\rho(x)$ and its second moment is $I(x) = 2\pi R^3\rho(x)$. By Euler, the potential energy of the column is

$$T = \int_0^h EI(x)z''(x)^2\,\mathrm{d}x - \lambda\int_0^h z'(x)^2\,\mathrm{d}x$$

where E is Young's modulus, $z(x)$ is the lateral deflection of the column at a point x, and λ is the magnitude of the axial load. Since the volume of the column is fixed we have $\int_0^h \rho(x)\,\mathrm{d}x = const$. Here we neglect the weight of the column. Since $z(0) = 0,\ z'(0) = 0$, the potential energy is positive for small values of λ for all $z \neq 0$. The buckling load of the column λ_{cr}, is the supremum of the values λ such that $T \geqslant 0$ for any z. The problem considered is to find the form of the strongest column, i.e., the function $I(x)$, for which the value λ_{cr} is maximal and $\int_0^h I(x)\,\mathrm{d}x = V$.

After rescaling and the passage from z to $y = z'$ the mathematical problem takes the following form:

Problem L. To find a positive function $Q(x) \in C[0,1]$ such that

$$\int_0^1 Q(x)\,\mathrm{d}x = 1 \tag{1}$$

and for which the minimal value λ of the functional

$$L_1[Q,y] \equiv \frac{\int_0^1 Q(x)y'(x)^2\,\mathrm{d}x}{\int_0^1 y^2(x)\,\mathrm{d}x} \tag{2}$$

in the class of functions $y \in C^1(0,1)$, satisfying the conditions

$$y(0)=0, \quad y(1)=0, \quad \int_0^1 y(x)\,\mathrm{d}x = 0 \tag{3}$$

is maximal.

We propose here a new approach based on two-dimensional variation of the functional L_1, allowing us to prove the existence of the solution and to find the optimal form of the column. We are giving also an algorithm allowing us to find the shape of the optimal column.

Definition 1.1. $\mathcal{A}$ is the set of positive continuous functions Q satisfying (1).

S is the set of functions y of the class $C^1(0,1)$ satisfying (3).

S_0 is the set of the pairs of functions u, v of the class S such that $u'(0) > 0$, $v'(0) > 0$, $u'(1) > 0$, $v'(1) < 0$, the function $\theta(x) = \arctan(v'(x)/u'(x))$ is monotone decreasing with $\theta(0) - \theta(1) \leqslant 3\pi$ and the function $r(x) = u'(x)^2 + v'(x)^2$ is equal to 1 identically in]0, 1[.

We consider a related auxiliary problem: to find the minimal value of the functional

$$F[u,v] \equiv \int_0^1 \bigl(u(x)^2 + v(x)^2\bigr)\,\mathrm{d}x; \quad u, v \in S$$

It is easy to see that the minimal value of this functional is 0 and this problem is badly posed and cannot help us, at least if we do not put some supplementary restrictions on u and v.

The principal idea of our method is to find the minimal value of the functional $F[u,v]$ in S_0 and to show that the optimal functions u, v define the optimal shape of the most solid column. The functional F has actually an infinite set of the points of local minimum (u_k, v_k) in the space $S \times S$ and $F[u_k, v_k] \to 0$ as $k \to \infty$. The most interesting for us here is the first point (u_1, v_1) such that $F[u_1, v_1] = \max_k F[u_k, v_k]$. However, the other points are also interesting for the Lagrange problem. These points are characterized by the number of rotation of the vector $(u_k'(x), v_k'(x))$ when x is going from 0 to 1.

Our main result is:

Theorem 1.2. *There exists a unique solution of the Problem* L. *The optimal function Q_0 can be found from the relation*

$$Q_0(x) = Mr(x), \quad r(x) = \bigl(b - w(x)\bigr)^2 + \bigl(z(x) + a(2x-1)^2\bigr) \tag{4}$$

where M, a, b are constants and w, z are the solutions to the equations

$$w'' = \frac{b - w(x)}{r(x)}; \quad w(0) = w'(0) = 0; \; w(x) = w(1-x) \tag{5}$$

$$z''(x) = \frac{a(1-2x) - z(x)}{r(x)}; \quad z(0) = z'(0) = 0; \; z(x) = -z(1-x) \tag{6}$$

