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Nonlinear models of vehicular traffic flow – new frameworks of the mathematical kinetic theory

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Abstract

This paper deals with the design of mathematical frameworks for the modeling of traffic flow phenomena by suitable developments of classical models of the kinetic theory. Various types of evolution equations are deduced, and different mathematical structures are proposed toward conceivable applications. ***To cite this article: M. Delitala, C. R. Mecanique 331 (2003).***

Résumé

Modèles non-linéaires de trafic véhiculaire. Nouvelles structures de la théorie mathématique cinétique. Ce travail est consacrè à la construction des structures mathématiques pour modéliser des phénomènes de traffic véhiculaire en utilisant des développements appropriés des équations classiques de la théorie cinétique. La dérivation de divers types d'équations d'évolution et diverses structures mathématiques vers des applications appropriées sont proposées. ***Pour citer cet article : M. Delitala, C. R. Mecanique 331 (2003).***

Keywords: Dynamical systems; Traffic flow; Kinetic theory; Nonlinear sciences

Mots-clés : Systèmes dynamiques ; Flux du traffic ; Théorie cinétique ; Sciences non-linéaires

1. Introduction

The mathematical description of traffic flow phenomena can be obtained by methods of the mathematical kinetic theory exploiting generalizations of the classical models of the kinetic theory of gases toward modeling large complex systems of interacting objects-individuals, [1,2]. The generalization of the Boltzmann equation to traffic flow modeling was first proposed by Prigogine and Hermann [3], and then developed by various authors. The interested reader can recover the pertinent literature in various reviews, among others Helbing [4], and Bellomo, Coscia and Delitala [5], which provide the background for the contents of this paper.

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As known, the Boltzmann equation applied to traffic flow modeling shows some contradictions which limit its effective applicability. For instance, particles in a gas move in all directions in the space, while vehicles move in one direction only. This means that perturbations are transmitted only in the direction of the motion. Moreover, the Boltzmann equation is a model for a diluted gas, while relevant traffic flow phenomena are observed in dense traffic flow conditions, as carefully analyzed by Kerner and coworkers [6]. Additional criticisms can be recovered in various papers, e.g., [7,8], which motivate a deep insight into the design of the structure of equations which can be effectively used toward modeling. Suitable modifications of the classical Boltzmann or Vlasov equations may generate new classes of models, hopefully closer to physical reality.

The aim of this paper consists in suggesting some developments of the above mentioned classical models of the kinetic theory in order to obtain a description of traffic flow phenomena suitable to overcome, at least in part, the above contradictions. The contents are developed through three sections which follow this Introduction. Section 2 deals with the modeling of microscopic interactions. Section 3 defines the frameworks, the class of kinetic equations used toward modeling; the derivation of evolution equations is essentially based on different ways of modeling the microscopic interactions. The last section develops some reasoning about new approaches to mathematical modeling.

2. Modeling microscopic interactions

Consider the one dimensional flow of vehicles along a road with length ℓ. The microscopic state of vehicles is assumed to be identified, at each instant of time, by dimensionless position and velocity of each vehicle. Dimensionless quantities are obtained dividing the physical variables by suitable characteristic ones. Consequently, it can be defined:

- $x = x_R/\ell$: the dimensionless space variable which identifies the position of the vehicle located in the point x_R.
- $v = v_R/v_M$: the dimensionless velocity of each vehicle referred to v_M, where v_R is the real velocity of the single vehicle and v_M is the maximum mean velocity which may be reached by vehicles in the empty road. Considering that a fast isolated vehicle can reach velocities larger than v_M, a limit velocity can be defined as follows: $v_\ell = (1+\mu)v_M$, with $\mu > 0$. Both v_M and μ may depend on the characteristics of the lane, say a country lane or an highway, as well as to the type of vehicles, say slow and fast cars.
- $t = t_R/T$: the dimensionless time variable referred to T, where t_R is the real time and T is the characteristic time, generally assumed to be the time necessary to cover the whole road length at the maximum mean velocity v_M ($T = \ell/v_M$).

In the *kinetic* (*Boltzmann*) *description*, the microscopic state of each element of the system is still identified by position and velocity; however the identification of the whole system refers to a suitable probability distribution function over the above microscopic state: $f = f(t, x, v)$, where $f\,\mathrm{d}x\,\mathrm{d}v$ is the number of vehicles which at the time t are in the phase domain $[x, x+\mathrm{d}x] \times [v, v+\mathrm{d}v]$. Macroscopic observable quantities can be obtained, under suitable integrability assumptions, as momenta of the distribution f. The distribution function can be divided by u_M, the maximum density corresponding to bump-to-bump traffic jam. In this way, the first order momentum of the distribution function, the density $u = u(t, x)$, results to be dimensionless.

