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An approximate solution to the integral radiative transfer equation in an optically thick slab

Andrey Amosov [a,*], Grigori Panasenko [b], Bernard Rutily [c]

[a] *Department of Mathematical Modelling, Moscow Power Engineering Institute, Krasnokazarmennaja 14, 111250 Moscow, Russia*
[b] *Équipe d'analyse numérique, Université Jean Monnet, 23, rue P. Michelon, 42023 Saint-Etienne, France*
[c] *Observatoire de Lyon 9, avenue Charles André, 69561 Saint-Genis-Laval cedex, France*

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Abstract

We consider the problem of solving the integral form of the radiative transfer equation in an atmosphere with optical thickness $\tau_0 \gg 1$. We propose a method transforming this problem in the same problem posed in an atmosphere with optical thickness $\tau_1 \ll \tau_0$. An error over- estimation is derived. ***To cite this article: A. Amosov et al., C. R. Mecanique 331 (2003).***

Résumé

Solution approchée de l'équation de transfert intégrale dans une couche optiquement épaisse. On s'intéresse à la résolution de la forme intégrale de l'équation de transfert dans une atmosphère d'épaisseur optique $\tau_0 \gg 1$. Nous proposons une méthode ramenant ce problème au même problème posé dans une atmosphère d'épaisseur optique $\tau_1 \ll \tau_0$. Une majoration de l'erreur est donnée. ***Pour citer cet article : A. Amosov et al., C. R. Mecanique 331 (2003).***

Keywords: Waves; Radiative transfer equation; Optical thickness

Mots-clés : Ondes ; Équation de transfert ; Épaisseur optique

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On considére l'équation intégrale

$$S(\tau) = \frac{\varpi(\tau)}{2} \int_0^{\tau_0} E_1\big(|\tau - \tau'|\big) S(\tau')\,\mathrm{d}\tau' + F(\tau), \quad \tau \in [0, \tau_0]$$

* Corresponding author.
E-mail addresses: AmosovAA@mpei.ru (A. Amosov), grigory.panasenko@univ-st-etienne.fr (G. Panasenko), rutily@obs.univ-lyon1.fr (B. Rutily).

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qui décrit la diffusion isotrope du rayonnement dans une atmosphère d'épaisseur optique τ_0 et d'albédo $\varpi(\tau)$. La théorie de cette équation a été développée en détail [1–7]. Néanmoins, sa résolution pour une épaisseur optique τ_0 élevée et des albédos ϖ_0 proches de l'unité est très difficile.

Le présent article décrit une méthode permettant de ramener le problème avec $\tau_0 \gg 1$ au même problème posé dans une atmosphère d'épaisseur optique $\tau_1 \ll \tau_0$. Cette méthode permet donc de réduire de façon considérable le volume des calculs numériques.

Nous supposons que $0 \leqslant \varpi(\tau) \leqslant \varpi_0 < 1$, $\varpi \in W^{2m,\infty}(0,\tau_0)$ pour un $m \geqslant 0$. Le terme libre est de la forme $F = \varpi F_\ell + (1-\varpi)F_0 + \varpi F_r$, où $F_0 \in W^{2m+2,\infty}(0,\tau_0)$, F_ℓ, $F_r \in L^\infty(0,\tau_0)$. Les fonctions F_ℓ, F_r sont supposées à décroissance exponentielle : $|F_\ell(\tau)| \leqslant M\mathrm{e}^{-\tau}$, $|F_r(\tau)| \leqslant M\mathrm{e}^{-(\tau_0-\tau)}$. Les fonctions F_0 et $\gamma = \frac{\varpi}{1-\varpi}$ sont censées varier lentement : $\|F_0^{(j)}\|_{L^\infty(0,\tau_0)} \leqslant M\delta^j$ pour $0 \leqslant j \leqslant 2m+2$; $\|\gamma^{(j)}\|_{L^\infty(0,\tau_0)} \leqslant \frac{N}{1-\varpi_0}\delta^j$ pour $0 \leqslant j \leqslant 2m$, où $\delta \in (0,1)$ et $M \geqslant 0$, $N \geqslant 1$. On suppose également que $\varepsilon = \frac{N\delta^2}{1-\varpi_0} < 1$.