The function Q_0 is symmetric, $Q_0(x) = Q_0(1-x)$, and r can be found also as the solution to the problem

$$r(x)^2 r'(x)^2 = cr(x)^2 - 2r(x)^3 - 4a^2b^2, \quad r(0) = \sqrt{a^2+b^2} \tag{7}$$

which is not constant on any subinterval, and

$$c = 4a^2 + 2\sqrt{a^2 + b^2}$$

If $P(r) = (c - 2r)r^2 - a^2b^2$, *and* $r_1,\ r_2$ *are the real roots of* P, *then* $0 < r_1 < r_0 = \sqrt{a^2 + b^2} < r_2 < c$ *and*

$$2\int_{r_1}^{r_0} \frac{r\,\mathrm{d}r}{\sqrt{P(r)}} + \int_{4m}^{r_2} \frac{r\,\mathrm{d}r}{\sqrt{P(r)}} = \frac{1}{2} \tag{8}$$

$$2\int_{r_1}^{r_0} \frac{\mathrm{d}r}{r\sqrt{P(r)}} + \int_{r_0}^{r_2} \frac{\mathrm{d}r}{r\sqrt{P(r)}} = \frac{2\pi}{ab} \tag{9}$$

The latter system of two equations for the unknown a, b can be solved numerically. The calculation shows that $m = 0.019100,\ 0.0519 \leqslant Q(x) \leqslant 1.7824,\ k = 0.27321$. The critical load M can be found from condition (1).

The optimal column with circular sections is formed by rotation of the curve $y = 1 + R(x)$, where $R(x) = Q_0(x)^{1/2}/\sqrt{\pi}$.

We have $R(0) = R(1) = 0.65140$, the minimal value of R is 0.2694 and is attained at $x = 0.2449$ and $x = 0.7551$, the maximal value of R at $x = 1/2$ is 0.6519.

Our proof uses the following:

Lemma 1.3. *There exists a pair* $u_0, v_0 \in S_0$ *giving the minimal value* $m \neq 0$ *to the functional*

$$F[u, v] = \int_0^1 \left[u(x)^2 + v(x)^2\right] \mathrm{d}x$$

in S_0.

Proof of Theorem 1.1 (*Existence*). Set

$$L[Q, u, v] = \frac{\int_0^1 Q(x)(u'(x)^2 + v'(x)^2)\,\mathrm{d}x}{\int_0^1 (u(x)^2 + v(x)^2)\,\mathrm{d}x}$$

Let $u_0 = u, v_0 = v$ be the functions found in Lemma 1.3 such that $u_0'(x)^2 + v_0'(x)^2 = 1$. Put $Q_0(x) = M[C_1 - u_0(x)^2 - (v_0(x) + 2a)^2]$. We can verify that $Q_0 \in C^\infty[0, 1]$ and that $Q_0(x) = Q_0(1 - x)$.

We have

$$L[Q_0, u_0, v_0] = M = \frac{1}{m}$$

On the other hand, if Q is a positive function from $C^1(I)$, and $\int_0^1 Q(x)\,\mathrm{d}x = 1$, then,

$$\inf_{u \in S,\ v \in S} L[Q, u, v] \leqslant L[Q, u_0, v_0] = \frac{1}{F[u_0, v_0]} = \frac{1}{m} \equiv M$$

Thus,

$$\sup_Q \inf_{u \in S,\ v \in S} L[Q, u, v] \leqslant M$$

and

$$\sup_Q \inf_{y \in S} L_1[Q, y] \leqslant M$$

Now we will show that

$$\inf_{y\in S} L_1[Q_0,y]=\inf_{u\in S,\ v\in S} L[Q_0,u,v]=\inf_{u\in S_1,\ v\in S_2} L[Q_0,u,v]=M$$

Let

$$\inf_{u\in S_{\mathrm{odd}}} L[Q_0,u,0]=\mu$$

This minimal value is attained on a function u_1 which is positive in $]0,1/2[$. It satisfies the Lagrange equation

$$\big(Q_0u_1'(x)\big)'+\mu u_1(x)=0$$

and $u_1(0)=0, u_1(1/2)=0$.

The function u_0 is odd, positive in $]0,1/2[$ and $(Q_0(x)u_0'(x))'+Mu_0(x)=0$. Therefore,

$$(M-\mu)\int_0^{1/2} u_1(x)u_0(x)\,\mathrm{d}x=0$$

Both the functions u_0 and u_1 are positive in $]0,1/2[$. Therefore, $M=\mu$ and $u_1=ku_0$.

Let now

$$\inf_{v\in S_{\mathrm{even}}} L[Q_0,0,v]=\mu_1$$

This minimal value is attained on a function v_1 which is positive in $]0,p[$ and negative in $]p,1/2[$. It satisfies the Lagrange equation

$$\big(Q_0v_1'(x)\big)'+\mu_1v_1(x)=C'$$

and $v_1(0)=0, v_1'(1/2)=0$. Here $C'=-2Q_0(0)v_1'(0)<0$.

