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On a distributed derivative model of a viscoelastic body

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Abstract

We study a viscoelastic body, in a linear stress state with fractional derivative type of dissipation. The model was formulated so that it takes into account, with a weighting factor, all derivatives of stress and strain between zero and one. We derive restrictions on the model that follow from Clausius–Duhem inequality. Several known constitutive equations are derived as special cases of the model proposed here. Two examples are discussed. ***To cite this article: T.M. Atanackovic, C. R. Mecanique 331 (2003).***

Résumé

Sur un model de corps viscoélastique de dérivé distribué. Nous étudions un corps viscoélastique dans un état de tension linéaire avec une sorte dissipation à la derivée fractionnelle. Le model a été formulé de façon à ce qu'il prenne en compte, avec le facteur de poids, toutes les derivées de la tension et des déformations entre zéro et un. Nous dérivons des restrictions posées sur le model qui suivent de l'inégalite de Clausius–Duhem. Plusieurs équations constitutives connues sont derivées comme les cas spéciaux du model proposé ici. Deux examples sont discutées. ***Pour citer cet article : T.M. Atanackovic, C. R. Mecanique 331 (2003).***

Keywords: Solids and structures; Fractional derivative

Mots-clés : Solides et structures ; Dérivé fractionnaire

1. Introduction

We shall analyze a model of a viscoelastic body proposed in [1]. We first recall few results from [1]. Suppose that the stress $\sigma(t)$ and its derivatives of real (not necessarily integer) order at time instant t in a linear stress state depend on a strain $\varepsilon(t)$ and its derivatives of real order. Then, we may write

$$b_0\sigma + b_1\sigma^{(\alpha_1)} + \cdots + b_M\sigma^{(\alpha_M)} = a_0\varepsilon + a_1\varepsilon^{(\alpha_1)} + \cdots + a_N\varepsilon^{(\alpha_N)} \tag{1}$$

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where $a_0,\dots,a_N,b_0,\dots,b_M$ and $\alpha_1,\dots,\alpha_N$ are real constants and we used $\varepsilon^{(\alpha)}$ and $\sigma^{(\alpha)}$ to denote the α-th derivative of $\varepsilon(t)$ and $\sigma(t)$, respectively defined as (see [2])

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\varepsilon(t)=\varepsilon^{(\alpha)}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\varepsilon(\xi)\,\mathrm{d}\xi}{(t-\xi)^\alpha} \tag{2}$$

where Γ is the Euler gamma function. For the case when $b_0=1,\ b_i=0,\ i=1,\dots,M,\ N=1,\ \alpha_1=1$ we obtain Kelvin–Voigt model of viscoelastic body. Recently distributed order differential equations are introduced, as a generalization of (1) (see [3–5] and references given there) that lead to the following type of $\sigma-\varepsilon$ relation proposed in [1]

$$\int_0^1\phi_\sigma(\gamma)\sigma^{(\gamma)}\,\mathrm{d}\gamma=\int_0^1\phi_\varepsilon(\gamma)\varepsilon^{(\gamma)}\,\mathrm{d}\gamma \tag{3}$$

where $\phi_\sigma(\gamma)$ and $\phi_\varepsilon(\gamma)$ are constitutive functions. Note that in the special case when $\phi_\sigma(\gamma,t)=\delta(\gamma)$, where δ is the Dirac distribution, (3) becomes

$$\sigma(t)=\int_0^1\phi_\varepsilon(\alpha)\varepsilon^{(\alpha)}(t)\,\mathrm{d}t \tag{4}$$

with $\phi_\varepsilon(\alpha)$ given. Of course the stress σ could depend on integer order derivatives, so that Eq. (4) may read

$$\sigma(t)=\sum_{i=0}^m a_i\varepsilon^{(i)}+\int_0^1\phi(\alpha)\varepsilon^{(\alpha)}(t)\,\mathrm{d}t \tag{5}$$

where $m\geqslant 1$.

The functions $\phi_\sigma(\gamma,t)$ and $\phi_\varepsilon(\gamma,t)$ in (3) characterize the material under consideration and must satisfy restrictions that follow from the entropy inequality. We proceed now to derive those restrictions.