Nous construisons une solution approchée de la forme $S^{[m]} = S^{[m]}_{\mathrm{reg}} + S^{[m]}_\ell + S^{[m]}_r$. Ici, $S^{[m]}_{\mathrm{reg}} = \sum_{k=0}^m S_{k,\mathrm{reg}}$ est une partie régulière de la solution, où $S_{0,\mathrm{reg}} = F_0$, $S_{k,\mathrm{reg}} = \gamma\sum_{\alpha=1}^k \frac{1}{2\alpha+1} S^{(2\alpha)}_{k-\alpha,\mathrm{reg}}$ pour $1 \leqslant k \leqslant m$. En outre, $S^{[m]}_\ell$ et $S^{[m]}_r$ sont des corrections dues aux conditions aux limites, qui doivent être calculées en résolvant les équations intégrales suivantes :

$$S^{[m]}_\ell(\tau) = \varpi(\tau)\left[\frac{1}{2}\int_0^{\tau_1} E_1\big(|\tau-\tau'|\big)S^{[m]}_\ell(\tau')\,\mathrm{d}\tau' + F_\ell(\tau) + R^{[m]}_\ell(\tau)\right], \quad \tau \in [0,\tau_1]$$

$$S^{[m]}_r(\tau) = \varpi(\tau)\left[\frac{1}{2}\int_{\tau_0-\tau_1}^{\tau_0} E_1\big(|\tau-\tau'|\big)S^{[m]}_r(\tau')\,\mathrm{d}\tau' + F_r(\tau) + R^{[m]}_r(\tau)\right], \quad \tau \in [\tau_0-\tau_1,\tau_0]$$

Ici, $\tau_1 = \frac{m+1}{k(\varpi_0)}\ln\frac{1}{\varepsilon}$, où $k(\varpi_0)$ est l'unique solution dans $]0,1[$ de l'équation caractéristique

$$1 - \frac{\varpi_0}{2}\frac{1}{k}\ln\frac{1+k}{1-k} = 0$$

et $R^{[m]}_\ell(\tau) = -\sum_{k=0}^m\sum_{j=0}^{2m-2k+1}(-1)^j E_{1,j}(\tau)S^{(j)}_{k,\mathrm{reg}}(\tau)$, $R^{[m]}_r(\tau) = -\sum_{k=0}^m\sum_{j=0}^{2m-2k+1} E_{1,j}(\tau_0-\tau)S^{(j)}_{k,\mathrm{reg}}(\tau)$, $E_{1,j}(\tau) = \frac{1}{2}\sum_{k=0}^j \frac{\tau^k}{k!}E_{j+2-k}(\tau)$. Nous posons $S^{[m]}_\ell(\tau) = 0$ sur $(\tau_1,\tau_0]$ et $S^{[m]}_r(\tau) = 0$ sur $[0,\tau_0-\tau_1)$.

Sous ces hypothèses, nous avons la majoration suivante de l'erreur :

$$\big\|S - S^{[m]}\big\|_{L^\infty(0,\tau_0)} \leqslant C_m M\varepsilon^{m+1}$$

1. Formulation of the problem and description of the method

Let us consider the integral equation

$$S(\tau) = \frac{\varpi(\tau)}{2}\int_0^{\tau_0} E_1\big(|\tau-\tau'|\big)S(\tau')\,\mathrm{d}\tau' + F(\tau), \quad \tau \in [0,\tau_0]$$

which describes isotropic scattering of radiation in an atmosphere with optical thickness τ_0 and albedo $\varpi(\tau)$. Here $E_k(\tau) = \int_0^1 \mu^{k-2}\mathrm{e}^{-\tau/\mu}\,\mathrm{d}\mu$ $(k \geqslant 1)$ is the integro-exponential function of order k.

