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Self adjoint extensions of differential operators in application to shape optimization

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Abstract

Two approaches are proposed for the modelling of problems with small geometrical defects. The first approach is based on the theory of self adjoint extensions of differential operators. In the second approach function spaces with separated asymptotics and point asymptotic conditions are introduced, and the variational formulation is established. For both approaches the accuracy estimates are derived. Finally, the spectral problems are considered and the error estimates for eigenvalues are given. ***To cite this article: S.A. Nazarov, J. Sokolowski, C. R. Mecanique 331 (2003).***

Résumé

Extensions autoadjointes d'opérateurs différentiels et application à l'optimisation de forme. On propose deux approches permettant de modéliser des problèmes avec des singularités géométriques. La première approche repose sur des extensions autoadjointes d'opérateurs différentiels avec conditions asymptotiques. Dans la seconde approche, on introduit des espaces fonctionnels avec développements asymptotiques séparés puis on établit la formulation variationnelle. On obtient des estimations montrant que la même précision est atteinte pour ces deux approches. Enfin, on considère des problèmes spectraux et on donne des estimations pour les valeurs propres. ***Pour citer cet article : S.A. Nazarov, J. Sokolowski, C. R. Mecanique 331 (2003).***

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1. Introduction

It seems that in the literature on shape optimisation there is a lack of a general numerical method or technique that can be applied in the process of optimisation of an arbitrary shape functional (SF) for simultaneous boundary and topology variations. In the paper [12] (see also [13]) the so-called topological derivative (TD) of an arbitrary SF is introduced. TD usually determines whether a change of topology by nucleation of a small hole, or in similar setting of a small inclusion at a given point $\boldsymbol{x} \in \Omega$, would result in improving the value $J(\Omega)$ of a given SF or not. In the note the *internal topology variations* are considered for mixed boundary value problem for Poisson equation. The singular perturbations of the geometrical domain Ω are defined by small openings $\omega_\varepsilon^1, \dots, \omega_\varepsilon^J$ of the diameters $\mathrm{O}(\varepsilon)$. We have selected such a model problem taking into account the particular features of solutions with respect to small parameter ε. First of all, at $\varepsilon = 0$ the openings disappear, and no Dirichlet boundary condition remains, so the limit Neumann problem looses the uniqueness of solutions. This allows for unbounded growth, with $\varepsilon \to 0+$, of the solution $u(\varepsilon, x)$ of the singularly perturbed problem. Beside that, the dependence of the solution $u(\varepsilon, x)$ on the geometrical parameter is quite complicated: even far from the boundaries $\partial\omega_\varepsilon^j$, where the influence of boundary layers can be neglected, the solution is approximated, with the precision $\mathrm{O}(\varepsilon^{1-\delta}),\ \delta > 0$, by a rational function of large parameter $\ln \varepsilon$. Finally, the leading terms of asymptotics, which are easily determined, do contain no information either on the openings ω_ε^j or on the interaction of the openings. All those properties make difficult the proper definition of topological derivatives and the appropriate derivation of necessary optimality conditions for minimization of specific shape functionals. Therefore, we restrict ourselves to the energy shape functional and to the modelling of topological variations.

We propose two efficient approaches to the modelling of topological variations. The first approach is developped in the framework of the selfadjoint extensions of differential operators, the second uses the function spaces with the separated asymptotics. In both cases, the main idea consists in modelling of small defects or inhomogenuities by concentrated *actions*, the so-called potentials of zero-radii. In this way, the solution $u(\varepsilon, x)$ with *singular* behaviour for $\varepsilon \to 0+$ is replaced by a function with the singularities at the centres $P^1, \dots, P^J$ of the defects. Such an approach is known in modelling of phisical processes in materials with defects, we refer the reader, e.g., to [7]. The modern framework of analysis of elliptic boundary value problems in non smooth domains [8] allows for the relatively complete theory of singular solutions and provides the techniques of derivation of error estimates in weighted spaces for asymptotic approximations. We can use the known results in this field for the solution of shape and topology optimization problems in an *inverse order*. First, the localization and integral attributes of openings are determined, followed by the appropriate changes of the topology of geometrical domains. The proposed two different approaches to topology optimization have some positive features. The first approach deals with selfadjoint operators, so can be extended to the evolution boundary value problems. The second approach, based on the *generalized Green's formulae*, results in the variational problem formulation with the solution given by a stationary point of an auxiliary functional close in its form to the energy functional.

