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Linear bending of star-like pyramids

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Abstract

A family of polyhedra possessing unusual deformation properties is found. On one hand, models of these polyhedra admit free continuous large reversible bending without visible distortions of the material. On the other hand, the polyhedra themselves are rigid and do not admit continuous bending in the sense of the Cauchy definition. The found polyhedra are called model flexors in order to distiguish them from theoretical flexsors of Connelly. Bendings of the models are asymptotically exactly approximated by linear bendings of polyhedra. They represent a nonrigid, soft or slow, loss of stability that corresponds to the loss of stability in small accordingly to Euler. This new phenomenon in mechanics of deformable solid bodies may be considered as an original geometric machine of catastrophe. ***To cite this article: A.D. Milka, C. R. Mecanique 331 (2003).***

Résumé

Flexions linéaires des pyramides etoilées. On exhibe une famille de polyèdres qui possèdent des propriétés inhabituelles de déformations. D'une part, les modèles de ces polyèdres admettent des flexions libres continues, grandes, réversibles, sans distorsions visibles du matériel. D'autre part, les polyèdres sont rigides et n'admettent pas des flexions continues dans le sens de la définition de Cauchy. Les polìedres décris sont appelés des flexors modèles pour les distinguer des flexors théoriques de Connelly. Des flexions de ces modèles sont approximées asymptotiquement par des flexions linéaires des polyèdres. Elles représentent une perte de stabilité, douce, qui correspond à la perte de stabilité « in small » conformement à la définition de Euler. Ce nouveau phénomène dans la mécanique de corps solides déformables peut être considéré comme l'origine d'un processus de catastrophe géométrique. ***Pour citer cet article : A.D. Milka, C. R. Mecanique 331 (2003).***

Keywords: Dynamics of rigid or flexible systems; Rigidity and bendings of polyhedra; Linear bending; Nonrigid, soft or slow, loss of stability; Supercritical deformations of shells

Mots-clés : Dynamique des systèmes rigides ou flexibles ; Rigidité et flexions de polyèdres ; Flexion linéaire ; Perte de stabilité non-rigide, douce ou ralentie ; Déformations supercritiques d'enveloppes

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1. General representation of results

In 1978 the American mathematician Connelly discovered a nonrigid polyhedron, a flexor – a simple closed polyhedron which admits a continuous bending in the sense of the classical definition given by Cauchi [1]. The physical polyhedral model of this polyhedron admits a free geometrical bending without visible distortions of material. Naturally, a reason for the model to admit such free bending is that the polyhedron admits a classical mathematical bending. In 2000 the author discovered a curious family of rigid polyhedra, star-like bipyramids of Alexandrov and Vladimirova [2], called *model flexors* [3–5]. A model flexor is a polyhedron which is rigid, i.e. it does not admit any continuous bending in the sense of Cauchi's definition; however its physical polyhedral model admits free bendings without visible distortions of material, as well as Connelly's model. In this case the reason for the physical model to be flexible is the existence of mathematical *linear bendings* of the polyhedron. A linear bending of a polyhedron is its isometric continuous deformation in the class of polyhedra with change of sizes and forms of some faces. Such kind of isometric deformations of polyhedra was introduced earlier by the author, Zalgaller and Burago, Bleeker (1994–1996). The linear bending of bipyramids may be qualified in terms of the theory of dynamical systems as a nonrigid, soft or slow, loss of stability with large supercritical deformations [6].

In this Note we present the construction of model flexors and carry out geometrical, analytical, numerical and graphic descriptions of linear bendings of star-like pyramids with sliding of boundary edges in a plane. Relations of deformations of physical models of bipyramids with the mechanics of deformations of thin shells are found out at a level of asymptotically exact mathematical simulations and based on known general principles produced in wide technical practice and in numerous laboratory experiments. Accordingly to the geometrical theory, these deformations are numerically identified with continuous bendings of mean surfaces of shells [7]. Considered as a two-parametrical dynamic system, the linear bending of a star-like pyramid represents an original geometrical machine of catastrophe supplementing the well studied physical models of Ziman and Poston [8].

It should be noticed that the result of Connelly denies the Euler hypothesis about the non-existence of continuous bendings of surfaces such as spheres and of their models, in certain classes of regularity. On the other hand, the discussed results of the author confirm Euler's hypothesis that the general type of loss of stability of a surface and of its model may be viewed as a transition to statically possible equilibrium forms indefinitely close to initial ones. Expanding Connelly's discovery, these results establish a new phenomenon in the theory of large supercritical deformations of solid bodies and represent a significant interest for geometry, theory of dynamical systems, mechanics, for various applications, technical and geophysical ones in particular.