As known in kinetic theory, the derivation of evolution equations needs the modeling of pair interactions at the microscopic level. Let consider pair interactions between a *test* vehicle with state $\{x, v\}$ and a *field* vehicle with state $\{y, w\}$. Following two classical models of the kinetic theory, see, e.g., [9], different types of interactions can be distinguished: *localized interactions*, referred to the Boltzmann or Enskog equation, which occur when both vehicles are at a minimal distance, and *mean interactions*, referred to the Vlasov equation, which occur when the field vehicle is within the interaction domain D_x of the test vehicle. The size of D_x depends on the “visibility area” of the test driver.

The microscopic models may be defined by the following quantities:

- $\eta(v, w)$ is the *encounter rate*, the number of interactions between pair of vehicles per unit time in the unit volume.
- $A(v^*, w^*; v)$ is the *transition probability density* that a vehicle with velocity v^* interacting with a vehicle with velocity w^* ends up into the velocity v. The density A must be equal to zero for $v \geqslant 1 + \mu$. Interactions are localized either in the point x of the test vehicle or at a fixed distance on its front.
- $\mathcal{F}(x, y, v, w^*)$ is the *positional acceleration* applied to the vehicle in x with velocity v by the one in y with velocity w^*.
- $\varphi(x, y)$ models the *weight of the action* on the driver of the test vehicle in x due to the interactions with the field vehicle in y within the "visibility area" $D_x = [x - \Delta_r, x + \Delta_f]$ of the test vehicle, where Δ_r and Δ_f are respectively the rear and frontal visibility distance of the test vehicle. For $y \in D_x$ the weight $\varphi(x, y)$ must be such that $|x - y| \uparrow \Rightarrow \varphi \downarrow$ and its integral in dy over the domain D_x is equal to 1. Considering that frontal stimuli are relatively more relevant then rear ones, the approximation $\Delta_r \simeq 0$ can be possibly adopted.

In the above description, the microscopic interactions are the same for all vehicles. On the other hand, as observed in [7], different types of drivers-vehicles have to be considered: fast, aggressive, slow, shine, and so on. The simplest way of modeling this aspect consists in assuming that the specificity of drivers is related to a certain random variable ω in a suitable domain D_ω linked to a suitable probability density $P(\omega)$. Microscopic interactions depend also on the values ω and ω^* of the interacting pair. In this case the distribution function has to be parameterized, while the averaged distribution is:

$$f(t, x, v) = \int_{D_\omega} f(t, x, v; \omega) P(\omega)\, d\omega. \tag{1}$$

3. Mathematical frameworks toward kinetic modeling

In the kinetic approach, traffic flow models are derived, exploiting the microscopic modeling, by suitable balance relations in the phase-space volume $dx\, dv$, which are obtained equating the total derivative of the distribution function to the difference between the inlet and the outlet of vehicles in the said volume. Therefore the development of a proper mathematical methodology for different classes of models may be useful.

3.1. Boltzmann-like models with binary interactions

Boltzmann-like models with binary interactions are deduced referring to the microscopic modeling of binary interactions between test and field vehicles, as described by the terms η and A defined in Section 2. If the interactions are localized in the point x for both field and test vehicles, then the formal structure of the evolution equation writes as follows:

$$\begin{aligned}\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = &\int_0^{1+\mu}\int_0^{1+\mu} \eta(v^*, w^*) A(v^*, w^*; v) f(t, x, v^*) f(t, x, w^*)\, dv^*\, dw^* \\ &- f(t, x, v) \int_0^{1+\mu} \eta(v, w^*) f(t, x, w^*)\, dw^* \end{aligned} \tag{2}$$

where on the right-hand side there is the difference between the inflow (gain) and outflow (loss) of vehicles in the control volume of the phase space. These terms are generally integral operators on f.

The above description may lead to good results in homogeneous traffic flow situations; of course the description is satisfactory if the microscopic modeling is correct. Stationary solutions and fundamental diagram can be well fitted. On the other hand, non-homogeneous traffic flow situations generate technical problems: traffic flow models have only positive velocities, and backwards propagation of a perturbation is not described, in evident contrast with the real traffic flow in which a jam moves also backwards.

3.2. Models with Enskog-like interactions

Enskog-like models have a structure analogous to the Boltzmann-like models with binary interactions. The main difference is that the effects of the finite size of the vehicles are taken into account. Namely, in the Boltzmann-like approach, to obtain a closed equation for $f(t,x,v)$, the chaos approximation is made for the joint distribution function of the test vehicle and the field one. It is assumed the statistical independence: the number of pairs of test vehicle with velocity v at the position x and field vehicle in y with velocity w is simply the product of the single vehicle distribution functions. Of course, this assumption is reasonable only for low density of traffic, when the drivers behave independently. At higher densities, the velocities of the cars must be correlated.