2. Problem formulation

Let Ω and $\omega^1, \dots, \omega^I$ be bounded domains in $\mathbb{R}^2$ with the smooth boundaries $\partial\Omega$ and $\partial\omega^1, \dots, \partial\omega^I$. For fixed points $P^1, \dots, P^I$ inside Ω the following sets are introduced

$$\omega_\varepsilon^i = \left\{x \in \mathbb{R}^2\colon\ \xi^i := \varepsilon^{-1} x^i \in \omega^i\right\}, \quad x^i = x - P^i,\ i = 1, \dots, I; \qquad \Omega(\varepsilon) = \Omega \setminus \left\{\overline{\omega_\varepsilon^1} \cup \dots \cup \overline{\omega_\varepsilon^I}\right\} \tag{1}$$

Choose a small $\varepsilon \in (0, \varepsilon_0]$ such that $\overline{\omega_\varepsilon^i} \subset \Omega,\ i = 1, \dots, I$, and $\overline{\omega_\varepsilon^i} \cap \overline{\omega_\varepsilon^j} = \emptyset$ for $i \neq j$. The sets ω_ε^i imply holes, or openings, in the geometrical domain $\Omega(\varepsilon)$. Let us consider the functional

$$\mathcal{J}(u; \varepsilon) = \int\limits_{\Omega(\varepsilon)} J\big(x; u(\varepsilon, x)\big)\, \mathrm{d}x \tag{2}$$

defined for functions $u \in L_q(\Omega(\varepsilon))$ for a given $q \in [1, +\infty)$. Furthermore, we assume that for all $u, v \in L_q(\Omega(\varepsilon))$ and $\varepsilon \in [0, \varepsilon_0]$ the following inequality holds

$$\bigl|\mathcal{J}(u;\varepsilon) - \mathcal{J}(v;\varepsilon)\bigr| \leqslant c\bigl\|u - v; L_q\bigl(\Omega(\varepsilon)\bigr)\bigr\|\bigl(\bigl\|u; L_q\bigl(\Omega(\varepsilon)\bigr)\bigr\|^{q-1} + \bigl\|v; L_q\bigl(\Omega(\varepsilon)\bigr)\bigr\|^{q-1}\bigr) \tag{3}$$

where the constant c depends on Ω, but it is independent of $\varepsilon \in [0, \varepsilon_0]$ and of u, v while $\Omega(0) = \Omega$. Functional (2) has twofold dependence on the small parameter ε, first of all, by the domain of integration $\Omega(\varepsilon)$, and also by means of the integrand which depends on the solution to the mixed boundary value problem

$$-\Delta u(\varepsilon, x) = f(x), \quad x \in \Omega(\varepsilon) \tag{4}$$

$$\partial_n u(\varepsilon, x) = 0, \quad x \in \partial\Omega, \quad u(\varepsilon, x) = 0, \quad x \in \partial\omega_\varepsilon^i,\ i = 1, \ldots, I \tag{5}$$

Here $f \in L_2(\Omega)$ is a given function, independent of the parameter ε, and ∂_n stands for the normal derivative in the direction of the outward normal vector n. Problem (4), (5) admits the unique solution $u(\varepsilon, \cdot) \in H^2(\Omega(\varepsilon))$ for any $\varepsilon \in (0, \varepsilon_0]$ such that $\|u(\varepsilon, \cdot); H^2(\Omega(\varepsilon))\| \leqslant c(\varepsilon)\|f; L_2(\Omega(\varepsilon))\|$, where the constant $c(\varepsilon)$ depends on the domain $\Omega(\varepsilon)$, i.e., the parameter ε, but it is not dependent on the right-hand side f. In general the solution $u(\varepsilon, \cdot)$ to problem (4), (5) admits no limit as $\varepsilon \to 0+$. For $\varepsilon \to 0+$ the functional (2) may also have unbounded growth, since it behaves like the integral

$$\int_\Omega J\left(x, \frac{1}{2\pi}|\ln\varepsilon| \int_\Omega f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)\mathrm{d}x \tag{6}$$

Using the methods developed in [1–3] asymptotic expansions can be constructed for solutions $u(\varepsilon, x)$ with the prescribed precision $\mathrm{O}(\varepsilon^N)$. However, such expansions are relatively complex and, therefore, of small practical interest for the analysis of functional (2). There are some particular features of the problem under consideration. Beside the presence of boundary layers near the openings, the coefficients of asymptotic expansions are rational functions of the large parameter $|\ln\varepsilon|$. The latter property is established for the first time in [4]. Instead of constructing asymptotics there are presented two approaches for the modelling of problem (4), (5) and, as a result, to the modelling of functional (2).