The quantities involved in the equation have the following physical meaning: τ is the optical depth, $S(\tau)$ is the source function, $F(\tau)$ is the primary source function and $\varpi(\tau)$ is the albedo for single scattering.

We will use the notation $(\Lambda_{a,b}S)(\tau) = \frac{1}{2}\int_a^b E_1(|\tau - \tau'|)S(\tau')\,\mathrm{d}\tau'$ (where $-\infty \leqslant a < b \leqslant +\infty$) for the Hopf integral operator. The above equation may then be rewritten in the following form

$$S = \varpi \Lambda_{0,\tau_0} S + F \quad \text{on } [0, \tau_0] \tag{1}$$

The theory of Eq. (1) is developed in detail [1–7]. However, the problem of solving it for a large optical depth τ_0 and albedo values close to unity, is very difficult.

In the present paper we develop a method [8,9] transforming the problem (1) with $\tau_0 \gg 1$ in the same problem posed in an atmosphere of optical thickness $\tau_1 \ll \tau_0$. Thus, this method is able to decrease the volume of calculations very essentially. We suppose that

$$0 \leqslant \varpi(\tau) \leqslant \varpi_0 \quad \forall \tau \in [0, \tau_0], \text{ where } \varpi_0 \in (0, 1); \quad \varpi \in W^{2m,\infty}(0, \tau_0) \tag{2}$$

for some $m \geqslant 0$. The free term F has the following form:

$$F = \varpi F_\ell + (1 - \varpi)F_0 + \varpi F_r, \quad \text{where } F_0 \in W^{2m+2,\infty}(0, \tau_0),\ F_\ell, F_r \in L^\infty(0, \tau_0) \tag{3}$$

Here F_0 and F_ℓ, F_r describe the radiation due to internal and external sources. The functions F_ℓ, F_r are exponentially decaying:

$$\left|F_\ell(\tau)\right| \leqslant M\mathrm{e}^{-\tau}, \quad \left|F_r(\tau)\right| \leqslant M\mathrm{e}^{-(\tau_0-\tau)} \quad \forall \tau \in [0, \tau_0] \tag{4}$$

and F_0 and $\gamma = \varpi/(1 - \varpi)$ are supposed to verify

$$\left\|F_0^{(j)}\right\|_{L^\infty(0,\tau_0)} \leqslant M\delta^j, \quad 0 \leqslant j \leqslant 2m+2; \qquad \|\gamma^{(j)}\|_{L^\infty(0,\tau_0)} \leqslant \frac{N}{1-\varpi_0}\delta^j, \quad 0 \leqslant j \leqslant 2m \tag{5}$$

for some $\delta \in (0, 1)$ and $M \geqslant 0$, $N \geqslant 1$. (When $m = 0$, the inequalities are fulfilled with $N = 1$.) Note that γ is the ratio of the absorbtion coefficient to the scattering coefficient. In fact, the assumptions (5) mean that the functions F_0 and γ are slowly varying ones, with typical scale of variation $L = 1/\delta$. Moreover, we assume that

$$\varepsilon = \frac{N\delta^2}{1-\varpi_0} < 1 \tag{6}$$

Let us construct an approximate solution $S^{[m]}$ to Eq. (1) in the form $S^{[m]} = S^{[m]}_{\mathrm{reg}} + S^{[m]}_\ell + S^{[m]}_r$, where $S^{[m]}_{\mathrm{reg}} = \sum_{k=0}^m S_{k,\mathrm{reg}}$ is a regular part of the solution, which is analytically calculated below. Here $S_{0,\mathrm{reg}} = F_0$, $S_{k,\mathrm{reg}} = \gamma \sum_{\alpha=1}^k \frac{1}{2\alpha+1} S^{(2\alpha)}_{k-\alpha,\mathrm{reg}}$, $1 \leqslant k \leqslant m$, and $S^{[m]}_\ell$ and $S^{[m]}_r$ are boundary layer corrections, which must be calculated as solutions of the following integral equations