2. Analogies in classical mechanics

It is known that the rigidity of the middle surface of a thin elastic shell is accepted as a criterion of the stability of the shell itself. In building mechanics this principle of rigidity of shell-like forms is formulated for polyhedral models by a group of Canadian mathematicians and architects in 1978 [9]. The discovery of model flexors shows that the principle of rigidity of shell-like forms is insufficient; we should also take into account how a shell may lost its stability. Intuitively this conclusion seems to have been known by Connelly in 1974. Relying on a statement similar to a Gluck theorem, Connelly in fact stated and discussed a hypothesis about the existence of model flexors [10,11]. For regular shells, a similar conclusion was made by Goldenveizer in 1979 [12]. The matter is that in classical mechanics, where small deformations of shells are examined, a theorem of non-existence of bending was considered as a principle of rigidity. This principle stated that if a surface is rigid, i.e. if it does not admit non-trivial infinitesimal bending, then its physical model is a rigid shell. However, contrary to the stated principle, some non-convex flexible shells, such as torii and bent pipes, with rigid mean surfaces were discovered. The reasons for such interesting phenomenon, which is inconsistent for the given class of shells, is the *tangential bending* of surfaces which has been missed by mechanics. Called a *pseudo-bending*, it is accompanied by the loss of continuity, either for displacement fields, or for rotation fields, along some lines. The phenomenon of pseudo-bending was studied by Goldenveizer's school with help of mathematical simulations, by geometrical methods and in a vein of the mathematical theory of stability of elastic shells. Essential amendments were brought into the theory, in particular a new fundamental notion, *a physically nonrigid shell*, i.e. a mathematically rigid shell which admits a pseudo-bending, was introduced. It was also found that analogous shells have been already applied for a long time in technics as various equalisers of deformations. The notion of model flexors introduced by the author for characterise supercritical deformations of an elastic shell is an analogue of the notion of physically nonrigid shells applied by Goldenveizer for the characterisation of pre-critical deformations. From the geometric point of view, the two kind of bending, which correspond to the discussed unusual deformations of shells, were provided by Kon–Vossen in the theory of surfaces, when he specially allocated the bendings with lines of discontinuity and with singular floating edges. Probably these variants of deformation of shells with singularities, caused by the loss of stability, were meant also by Euler to be what the formulation of his well-known hypothesis testifies: "the closed spatial figure does not suppose changes *while it is not torn...* ".

3. Geometry and dynamics of bendings of pyramids

The discussed flexors-bipyramids are constructed with the help of right star-like pyramids which do not admit classical continuous bending, with sliding of boundary edges in a plane. Now we will present the precise description of model flexsors. For this aim, particular star-like pyramids and their special isometric deformations within the class of the polyhedra, linear bendings, are examined. Each of the pyramids represents a cyclic-symmetric polyhedron with a right star as the base.

Consider a cyclically repeating element of the star, referred to as *a petal of the star*; it is a convex quadrangle made of two equal triangles. For every component triangle, the inner and exterior doubled angles adjacent to the boundary of star are equal to $\pi/2 - \alpha$ and $\pi/2 + \alpha$, where α is the angle of the triangle at the center of the star. Star-like pyramids with such particular bases were investigated for the first time by Aleksandrov and Vladimirova in order to construct classical discret bendings of polyhedra. A petal of the pyramid is also formed from two equal triangles, faces of the pyramid. A face of the pyramid is orthogonally projected onto a component triangle of the star. Let β and γ stand for the angles of the face which correspond under orthogonal projection to the angle α and to another acute angle of the component triangle. Denote by H the height of the pyramid and by η the angle between a concave edge and the axis of the pyramid. Fig. 1 represents the base star of a triangular pyramid, which is normalized to avoid the homothety, the lengths of sides of the component triangle are indicated. The written formulae concern an arbitrary n-angular normalized pyramid, the corresponding angle is $\alpha = \pi/n$. The considered star-like pyramid admits the following linear bending. The top vertex of the pyramid is displaced vertically along the axis. Other vertices, i.e. the vertices of the base star, are displaced along the fixed rays going out from the center of the star. The faces of the pyramid are broken along 'floating' edges as is shown in Fig. 1 where the projection of edges of break are marked by dashed lines. Each petal of the pyramid contains two 'floating' edges which meet at a 'floating' point of break in the original edge of petal. During the bending the point of break runs all the edge of the petal and always remains in the corresponding plane of symmetry of this petal. Cyclic symmetry of the pyramid during deformation is naturally preserved too. We designate u the displacement of the top vertex of the pyramid, supposing that it is positive when the displacement is toward the base star. Let z be the deviation of the point of break from the axis of the pyramid; it is assumed to be positive in the case when the broken original edge does not cross the axis. Let s denote the length of a piece of the broken edge between the top vertex and the point of break, and ν stand for the inclination angle between other piece of the broken edge and the base star plane; it is taken to be positive if the point of break is above the base star plane. Take the variable u as a parameter of the deformation. For different values of u correspond different polyhedra, each of which is an isometrical deformation of the original star-like pyramid. Thus we obtain a continuous sequence of isometric polyhedra, the variable u being considered as a parameter of this sequence. Remark that u varies within an interval where the angle ν as a function of u is well-defined, without taking into account whether ν is positive or negative.