Several different approaches have been proposed, here we report the one proposed in [10]. The joint distribution function, denoted by $f^{(2)}$, is assumed to be:

$$f^{(2)}(t,x,v,y,w) \sim c\big(d_i(v,w)\big) f(t,y,w) f(t,x,v)/u(x) \tag{3}$$

where the function $c = c(d_i(v,w))$ is the correlation function between test and field vehicle depending, in a phenomenological way, on the reaction thresholds d_i of the driver, at least braking and accelerating thresholds. Interactions of the test vehicle are assumed to happen only when a threshold distance is crossed, and are supposed to be localized with a field vehicle in the position $y_i = y_i(v,w) = x + d_i(v,w)$.

The framework (2) can be rewritten in the following form:

$$\begin{aligned}\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = {} & \sum_{i=1}^{2} \int_0^{1+\mu} \int_0^{1+\mu} \eta(v^*,w^*) A_i(v^*,w^*;v) f^{(2)}\big(t,x,v^*,\big(x+d_i(v^*,w^*)\big),w^*\big)\,\mathrm{d}v^*\,\mathrm{d}w^* \\ & - \sum_{i=1}^{2} \int_0^{1+\mu} \eta(v,w^*) f^{(2)}\big(t,x,v,\big(x+d_i(v,w^*)\big),w^*\big)\,\mathrm{d}w^* \end{aligned} \tag{4}$$

which allows backwards propagation of the information.

3.3. Boltzmann models with averaged binary interactions

In the kinetic (Boltzmann) models with averaged binary interactions, binary microscopic interactions are weighted in the visibility area of the test vehicle. Then, the structure defined in (2) can be used with the introduction of the weight function $\varphi(x,y)$ defined in Section 2 as follows:

$$\begin{aligned}\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = {} & \int_{D_x} \int_0^{1+\mu} \int_0^{1+\mu} \varphi(x,y)\eta(v^*,w^*) A(v^*,w^*;v) f(t,x,v^*) f(t,y,w^*)\,\mathrm{d}v^*\,\mathrm{d}w^*\,\mathrm{d}y \\ & - f(t,x,v) \int_{D_x} \int_0^{1+\mu} \varphi(x,y)\eta(v,w^*) f(t,y,w^*)\,\mathrm{d}w^*\,\mathrm{d}y \end{aligned} \tag{5}$$

It is immediate to show that Eq. (5), assuming $\varphi(y) = \delta(y-x)$ where δ denotes Dirac's delta function, gives a localized interaction model. Analogous reasoning can be applied to Enskog-type models.

3.4. Mean field kinetic models

Mean field models are derived under quite different ideas. Similarly to Vlasov type models, it is defined a mean field action on the test vehicles due to the field vehicles, in principle more than one. The structure of the evolution equation is as follows:

$$\frac{\partial f}{\partial t} + v\,\frac{\partial f}{\partial x} + \frac{\partial}{\partial v}\big(\mathcal{A}[f]f\big) = 0 \tag{6}$$

where $\mathcal{A}[f]$ is the acceleration/deceleration due to the interactions with the field vehicles which affect the test one. The mean field description gives $\mathcal{A}[f]$ by means of a suitable interaction term which generates the action

$$\mathcal{A}[f](t,x,v) = \int_{D_x} \int_0^{1+\mu} \mathcal{F}(x,y,v,w^*) f(t,y,w^*)\,\mathrm{d}y\,\mathrm{d}w^* \tag{7}$$

where $\mathcal{F}$, according to the definition given in Section 2, is the positional acceleration applied by the field vehicle in y with velocity w^* to the test one in x with velocity v.

3.5. Discrete velocity models

Discrete velocity models in kinetic theory are based on the assumption that particles can attain a finite number of velocities. The interest of applied mathematicians to the above class of models is documented in the book edited by Bellomo and Gatignol [11] and in various recent papers, e.g., D'Almeida and Gatignol [12] on moving boundary problems, and Bellouquid [13] on the asymptotic theory toward macroscopic models.

Developing discrete velocity models in kinetic theory appears to be particularly interesting considering that vehicles are often observed to move along highways with group velocities, thus creating "clusters" of vehicles related to certain sets of velocities. Technically, developing a discrete velocity model of traffic flow means selecting a discrete number of velocities $I_v = \{v_0 = 0, \ldots, v_i, \ldots, v_{n+1} = v_\ell\}$ and linking to each velocity a density $f_i = f_i(t,x)$ for $i = 0, 1, \ldots, n+1$, such that $f_0 = f_{n+1} = 0$. The mathematical model is a set of evolution equations for the densities which can be formally written as follows:

$$\frac{\partial f_i}{\partial t} + v_i\,\frac{\partial f_i}{\partial x} = G_i[f] - L_i[f], \quad i = 1, \ldots, n \tag{8}$$

where $f = \{f_1, \ldots, f_n\}$ and where the collision term has been split into gain and loss terms. Specific models can be obtained by suitable modeling of microscopic interactions within a specialization of the frameworks we have seen above.