3. Self adjoint extensions

The approach renewed in this section was initiated in [6] (see also the bibliography in review [7]). Let us introduce the *unbounded* operator $\mathcal{A}_0$ in $L_2(\Omega)$ with the differential expression $-\Delta$ and with the domain which includes smooth functions, vanishing near the points $P^1, \ldots, P^I$,

$$\mathcal{D}(\mathcal{A}_0) = \bigl\{v \in C_0^\infty\bigl(\overline{\Omega} \setminus \bigl\{P^1 \cup \cdots \cup P^I\bigr\}\bigr)\colon\ \partial_n v = 0 \text{ on } \partial\Omega\bigr\} \tag{7}$$

We emphasize that the inclusion $v \in \mathcal{D}(\mathcal{A}_0)$ provides that the function v is smooth in $\overline{\Omega}$, satisfies the Neumann boundary conditions on $\partial\Omega$ and equals zero in vicinity of P^i, the latter condition imitates the Dirichlet boundary condition (5). The adjoint $\mathcal{A}_0^*$ for the operator $\mathcal{A}_0$ are given by the differential expression $-\Delta$, with the domains of definition:

$$\mathcal{D}\bigl(\mathcal{A}_0^*\bigr) = \left\{v\colon\ v(x) = \sum_{i=1}^{I} \chi_i(x)\left\{-\frac{1}{2\pi}\mathbf{a}_i \ln|x^i| + \mathbf{b}_i\right\} + v_0(x),\right.$$
$$\left. v_0 \in \mathcal{D}(\overline{\mathcal{A}_0});\ \mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_I)^\top,\ \mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_I)^\top \in \mathbb{R}^I\right\} \tag{8}$$

where $\chi_j \in C_0^\infty(\Omega)$ are cut-off functions such that $\chi_j(x) = 1$ near P^j and $\chi_j\chi_k = 0$ for $j \neq k$.

Proposition 3.1. *Let* $\mathbf{S} = \mathbf{S}(\varepsilon) := (2\pi)^{-1}(\mathbb{L} - |\ln\varepsilon|\mathbb{I})$ *be a diagonal* $(I \times I)$*-matrix, negative definite for* $\varepsilon \in (0, \varepsilon_0]$. *Here* $\mathbb{I}$ *is the unit matrix,* $\mathbb{L} = \mathrm{diag}\{l_1, \dots, L_I\}$ *the matrix with the entries* L_j *equal to logarithmic capacities of the sets* ω^j *(see, e.g.,* [5]*).*

(i) *The restriction* $\mathbf{A}$ *of the operator* $\mathcal{A}_0^*$ *on the linear space*

$$\mathcal{D}(\mathbf{A}) = \{v \in \mathcal{D}(\mathcal{A}_0^*)\colon\ \mathbf{b} = \mathbf{Sa}\} \tag{9}$$

is a self adjoint extension of the operator $\mathcal{A}_0$.

(ii) *For any* $f \in L_2(\Omega)$, *there exists a unique solution* $\mathbf{v} \in \mathcal{D}(\mathbf{A})$ *of the equation*

$$\mathbf{Av} = f \in L_2(\Omega) \tag{10}$$

Theorem 3.2. *If* u *and* $\mathbf{v}$ *are solutions to the problems* (4), (5) *and* (10), *respectively, with the same right-hand side* $f \in L_2(\Omega)$, *then*

$$\|u - \mathbf{v}; L_q(\Omega(\varepsilon))\| \leqslant c_\varkappa \varepsilon |\ln\varepsilon|^{\varkappa+5/2} \|f; L_2(\Omega)\| \tag{11}$$

For the functional (2) *we have the relation*

$$\left|\mathcal{J}(u;\varepsilon) - \int_\Omega J(x; \mathbf{v}(\ln\varepsilon, x))\,\mathrm{d}x\right| \leqslant C_\varkappa \mu_q(\varepsilon) \|f; L_2(\Omega)\|^q \tag{12}$$

where $\varkappa$ *is arbitrary positive but the constants* $c_\varkappa,\ C_\varkappa$ *are independent of* f *and* $\varepsilon \in (0, \varepsilon_0]$, *and*

$$\mu_q(\varepsilon) = \varepsilon|\ln\varepsilon|^{q(\varkappa+5/2)} \quad \text{for } q \in [1,2] \text{ and } \mu_q(\varepsilon) = \varepsilon^{2/q} \text{ for } q > 2 \tag{13}$$

Remark 1. The energy functional

$$\mathbf{E}(\mathbf{v}; f) = \frac{1}{2}(\mathbf{Av}, \mathbf{v})_\Omega - (f, \mathbf{v})_\Omega \tag{14}$$

associated to the self adjoint operator $\mathbf{A}$ given in Proposition 3.1 is an approximation of the energy functional for problem (4), (5)

$$\mathcal{E}_\varepsilon(u; f) = \frac{1}{2}(\nabla u, \nabla u)_{\Omega(\varepsilon)} - (f, u)_{\Omega(\varepsilon)} = -\frac{1}{2}(f, u)_{\Omega(\varepsilon)} \tag{15}$$

Due to the latter representation, functionals (14) and (15) are in relation (12) with $q = 1$.