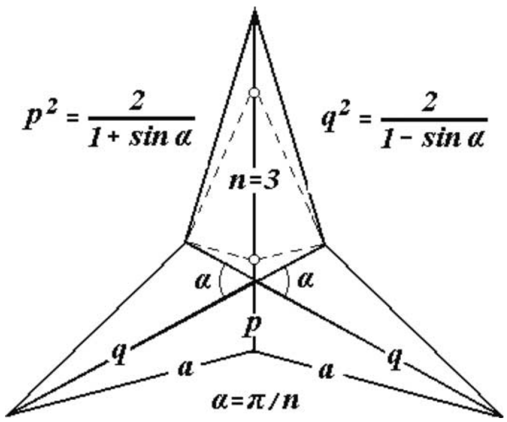

Fig. 1. The base of a triangular star-like normalized pyramid.

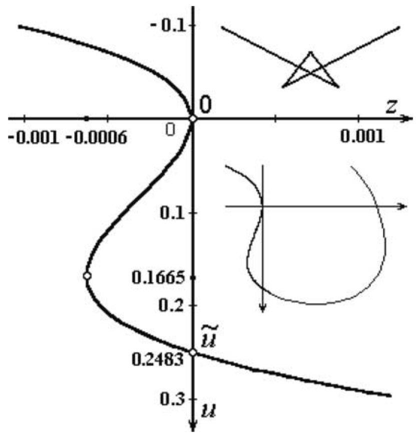

Fig. 2. The phase plane for the linear bending of the triangular pyramid.

Lemma 3.1. *The inclination angle ν, the phase z and the length s for the normalized pyramid are represented in term of the parameter u by the following expressions*:

$$\sin^2\nu = \frac{(H^2+p^2)\sin^2\beta - (p^2+v)\sin^2\alpha}{a^2-(p^2+v)\sin^2\alpha}, \qquad z = Q - \left(\sqrt{H^2+q^2}-s\right)\cos\nu$$

$$s^2 = \left(Q - \left(\sqrt{H^2+q^2}-s\right)\cos\nu\right)^2 + \left(H-u-\left(\sqrt{H^2+q^2}-s\right)\sin\nu\right)^2$$

where

$$\sin^2\beta = 1-\left(H^2+2\right)^2/\left(\left(H^2+p^2\right)\left(H^2+q^2\right)\right), \qquad v = H^2-(H-u)^2$$

$$Q = \sqrt{p^2+v}\cos\alpha + \sqrt{a^2-(p^2+v)\sin^2\alpha}$$

$$H^4/\left(H^2+q^2\right) \leqslant (H-u)^2 \leqslant H^2+p^2.$$

Note that these formulae and the geometrical description of the constructed linear bending of the pyramid are still valid for the degenerate case when the height vanishes, $H=0$. In Fig. 2, in the phase plane (u,z) it is shown the phase curve of deformation, $z=z(u)$, for the triangular star-like pyramid with $H=1$ when the break point appears near the top vertex of the pyramid. The interval $0<u<\tilde{u}=0.2483\ldots$ corresponds to the period of the slow loss of stability when the flexed pyramid has butterfly-like self-crossings along the concave edges. The thin line represents the complet phase curve.

In Fig. 3, the plane of managing parameters (H,α) for the family of normalized pyramids is presented. The separatrix S, i.e. the line of bifurcation given by $\beta=\eta$, is shown; it separates two classes of pyramids which are distinguished by the kind of the slow loss of stability. The dashed line shows another separatrix which is given by $\beta=\gamma$. It separates another two classes of pyramids whose physical model may be distinguished accordingly to how the linear bending starts, either at the top vertex or at the vertices of the base star. The formulae for the separatrices, for the point of phase change, $\tilde{u}$, and for the corresponding length $\tilde{s}$ are:

$$1/H^2 = q^2/4-1, \qquad H^2/4 = q^2/4-1, \qquad \sqrt{H^2+p^2}\cos\beta = H-\tilde{u}$$

$$\left(\sqrt{H^2+q^2}-(H-\tilde{u})\right)\tilde{s} = \sin\alpha\left(\tilde{v}\sin\alpha+2-\sqrt{(\tilde{v}\sin\alpha+2)^2-\tilde{v}^2}\right)$$