2. The restriction on functions $\phi_\sigma(\gamma)$ and $\phi_\varepsilon(\gamma)$

Constitutive equations must satisfy the restrictions that follow from the Second law of thermodynamics [7,8]. These restrictions require that the tangent of the mechanical loss angle is non-negative. By applying Fourier transform to (3) we obtain

$$\hat\sigma(\mathrm{i}\omega)\int_0^1\phi_\sigma(\gamma)(\mathrm{i}\omega)^\gamma\,\mathrm{d}\gamma=\hat\varepsilon(\mathrm{i}\omega)\int_0^1\phi_\varepsilon(\gamma)(\mathrm{i}\omega)^\gamma\,\mathrm{d}\gamma \tag{6}$$

where $\hat\sigma(\mathrm{i}\omega)=\mathcal{F}(\sigma)=\int_{-\infty}^{\infty}\sigma(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$ is the Fourier transform of $\sigma(t)$. From (6) we obtain

$$E^*(\mathrm{i}\omega)=\frac{\int_0^1\phi_\varepsilon(\gamma)(\mathrm{i}\omega)^\gamma\,\mathrm{d}\gamma}{\int_0^1\phi_\sigma(\gamma)(\mathrm{i}\omega)^\gamma\,\mathrm{d}\gamma} \tag{7}$$

Writing $E^*(\mathrm{i}\omega)=E'+\mathrm{i}E''$ and using $(\mathrm{i}\omega)^\gamma=\omega^\gamma(\cos\frac{\pi}{2}\gamma+\mathrm{i}\sin\frac{\pi}{2}\gamma)$ in (7) it follows

$$E'=\frac{C_1C_2+S_1S_2}{(C_2)^2+(S_2)^2};\qquad E''=\frac{S_1C_2-C_1S_2}{(C_2)^2+(S_2)^2} \tag{8}$$

where

$$C_1 = \int_0^1 \phi_\varepsilon(\gamma)\omega^\gamma \cos\frac{\pi}{2}\gamma \,\mathrm{d}\gamma; \qquad C_2 = \int_0^1 \phi_\sigma(\gamma)\omega^\gamma \cos\frac{\pi}{2}\gamma \,\mathrm{d}\gamma$$

$$S_1 = \int_0^1 \phi_\varepsilon(\gamma)\omega^\gamma \sin\frac{\pi}{2}\gamma \,\mathrm{d}\gamma; \qquad S_2 = \int_0^1 \phi_\sigma(\gamma)\omega^\gamma \sin\frac{\pi}{2}\gamma \,\mathrm{d}\gamma \tag{9}$$

Therefore the tangent of the mechanical loss angle is

$$\tan\delta = \frac{E''}{E'} = \frac{S_1C_2 - C_1S_2}{C_1C_2 + S_1S_2} \tag{10}$$

The restricition following from second law of thermodynamics implies

$$\tan\delta \geqslant 0 \tag{11}$$

and that *both* E' and E'' are positive for all values of ω (see [7], p. 140). Thus

$$C_1C_2 + S_1S_2 \geqslant 0; \quad S_1C_2 - C_1S_2 \geqslant 0; \quad \text{for all } 0 \leqslant \omega \leqslant \infty. \tag{12}$$

We write the conditions (12) in expanded form by using definitions of $S_1, \ldots, C_2$,