The function v_0 is positive in $]0,p_0[$, negative in $]p_0,1/2[$ and $(Q_0(x)v_0'(x))'+Mv_0(x)=C$ with $C=-2Q_0(0)v_0'(0)<0$. Therefore, the functions $w_0=v_0-C/M$ and $w_1=v_1-C'/\mu_1$ satisfy the equations

$$(Q_0w_0')'+Mw_0=0,\qquad (Q_0w_1')'+\mu w_1=0$$

Moreover,

$$w_0(0)=-\frac{C}{M},\quad M\int_0^{1/2} w_0(x)\,\mathrm{d}x=Q_0(0)w_0'(0),\quad \int_0^{1/2} w_0(x)\,\mathrm{d}x=-\frac{C}{2M}$$

and

$$w_1(0)=-\frac{C'}{\mu_1},\quad \mu_1\int_0^{1/2} w_1(x)\,\mathrm{d}x=Q_0(0)w_1'(0),\quad \int_0^{1/2} w_1(x)\,\mathrm{d}x=-\frac{C'}{2\mu_1}$$

Since both w_0 and w_1 have exactly one zero in $[0,1/2]$, we can verify that they are linearly dependent, i.e., $w_1(x)=k'w_0(x)$. Therefore, $v_1(x)=k'v_0(x)+C''$. But $v_0(0)=v_1(0)=0$, so that $C''=0$ and $v_1=k'v_0,\ \mu_1=M$.

If $y\in S$ then $y=u+v$, where $u\in S_{\mathrm{odd}}, v\in S_{\mathrm{even}}$. Therefore,

$$L_1[Q_0,y]=\frac{\int_0^{1/2} Q_0(x)[u'(x)^2+v'(x)^2]\,\mathrm{d}x}{\int_0^{1/2}[u(x)^2+v(x)^2]\,\mathrm{d}x}\geqslant M$$

because

$$\int_0^{1/2} Q_0(x)u'(x)^2\,\mathrm{d}x\geqslant M\int_0^{1/2} u(x)^2\,\mathrm{d}x,\qquad \int_0^{1/2} Q_0(x)v'(x)^2\,\mathrm{d}x\geqslant M\int_0^{1/2} v(x)^2\,\mathrm{d}x$$

as we have shown. Summing, we obtain

$$\inf_{y\in S} L_1[Q_0, y] = \inf_{u\in S_1,\ v\in S_2} L[Q_0, u, v] = L_1[Q_0, y_0] = M$$

and $y_0 = u_0 + v_0$.

2. Uniqueness

Lemma 2.1. *Let Q_0 be the solution of the Lagrange problem. Then there are two functions $u_0 \in S_1,\ v_0 \in S_2$ satisfying Eqs.* $(7),\ (8)$ *such that*

$$Q_0(x) = M\bigl[C_1 - u_0(x)^2 - \bigl(v_0(x) + 2a\bigr)^2\bigr], \quad M = 1/m$$

Suppose that the function $\widetilde{Q}$ is such that

$$\inf_{u,v\in S} L_1[\widetilde{Q}, u, v] = M$$

and that minimal value is attained when $u = \tilde{u},\ v = \tilde{v}$. These functions $\tilde{u}, \tilde{v}$ satisfy Eqs. (5), (6).

Let $u_0,\ v_0$ be the functions found in Lemma 1.3. These functions $u_0,\ v_0$ also satisfy Eqs. (5), (6). Remark that $u_0(0) = 0,\ v_0(0) = 0,\ u_0'(0)^2 + v_0'(0)^2 = 1$. We can choose now the orthogonal transformation

$$\hat{u}_0 = \alpha_1\tilde{u} + \alpha_2\tilde{v}, \qquad \hat{v}_0 = \alpha_3\tilde{u} + \alpha_4\tilde{v}$$

in such a way that the new solution $(\hat{u}, \hat{v})$ satisfies $\hat{u}(0) = 0,\ \hat{v}(0) = 0,\ \hat{u}'(0) = u_0'(0),\ \hat{v}'(0) = v_0'(0)$, i.e., coincides with the solution found in Lemma 2.1. as the solution of the Cauchy problem with the same initial data. So we obtain that

$$\widetilde{Q}(x) = Q_0(x) = M\bigl[C_1 - u_0(x)^2 - \bigl(v_0(x) + 2a\bigr)^2\bigr]$$

Therefore, any solution of the Lagrange problem coincides with the solution found above.

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