4. Function spaces with separated asymptotics

We introduce the Hilbert function space

$$\mathfrak{D} = \left\{\mathfrak{v}\colon\ \mathfrak{v}(x) = \sum_{i=1}^{I} \chi_i(x)\left\{-\frac{\mathfrak{a}_i}{2\pi}\ln(x^i) + \mathfrak{b}_i\right\} + \mathfrak{v}_0(x),\right.$$
$$\left.\mathfrak{v}_0 \in H^2(\Omega),\ \mathfrak{v}_0(P^1) = \cdots = \mathfrak{v}_0(P^I) = 0,\ \mathfrak{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_I)^\top,\ \mathfrak{b} = (\mathfrak{b}_1, \dots, \mathfrak{b}_I)^\top \in \mathbb{R}^I\right\} \tag{16}$$

equipped with the norm $\|\mathfrak{v}; \mathfrak{D}\| = (\|\mathfrak{v}_0; H^2(\Omega)\|^2 + \|\mathfrak{a}; \mathbb{R}^I\|^2 + \|\mathfrak{b}; \mathbb{R}^I\|^2)^{1/2}$. For a function $\mathfrak{v}$ in space (16) we set $\pi_+\mathfrak{v} = \mathfrak{b},\ \pi_-\mathfrak{v} = \mathfrak{a}$, whilst $\pi_\pm : \mathfrak{D} \mapsto \mathbb{R}^I$ are projections.

Lemma 4.1. *For functions* $\mathfrak{v}, \mathfrak{w} \in \mathfrak{D}$ *the generalized Green's formula*

$$(-\Delta\mathfrak{v}, \mathfrak{w})_\Omega + (\partial_n \mathfrak{v}, \mathfrak{w})_{\partial\Omega} - (\mathfrak{v}, -\Delta\mathfrak{w})_\Omega - (\mathfrak{v}, \partial_n \mathfrak{w})_{\partial\Omega} = \langle \pi_+\mathfrak{v}, \pi_-\mathfrak{w}\rangle - \langle \pi_-\mathfrak{v}, \pi_+\mathfrak{w}\rangle \tag{17}$$

is valid, where $\langle \mathfrak{a}, \mathfrak{b}\rangle = \mathfrak{a}^\top \mathfrak{b}$ *is the scalar product in* $\mathbb{R}^I$.

Let $f \in L_2(\Omega)$, $g \in H^{3/2}(\partial\Omega)$ and $h \in \mathbb{R}^J$. We consider the boundary value problem with *asymptotic conditions* at the points $P^1, \ldots, P^J$ (cf. [9])

$$-\Delta\mathfrak{v}(x) = f(x), \quad x \in \Omega \setminus \{P^1 \cup \cdots \cup P^I\}, \quad \partial_n \mathfrak{v} = g(x), \quad x \in \partial\Omega, \quad \mathbf{S}\pi_-\mathfrak{v} - \pi_+\mathfrak{v} = h \tag{18}$$

After selecting the symmetric nonsingular $(I \times I)$-matrix $\mathbf{S}$, relation (17) is adapted to problem (18) in the following way:

$$\begin{aligned}&(-\Delta\mathfrak{v}, \mathfrak{w})_\Omega + (\partial_n \mathfrak{v}, \mathfrak{w})_{\partial\Omega} + \langle \mathbf{S}\pi_-\mathfrak{v} - \pi_+\mathfrak{v}, \pi_-\mathfrak{w}\rangle \\ &\quad = (\mathfrak{v}, -\Delta\mathfrak{w})_\Omega + (\mathfrak{v}, \partial_n \mathfrak{w})_{\partial\Omega} + \langle \pi_-\mathfrak{v}, \mathbf{S}\pi_-\mathfrak{w} - \pi_+\mathfrak{w}\rangle\end{aligned} \tag{19}$$