$$S^{[m]}_\ell = \varpi\left[\Lambda_{0,\tau_1} S^{[m]}_\ell + F_\ell + R^{[m]}_\ell\right] \qquad \text{on } [0, \tau_1] \tag{7}$$

$$S^{[m]}_r = \varpi\left[\Lambda_{\tau_0-\tau_1,\tau_0} S^{[m]}_r + F_r + R^{[m]}_r\right] \quad \text{on } [\tau_0 - \tau_1, \tau_0] \tag{8}$$

Here $R^{[m]}_\ell(\tau) = -\sum_{k=0}^m \sum_{j=0}^{2m-2k+1} (-1)^j E_{1,j}(\tau) S^{(j)}_{k,\mathrm{reg}}(\tau)$, $R^{[m]}_r(\tau) = -\sum_{k=0}^m \sum_{j=0}^{2m-2k+1} E_{1,j}(\tau_0 - \tau) S^{(j)}_{k,\mathrm{reg}}(\tau)$, $E_{1,j}(\tau) = \frac{1}{2}\frac{1}{j!}\int_\tau^{+\infty} E_1(t)t^j\,\mathrm{d}t = \frac{1}{2}\sum_{k=0}^j \frac{\tau^k}{k!} E_{j+2-k}(\tau)$. We put $S^{[m]}_\ell(\tau) = 0$ on $(\tau_1, \tau_0]$ and $S^{[m]}_r(\tau) = 0$ on $[0, \tau_0 - \tau_1)$.

We propose the following choice of τ_1:

$$\tau_1 = \frac{m+1}{k(\varpi_0)} \ln \frac{1}{\varepsilon}$$

Here $0 < k(\varpi_0) < 1$ is the only solution in $(0, 1)$ of the characteristic equation

$$1 - \frac{\varpi_0}{2}\frac{1}{k}\ln\frac{1+k}{1-k} = 0$$

It is known [5–7], that the function $k(\varpi_0)$ is continuous and decreases on $(0,1)$. Moreover, $k(\varpi_0)\to 1$ when $\varpi_0\to 0$ and $k(\varpi_0)\to 0$ when $\varpi_0\to 1$, and we have

$$1-k(\varpi_0)\sim 2\exp(-2/\varpi_0)\quad \text{as } \varpi_0\to 0;\qquad k(\varpi_0)\sim\sqrt{3(1-\varpi_0)}\quad \text{as } \varpi_0\to 1 \tag{9}$$

Now we are able to formulate the main result of the paper.

Theorem 1.1. *Suppose conditions* (2)–(6) *are valid and* $\tau_1<\tau_0$. *Then we have the following error upper-bound*:

$$\|S-S^{[m]}\|_{L^\infty(0,\tau_0)}\leqslant C_m M\left(\frac{N\delta^2}{1-\varpi_0}\right)^{m+1} \tag{10}$$

The scheme for the proof of this theorem is contained in Section 3. In Section 2, an upper-bound for the solution to Ambartsumian's equation is proposed.

Remark 1. The method with parameter $m=0$ is the simplest one. For it

$$S^{[0]}_{\text{reg}}(\tau)=F_0(\tau),\qquad R^{[0]}_\ell(\tau)=-\frac{1}{2}\bigl[E_2(\tau)F_0(\tau)-\bigl(E_3(\tau)+\tau E_2(\tau)\bigr)F_0'(\tau)\bigr]$$

$$R^{[0]}_r(\tau)=-\frac{1}{2}\bigl[E_2(\tau_0-\tau)F_0(\tau)+\bigl(E_3(\tau_0-\tau)+(\tau_0-\tau)E_2(\tau_0-\tau)\bigr)F_0'(\tau)\bigr]$$

and our upper-bound has the following form:

$$\bigl\|S-S^{[0]}\bigr\|_{L^\infty(0,\tau_0)}\leqslant C_0M\frac{\delta^2}{1-\varpi_0}$$