$$\begin{gathered}\left(\int_0^1 \phi_\varepsilon(\gamma)\omega^\gamma \cos\frac{\pi}{2}\gamma \,\mathrm{d}\gamma\right)\left(\int_0^1 \phi_\sigma(\gamma)\omega^\gamma \cos\frac{\pi}{2}\gamma \,\mathrm{d}\gamma\right) \\ + \left(\int_0^1 \phi_\varepsilon(\gamma)\omega^\gamma \sin\frac{\pi}{2}\gamma \,\mathrm{d}\gamma\right)\left(\int_0^1 \phi_\sigma(\gamma)\omega^\gamma \sin\frac{\pi}{2}\gamma \,\mathrm{d}\gamma\right) \geqslant 0; \quad \text{for all } 0 \leqslant \omega \leqslant \infty \\ \left(\int_0^1 \phi_\varepsilon(\gamma)\omega^\gamma \sin\frac{\pi}{2}\gamma \,\mathrm{d}\gamma\right)\left(\int_0^1 \phi_\sigma(\gamma)\omega^\gamma \cos\frac{\pi}{2}\gamma \,\mathrm{d}\gamma\right) \\ - \left(\int_0^1 \phi_\varepsilon(\gamma)\omega^\gamma \cos\frac{\pi}{2}\gamma \,\mathrm{d}\gamma\right)\left(\int_0^1 \phi_\sigma(\gamma)\omega^\gamma \sin\frac{\pi}{2}\gamma \,\mathrm{d}\gamma\right) \geqslant 0 \quad \text{for all } 0 \leqslant \omega \leqslant \infty\end{gathered} \tag{13}$$

We consider next several special cases of the restrictions imposed by (12).

1. Suppose that

$$\phi_\sigma = \delta(\gamma) + a\delta(\gamma - \alpha); \qquad \phi_\varepsilon = \delta(\gamma) + b\delta(\gamma - \alpha) \tag{14}$$

where $a,\ b$ and $0 < \alpha < 1$ are constants. This choice correspond to the generalized Zener model $\sigma + a\sigma^{(\alpha)} = \varepsilon + b\varepsilon^{(\alpha)}$. By substituting (14) into (9) the condition (12) becomes

$$\begin{gathered}b\omega^\alpha \sin\frac{\pi}{2}\alpha - a\omega^\alpha \sin\frac{\pi}{2}\alpha \geqslant 0 \\ \left(1 + b\omega^\alpha \cos\frac{\pi}{2}\alpha\right)\left(1 + a\omega^\alpha \cos\frac{\pi}{2}\alpha\right) + \left(b\omega^\alpha \sin\frac{\pi}{2}\alpha\right)\left(a\omega^\alpha \sin\frac{\pi}{2}\alpha\right) \geqslant 0\end{gathered} \tag{15}$$

From (15) it follows that

$$b > a > 0 \tag{16}$$

a well known result (see [7] and [8]).

2. Suppose that

$$\phi_\varepsilon(\gamma) = c\phi_\sigma(\gamma), \quad c > 0 \tag{17}$$

Then $\tan\delta = 0$ and we conclude that the body behaves as an elastic body.

3. Let us assume that

$$\phi_\sigma = \delta(\gamma); \qquad \phi_\varepsilon = E(\tau_0)^\gamma \tag{18}$$

where $E = \mathit{const}$. and $\tau_0 = \mathit{const}$. are known constants. Thus, the constitutive equation reads $\sigma = E\int_0^1 (\tau_0)^\gamma \varepsilon^{(\gamma)}\,\mathrm{d}\gamma$ which is of the type (4). The condition (12) becomes

$$E\int_0^1 (\tau_0\omega)^\gamma \sin\frac{\pi}{2}\gamma\,\mathrm{d}\gamma \geqslant 0; \qquad E\int_0^1 (\tau_0\omega)^\gamma \cos\frac{\pi}{2}\gamma\,\mathrm{d}\gamma \geqslant 0 \tag{19}$$

or

$$E > 0; \qquad \tau_0 > 0 \tag{20}$$

4. Next we suppose that

$$\phi_\varepsilon = \delta(\gamma); \qquad \phi_\sigma = C(\tau_1)^\gamma \tag{21}$$

where C and τ_1 are constants. The constitutive equation corresponding to (21) reads

$$C\int_0^1 (\tau_1)^\gamma \sigma^{(\gamma)}\,\mathrm{d}\gamma = \varepsilon \tag{22}$$