Green's formulae (17) and (19) exhibit the hierarchy of integrals on the plane: two-, one-, and zero-dimensional integrals appear as scalar products in $L_2(\Omega)$, $L_2(\partial\Omega)$, and $\mathbb{R}^J$, respectively. To the symmetric generalised Green's formula (19), there corresponds the following energy functional on the Hilbert space $\mathfrak{D}$ providing the variational formulation of problem (18),

$$\begin{aligned}&\mathfrak{E}(\mathfrak{v};\ f, g, h) \\ &\quad = \frac{1}{2}(-\Delta\mathfrak{v}, \mathfrak{v})_\Omega + \frac{1}{2}(\partial_n\mathfrak{v}, \mathfrak{v})_{\partial\Omega} + \frac{1}{2}\langle \mathbf{S}\pi_-\mathfrak{v} - \pi_+\mathfrak{v}, \pi_-\mathfrak{v}\rangle - (f, \mathfrak{v})_\Omega - (g, \mathfrak{v})_{\partial\Omega} - \langle h, \pi_-\mathfrak{v}\rangle\end{aligned} \tag{20}$$

Proposition 4.2. *The function* $\mathfrak{v}$ *is a solution to problem* (18) *if and only if it is a critical point of the functional* (20).

Theorem 4.3. *If* $f \in L_2(\Omega)$, $g = 0$, $h = 0$ *and the matrix* $\mathbf{S}$ *in the point asymptotic conditions is chosen in the same way as in* (9)*, then a solution* $\mathfrak{v} \in \mathfrak{D}$ *of problem* (18) *coincides with the solution* $\mathbf{v} \in \mathcal{D}(\mathbf{A})$ *of Eq.* (10)*. Therefore, the assertions of Theorem* 3.2 *and Remark* 1 *remain valid for* $\mathbf{v}$ *and* $\mathbf{E}(\mathbf{v}; f)$ *replaced with* $\mathfrak{v}$ *and* $\mathfrak{E}(\mathfrak{v}; f, 0, 0)$*, respectively.*

5. Spectral problems

The eigenvalue sequence $0 < \Lambda_1(\varepsilon) < \Lambda_2(\varepsilon) \leqslant \cdots \leqslant \Lambda_n(\varepsilon) \leqslant \cdots \to +\infty$ is written for the equation

$$-\Delta u(\varepsilon, x) = \Lambda(\varepsilon) u(\varepsilon, x), \quad x \in \Omega(\varepsilon) \tag{21}$$

supplied with the boundary conditions (5). The convention on repeated multiply eigenvalues is adopted in the paper. Asymptotic expansions of eigenvalues for mixed boundary value problem (21), (5) can be constructed and justified by employing the procedures developed in [10] and [1]. Our aim is the comparison of the sequence $\{\Lambda_n(\varepsilon),\ n = 1, 2, \ldots\}$, with the spectrum

$$\sigma(\mathbf{A}) = \{\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots, \lambda_n(\varepsilon), \ldots\} \tag{22}$$

of the self adjoint operator $\mathbf{A}$ defined in Proposition 3.1. In view of Theorem 4.3, (22) also implies the set of eigenvalues of the spectral problem with point conditions

$$-\Delta\mathfrak{v}(x) = \lambda(\varepsilon)\mathfrak{v}(x), \quad x \in \Omega \setminus \{P^1, \ldots, P^I\}, \quad \partial_n\mathfrak{v}(x) = 0, \quad x \in \partial\Omega, \quad \mathbf{S}\pi_\mathfrak{v} - \pi_+\mathfrak{v} = 0 \in \mathbb{R}^J \tag{23}$$

The space (16) is compactly embedded into $L_2(\Omega)$, hence the eigenvalues $\lambda_k(\varepsilon)$ are of finite multiplicity, with the only accumulation point of set (22) at the infinity. The operator $\mathbf{A}$ is not positive, it is only the case for the matrix $\mathbf{S}$

in (9) positive definite. Thus, spectrum (22) can deviate from the positive semi-axis $\mathbb{R}_+$. Nevertheless, using the approach proposed in [11] the following assertion is proved.

Theorem 5.1. *For any $T > 0$ there exists $\varepsilon_T > 0$ such that all eigenvalues $\lambda_1(\varepsilon), \ldots, \lambda_{N(T)}(\varepsilon) \in \sigma(\mathbf{A}) \cap (-T, T)$, for $\varepsilon \in (0, \varepsilon_T]$, become positive and satisfy the estimates*

$$\left|\lambda_n(\varepsilon) - \Lambda_n(\varepsilon)\right| \leqslant c_{n,\varkappa}\mu_2(\varepsilon) \tag{24}$$

where $\mu_2(\varepsilon)$ is defined in (13) *and the constant $c_{n,\varkappa}$ depends on the eigenvalue number $n = 1, \ldots, N(T)$ and $\varkappa > 0$ but it is independent of $\varepsilon \in (0, \varepsilon_T]$.*

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