2. An upper-bound for the solution to Ambartsumian's equation in a half-space

Let us consider the so-called Ambartsumian equation (exponentially decaying free term, $\tau_0=+\infty$, constant albedo $\varpi=\varpi_0$)

$$J(\tau,\varpi_0)=\frac{\varpi_0}{2}\int_0^{+\infty}E_1\bigl(|\tau-\tau'|\bigr)J(\tau',\varpi_0)\,\mathrm{d}\tau'+\varpi_0\,\mathrm{e}^{-\tau},\qquad \tau\in[0,\infty) \tag{11}$$

It is known [2], that the solution $J(\tau,\varpi_0)$ of this equation in $C[0,+\infty)\cap L_\infty(0,+\infty)$ exists and is unique.

The following result is essential for the proof of (10).

Lemma 2.1. *The function* $J(\tau,\varpi_0)$ *satisfies the estimate* $J(\tau,\varpi_0)\leqslant c\,\mathrm{e}^{-k(\varpi_0)\tau}$, *where* c *does not depend on* ϖ_0 *and* τ.

Proof. We use the exact formula [10,11]

$$J(\tau,\varpi_0)=H(1,\varpi_0)\left[A(\varpi_0)\,\mathrm{e}^{-k(\varpi_0)\tau}-\frac{\varpi_0^2}{2}\int_0^1\frac{\mathrm{e}^{-\tau/x}}{H(x,\varpi_0)Z(x,\varpi_0)}\frac{\mathrm{d}x}{1-x}\right] \tag{12}$$

Here $A(\varpi_0)=\frac{\varpi_0}{H(1/k(\varpi_0),\varpi_0)}\frac{k(\varpi_0)(1+k(\varpi_0))}{k^2(\varpi_0)+\varpi_0-1}$, $Z(x,\varpi_0)=[1-\frac{\varpi_0}{2}x\ln\frac{1+x}{1-x}]^2+\frac{\pi^2\varpi_0^2}{4}x^2$. Function $H(\mu,\varpi_0)$ is Ambartsumian–Chandrasekhar function [7]. It satisfies the inequalities $1\leqslant H(\mu,\varpi_0)\leqslant 3$ for $(\mu,\varpi_0)\in[0,1]\times[0,1]$. It may also be proved that

$$\frac{1}{H(1/k(\varpi_0),\varpi_0)}\to 1\quad \text{as } \varpi_0\to 0;\qquad \frac{1}{H(1/k(\varpi_0),\varpi_0)}\sim 2\sqrt{1-\varpi_0}\quad \text{as } \varpi_0\to 1 \tag{13}$$

From formula (12), we have the inequality $J(\tau,\varpi_0) \leqslant 3A(\varpi_0)e^{-k(\varpi_0)\tau}$. The function $A(\varpi_0)$ is continuous over $(0,1)$, and it follows from properties (9), (13) that $A(\varpi_0) \to 2$ when $\varpi_0 \to 0$; $A(\varpi_0) \to \sqrt{3}$ when $\varpi_0 \to 1$. Thus, Lemma 2.1 is proved.

3. The scheme for the proof of (10)

Lemma 3.1. *Let S be a solution to Eq.* (1) *and S^{τ_1} be a solution to equation*

$$S^{\tau_1} = \varpi \Lambda_{0,\tau_1} S^{\tau_1} + F \quad \text{on } [0,\tau_1]$$

with $\tau_1 < \tau_0$, and $S^{\tau_1} = 0$ on $(\tau_1,\tau_0]$. Then $|S(\tau) - S^{\tau_1}(\tau)| \leqslant \frac{1}{2}\|S\|_{L^\infty(\tau_1,\tau_0)} J(\tau_1-\tau,\varpi_0)$ for $\tau \in [0,\tau_1]$. As a consequence we have

$$\|S - S^{\tau_1}\|_{L^\infty(0,\tau_0)} \leqslant c\|S\|_{L^\infty(\tau_1,\tau_0)}$$