3.6. Models with stochastic interactions

We have seen in Section 2 that the specific behavior of drivers can be represented by a random variable ω linked to a suitable probability density $P(\omega)$: this aspect can be related to each one of the above mathematical frameworks, thus avoiding the hypothesis that all drivers behave in the same way.

This aspect can be related to each one of the above frameworks. For instance, it can be referred to models with localized interactions and, supposing that the above distribution is not modified by interactions, the evolution equation writes:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = \int_{D_\omega} \int_0^{1+\mu} \int_0^{1+\mu} \eta(v^*,w^*) A(v^*,w^*;v|\omega,\omega^*) f(t,x,v^*,\omega) f(t,x,w^*,\omega^*)\,\mathrm{d}v^*\,\mathrm{d}w^*\,\mathrm{d}\omega^*$$

$$- f(t,x,v) \int_{D_\omega} \int_0^{1+\mu} \eta(v,w^*) f(t,x,w^*,\omega^*)\,\mathrm{d}w^*\,\mathrm{d}\omega^*. \tag{9}$$

4. Perspectives

This paper has analyzed some conceptual aspects which are preliminary to the modeling of traffic flow phenomena. It has been shown how microscopic modeling can generate various classes of models. Actually not all of them have been exploited yet, therefore the contents of this paper provides a background toward modeling which is broader than the one available in the literature.

A crucial aspect is the selection of the specific framework. The choice has to be based not only on the interpretation of the phenomenology of the system, but also on the effective possibility of modeling microscopic interactions. Moreover, additional frameworks can be developed: the introduction of stochastic interactions in the discrete velocities framework is not only a method to reduce computational complexity, but also a way to describe interesting phenomena (clustering of vehicles), and taking into account the specificity of the drivers.

The author is aware that the specific models need to be based on further technical developments of the mathematical frameworks offered in the above brief presentation. Nevertheless, the essential message given in this paper is that modeling of traffic flow phenomena needs substantial development (and modifications) of the classical equations of the mathematical kinetic theory. Various proposals in this direction have been given in this paper.

References

[1] L. Arlotti, N. Bellomo, E. De Angelis, Generalized kinetic (Boltzmann) models: mathematical structures and applications, Math. Models Methods Appl. Sci. 12 (2002) 579–604.
[2] N. Bellomo, M. Pulvirenti (Eds.), Modelling in applied sciences: a kinetic theory approach, Math. Models Methods Appl. Sci. 12 (2002), Special Issue.
[3] I. Prigogine, R. Herman, Kinetic Theory of Vehicular Traffic, Elsevier, New York, 1971.
[4] D. Helbing, Traffic and related self-driven many-particle systems, Rev. Modern Phys. 73 (2001) 1067–1141.
[5] N. Bellomo, V. Coscia, M. Delitala, On the mathematical theory of vehicular traffic flow I – Fluid dynamic and kinetic modeling, Math. Models Methods Appl. Sci. 12 (2002) 1801–1844.
[6] B. Kerner, Sincronized flow as a new traffic phase and related problems of traffic flow, Math. Comp. Modelling 35 (2002) 481–508.
[7] C. Daganzo, Requiem for second order fluid approximations of traffic flow, Transportation Res. Part B 29 (1995) 277–286.
[8] S. Darbha, K.R. Rajagopal, Limit of a collection of dynamical systems: an application to modelling the flow of traffic, Math. Models Methods Appl. Sci. 12 (2002) 1381–1402.
[9] C. Cercignani, R. Illner, M. Pulvirenti, The Mathematical Theory of Dilute Gases, Springer, New York, 1994.
[10] A. Klar, R. Wegener, Enskog-like kinetic models for vehicular traffic, J. Statist. Phys. 87 (1/2) (1997) 91–114.
[11] N. Bellomo, R. Gatignol, Lecture Notes on the Discretization of the Boltzmann Equation, World Scientific, London, 2002.
[12] A. D'Almeida, R. Gatignol, The half-space problem in discrete kinetic theory, Math. Models Methods Appl. Sci. 13 (2003) 99–120.
[13] A. Bellouquid, Diffusive limit of the nonlinear discrete velocity models, Math. Models Methods Appl. Sci. 13 (2003) 33–58.