The constants (9) are

$$C_1 = 1; \quad C_2 = C\int_0^1 (\tau_1\omega)^\gamma \cos\frac{\pi}{2}\gamma\,\mathrm{d}\gamma; \quad S_1 = 0; \quad S_2 = C\int_0^1 (\tau_1\omega)^\gamma \sin\frac{\pi}{2}\gamma\,\mathrm{d}\gamma \tag{23}$$

so that (12) leads to

$$-C\int_0^1 (\tau_1\omega)^\gamma \sin\frac{\pi}{2}\gamma\,\mathrm{d}\gamma \geqslant 0; \qquad C\int_0^1 (\tau_1\omega)^\gamma \cos\frac{\pi}{2}\gamma\,\mathrm{d}\gamma \geqslant 0 \tag{24}$$

From (24) we conclude that (22) violates the second law of thermodynamics for any value of $C \neq 0$. This is an interesting fact and it generalizes the results of [7] and [8]; namely, suppose that we consider $\sigma + a\sigma^{(\alpha)} = \varepsilon + b\varepsilon^{(\alpha)}$ with $b = 0,\ a > 0$. This case is a special case of (21) with $\phi_\varepsilon = \delta(\gamma),\ \phi_\sigma = \delta(\gamma) + a\delta(\gamma - \alpha)$. Thus, the result (24) *forbids* the constitutive equation of the form $\sigma + a\sigma^{(\alpha)} = \varepsilon,\ a > 0$. This is in agreement with (16) that was obtained in [8] by using different arguments.

5. Consider the case when

$$\phi_\sigma = \delta(\gamma) + a\delta(\gamma - \alpha); \qquad \phi_\varepsilon = E_0\big[\delta(\gamma) + b\delta(\gamma - \alpha) + c\delta(\gamma - \beta)\big] \tag{25}$$

where $a,\ b,\ c,\ \alpha$ and β are constants with $0 < \alpha < 1,\ 0 < \beta < 1$. This choice correspond to the generalized model $\sigma + a\sigma^{(\alpha)} = E_0[\varepsilon + b\varepsilon^{(\alpha)} + c\varepsilon^{(\beta)}]$ recently used in [6,13] and [14]. As a matter of fact in [13] a special case of (25) was used where $a = \tau^\alpha,\ b = \tau^\alpha,\ c = \tau^{(\beta)}$, with $\tau = \mathit{const}$. (see Eq. (37) of [13]) while in [6] it was assumed that $a = \tau_\varepsilon^\alpha,\ b = \tau_\sigma^\alpha,\ c = \tau_\sigma^\beta$ with $\tau_\sigma = \mathit{const}.,\ \tau_\varepsilon = \mathit{const}$. From (9) and (26) we have

$$1+ab\omega^{2\alpha}+\omega^{\alpha}(a+b)\cos\frac{\pi}{2}\alpha+c\omega^{\beta}\cos\frac{\pi}{2}\beta+ac\omega^{\alpha+\beta}\cos\frac{\pi}{2}(\beta-\alpha)\geqslant 0$$
$$\omega^{\alpha}(b-a)\sin\frac{\pi}{2}\alpha+c\omega^{\beta}\sin\frac{\pi}{2}\alpha+ac\omega^{\alpha+\beta}\sin\frac{\pi}{2}(\beta-\alpha)\geqslant 0 \tag{26}$$

together with $E_0>0$. Thus, the constants in (25) must satisfy

$$E_0>0; \quad b\geqslant a>0; \quad c>0; \quad \alpha<\beta \tag{27}$$

The conditions (27) contain, as a special case, the results presented in [6,13].

3. Applications

We use the constitutive equation (3) to formulate govering equations for two specific problems.

3.1. Application I

Consider a mechanical system consisting of a body of mass m that moves translatory and is connected to one end of a viscoelastic rod. The other end of the rod is fixed to unmovable wall. A force $F=h\sin\Omega t$, where h and Ω are constants, is also acting on the body. The action line of the force coincides with the rod axis. Suppose that the initial (undeformed) length of the rod is l_0. In the deformed state the length is given as $l(t)=l_0+y(t)$ where $y(t)$ is the change of the length so that the strain is $\varepsilon=y/l_0$.