Now we may consider the proof of inequality (10). Expanding each component $S_{k,\mathrm{reg}}(\tau')$ of the regular part of the solution with the help of Taylor formula, we have: $S_{\mathrm{reg}}^{[m]}(\tau') = S_T^{[m]}(\tau',\tau) + r_T^{[m]}(\tau',\tau)$, where

$$S_T^{[m]}(\tau',\tau) = \sum_{k=0}^{m} \sum_{j=0}^{2m-2k+1} \frac{S_{k,\mathrm{reg}}^{(j)}(\tau)}{j!}(\tau'-\tau)^j$$

$$\left|r_T^{[m]}(\tau',\tau)\right| \leqslant \sum_{k=0}^{m} \frac{\|S_{k,\mathrm{reg}}^{(2m-2k+2)}\|_{L^\infty(0,\tau_0)}}{(2m-2k+2)!}(\tau'-\tau)^{2m-2k+2}$$

Notice that

$$\frac{1}{2}\int_0^{\tau_0} E_1\big(|\tau'-\tau|\big) S_T^{[m]}(\tau',\tau)\,\mathrm{d}\tau' = \sum_{k=0}^{m}\sum_{\alpha=0}^{m-k} \frac{1}{2\alpha+1} S_k^{(2\alpha)}(\tau) + R_\ell^{[m]}(\tau) + R_r^{[m]}(\tau)$$

Changing the order of summation and taking into account the definition of functions $S_{k,\ \mathrm{reg}}$, we obtain

$$\varpi \sum_{k=0}^{m}\sum_{\alpha=0}^{m-k} \frac{1}{2\alpha+1} S_{k,\mathrm{reg}}^{(2\alpha)} = \varpi \sum_{k=0}^{m} S_k + (1-\varpi)\sum_{k=1}^{m} \gamma \sum_{\alpha=1}^{k} \frac{1}{2\alpha+1} S_{k-\alpha,\mathrm{reg}}^{(2\alpha)} = (\varpi - 1)F_0 + S_{\mathrm{reg}}^{[m]}$$

Thus

$$S_{\mathrm{reg}}^{[m]} = \varpi \Lambda_{0,\tau_0} S_{\mathrm{reg}}^{[m]} + (1-\varpi)F_0 - \varpi R_\ell^{[m]} - \varpi R_r^{[m]} - \varpi \Delta^{[m]} \quad \text{on } [0,\tau_0]$$

where $\Delta^{[m]}(\tau) = \frac{1}{2}\int_0^{\tau_0} E_1(|\tau-\tau'|) r_T^{[m]}(\tau',\tau)\,\mathrm{d}\tau'$.

Put $\widetilde{S}^{[m]} = S_{\mathrm{reg}} + \widetilde{S}_\ell^{[m]} + \widetilde{S}_r^{[m]}$, where $\widetilde{S}_\ell^{[m]}$, $\widetilde{S}_r^{[m]}$ are such that

$$\widetilde{S}_\ell^{[m]} = \varpi\big[\Lambda_{0,\tau_0}\widetilde{S}_\ell^{[m]} + F_\ell + R_\ell^{[m]}\big], \quad \widetilde{S}_r^{[m]} = \varpi\big[\Lambda_{0,\tau_0}\widetilde{S}_r^{[m]} + F_r + R_r^{[m]}\big] \quad \text{on } [0,\tau_0]$$

It is easy to see that

$$S - \widetilde{S}^{[m]} = \varpi \Lambda_{0,\tau_0}(S - \widetilde{S}^{[m]}) + \varpi \Delta^{[m]} \quad \text{on } [0,\tau_0]$$

and as consequence

$$\big\|S - \widetilde{S}^{[m]}\big\|_{L^\infty(0,\tau_0)} \leqslant \frac{\varpi_0}{1-\varpi_0}\big\|\Delta^{[m]}\big\|_{L^\infty(0,\tau_0)}$$

Over-estimating $S_\ell^{[m]} - \widetilde{S}_\ell^{[m]}$ and $S_r^{[m]} - \widetilde{S}_r^{[m]}$ with the help of Lemma 3.1, we have:

$$\|S - S^{[m]}\|_{L^\infty(0,\tau_0)} \leqslant \frac{\varpi_0}{1-\varpi_0}\|\Delta^{[m]}\|_{L^\infty(0,\tau_0)} + c\|\widetilde{S}_\ell^{[m]}\|_{L^\infty(\tau_1,\tau_0)} + c\|\widetilde{S}_r^{[m]}\|_{L^\infty(0,\tau_0-\tau_1)}$$

To complete the proof of (10) we need the following lemmas.

Lemma 3.2. *The following inequality is valid*:

$$\|S_{k,\mathrm{reg}}^{(j)}\|_{L^\infty(0,\tau_0)} \leqslant C_{m,k}\frac{N^k M}{(1-\varpi_0)^k}\delta^{2k+j}, \quad 0 \leqslant k \leqslant m,\ 1 \leqslant j \leqslant 2m+2-2k$$

As a consequence, $\frac{\varpi_0}{1-\varpi_0}\|\Delta^{[m]}\|_{L^\infty(0,\tau_0)} \leqslant C_m M\varepsilon^{m+1}$.

Lemma 3.3. *The following inequality is valid*:

$$|\widetilde{S}_\ell^{[m]}(\tau)| + |\widetilde{S}_r^{[m]}(\tau_0 - \tau)| \leqslant C_m M\big(J(\tau, \varpi_0) + \varepsilon^{m+1}\big) \quad \forall\tau \in [0, \tau_0]$$

As a consequence, $\|\widetilde{S}_\ell^{[m]}\|_{L^\infty(\tau_1,\tau_0)} + \|\widetilde{S}_\ell^{[m]}\|_{L^\infty(0,\tau_0-\tau_1)} \leqslant C_m M\varepsilon^{m+1}$.

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References

[1] S. Chandrasekar, Radiative Transfer, Calderon Press, Oxford, 1950.
[2] I.W. Busbridge, The Mathematics of Radiative Transfer, University Press, Cambridge, 1960.
[3] V. Kourganoff, Basic Methods in Transfer Problems, Dover, New York, 1963.
[4] V.V. Sobolev, A Treatise on Radiative Transfer, Van Nostrand, Pricenton, NJ, 1963.
[5] K.M. Case, P.F. Zweifel, Linear Transport Theory, Addison-Wesley, Reading, MA, 1967.
[6] K.M. Case, E. de Hoffman, G.P. Plachek, Introduction to the Theory of Neutron Diffusion, Los Alamos Scientific Laboratory, US Goverment Printing Office, Washington, DC, 1953.
[7] V.V. Ivanov, Radiative Transfer and Spectra of Celestial Bodies, Nauka, Moscow, 1969; Traduction révisée: Transfer of Radiation in Spectral Lines, in: National Bureau of Standards Special Publication, Vol. 385, US Government Printing Office, Washington, DC, 1973.
[8] G.P. Panasenko, B. Rutily, O. Titaud, Asymptotic analysis of integral equations for great interval and its application to stellar radiative transfer, C. R. Mecanique 330 (2002) 735–740.
[9] A.A. Amosov, G.P. Panasenko, B. Rutily, An approximate solution to the integral radiative transfer equation in an optically thick slab, in: 7th International Conference on Integral Methods in Science and Engineering, Saint-Etienne, France, 2002, pp. 12–13.
[10] M.A. Heaslet, E.E. Warming, Radiative source-function predictions for finite and semi-infinite non-conservative atmospheres, Astrophys. Space Sci. 1 (1968) 460–490.
[11] T. Viik, On an exact solution of the Ambartsumian equation, Tartu Astr. Obs. Publ. 44 (1976) 47–57.