Suppose that the rod is made of a material described by (4), with (see [1])

$$\phi(\alpha)=E(\tau_\varepsilon)^{\alpha} \tag{28}$$

where $E>0$, $\tau_\varepsilon>0$ (see (20)) are constants. The equation of motion reads

$$my^{(2)}(t)+\frac{E}{l_0}\int_0^1(\tau_\varepsilon)^{\alpha}y^{(\alpha)}(t)\,\mathrm{d}\alpha=h\sin\Omega t \tag{29}$$

By applying Laplace transform $\mathcal{L}(f)(z)=\int_0^\infty \mathrm{e}^{\mathrm{i}tz}f(t)\,\mathrm{d}t=\bar{f}(z)$ to (28) we obtain (with $m=1,\ l_0=1$)

$$z^2\bar{y}(z)=h\frac{\Omega}{z^2-\Omega^2}-E\int_0^1(\tau_\varepsilon)^{\alpha}z^{\alpha}\bar{y}(z)\,\mathrm{d}\alpha+y^{(1)}(0)+zy(0) \tag{30}$$

where we used the fact that $\mathcal{L}[y^{(\alpha)}]=p^{\alpha}\bar{f}(p)-\big(\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{y(\tau)\,\mathrm{d}\tau}{(t-\tau)^{\alpha}}\big)_{t=0}$. The term $\big(\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{y(\tau)\,\mathrm{d}\tau}{(t-\tau)^{\alpha}}\big)_{t=0}$ vanishes if $y(t)$ is bounded for $t\to+0$, so that from (30) follows

$$\bar{y}(z)=\frac{y^{(1)}(0)+zy(0)}{z^2+E(\tau_\varepsilon z-1)/\ln(\tau_\varepsilon z)}+h\frac{\Omega/(z^2-\Omega^2)}{z^2+E(\tau_\varepsilon z-1)/\ln(\tau_\varepsilon z)} \tag{31}$$

We shall not go into the problem of finding inverse Laplace transform of (31).

3.2. Application II

On the basis of constitutive equation (3) we derive a moment curvature relation for a rod. Such relations may be used for the study of motion and stability of viscoelastic rods (see [9,10] and [12]). Following the standard procedure (plane cross section hypothesis), as was described in [11] we obtain

$$\int_0^1\phi_\sigma(\gamma,t)M^{(\gamma)}\,\mathrm{d}\gamma=I\int_0^1\phi_\varepsilon(\gamma,t)\left(\frac{1}{\rho}\right)^{(\gamma)}\mathrm{d}\gamma \tag{32}$$

where I is the moment of inertia of the rod's cross-section A, that is $I = \int_A y^2\,\mathrm{d}A$, where y is the distance from the neutral axis, and ρ is the radius of curvature of the rod axis.

Note that in the special case of elastic material $\phi_\sigma(\gamma, t) = \delta(\gamma)$, $\phi_\varepsilon(\gamma, t) = E\delta(\gamma)$ Eq. (32) becomes $M = EI(1/\rho)$, i.e., the moment curvature relation of classical Bernoulli–Euler rod theory. If we choose $\phi_\sigma = \delta(\gamma)$, $\phi_\varepsilon = E\delta(\gamma) + b\delta(\gamma - \alpha)$ Eq. (32) leads to

$$M = EI\left(\frac{1}{\rho}\right) + bI\left(\frac{1}{\rho}\right)^{(\alpha)} \tag{33}$$

The linearized version of (33) was used in [12].

For the case $\phi_\sigma = \delta(\gamma) + a\delta(\gamma - \alpha)$; $\phi_\varepsilon = \delta(\gamma) + b\delta(\gamma - \alpha) + c\delta(\gamma - \beta)$ (see (14)) we obtain

$$M + aM^{(\alpha)} = I\left(\frac{1}{\rho}\right) + bI\left(\frac{1}{\rho}\right)^{(\alpha)} + cI\left(\frac{1}{\rho}\right)^{(\beta)} \tag{34}$$

The constitutive equation of the type (34) was used in [9,10] and [11].

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