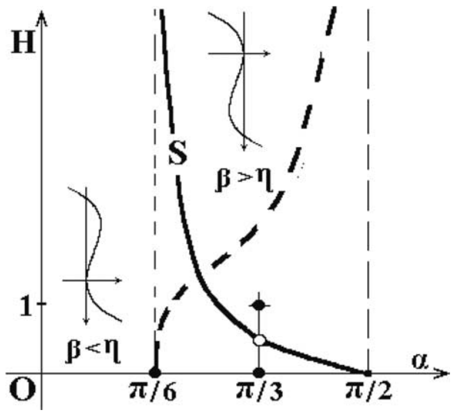

Fig. 3. The plane of managing parameters for the family of linearly flexible normalized star-like pyramids.

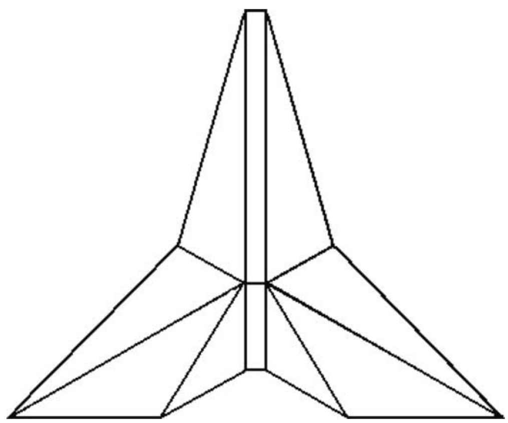

Fig. 4. The base of a complex model flexor with elements of triangular bipyramid and hexagonal ones.

4. Elementary physical models

The mathematical rigidity of the considered star-like pyramids and a sufficiently free flexibility of their physical models, the so-called model flexors, can be easily verified for the triangular star-like bypiramids formed by pyramids of equal heights. Their physical models are optimal in the sense that they have only *eight* vertices. For constructing two concrete models, let us take the following length of edges, in mm; the deviations do not exceed 0.1 mm:

$$87, 36, 100 \quad \text{and} \quad 87, 26, 97.$$

At the starting position, the heights of the pyramids are equal to 25 mm and 00 mm respectively. The models admit free bendings, with large amplitude and without apparent distortions of the material. Another substantial example of freely flexible model is the hexagonal star-like bipyramid with zero height. Real models of these bipyramids were constructed by the author from a carton of high quality with thickness 0.25 mm. The first of these models, which withstood hundreds of bending cycles, is working well during 5 years. In Fig. 3 the three pyramids mentioned are marked by dark dots. During the deformations, each of the big edges of the pyramids are one-point broken and the small ones are not broken. In a first period, the soft deformation, the length s of a piece of broken edge and the deviation of break points from the axis, z, are of the second order with respect to the deflection u of the top vertex. The slow loss of stability corresponds to deformations with a reduction by quarter of the height of pyramids in the first case, and with an increase from zero to 13 mm of the heights of pyramids in the second case. The relative changes of the lengths of edges during the deformations of the bipiramids does not exceed 0.00269 and 0.00137 respectively. For the examples given, these relative changes are comparable with the basic characteristics of construction materials, such as metals and their alloys. Formally, roughly speaking, bending similar to soft and slow loss of stability are typical for all star-like bipyramids examined, and for more complicated polyhedral models constructed with help of star-like pyramids; an example is shown in Fig. 4.

5. Conclusion

The results presented are based on geometrical beginnings, following Minkowski; they develop concrete mathematical and physical results by Aleksandrov and Connelly, Arnold, Pogorelov and Goldenveizer. They determine a priority direction of theoretical and applied study for the process of bending of surfaces and their models. They also stimulate non-standard research on the widespread, and frequently inexplicable, processes of the destruction of thin elastic shells at large deformation, connecting phenomena of nonrigid and rigid loss of stability.

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References

[1] R. Connelly, A flexible sphere, Math. Intell. 1 (3) (1978) 130–131.
[2] A.D. Aleksandrov, S.M. Vladimirova, About bending of polyhedrons with solid faces, Vestnik Leningr. Gos. Univ. Mat. Mekh. Astronom. 13 (3) (1962) 138–140.
[3] A.D. Milka, The star-like pyramids of A.D. Alexandrov, S.M. Vladimirova, Siberian Adv. Math. 12 (2) (2002) 56–72.
[4] A.D. Milka, Bendings of Surfaces, Bifurcation, dynamical systems and stability of shells, Abstracts Intern. Congress of Mathematicians, Beijing, China, 2002.
[5] A.D. Milka, Bendings of surfaces and stability of shells, Mat. Trudy (Siberian Branch of the Russian Acad. Sci.), 2002, submitted for publication.
[6] V.I. Arnold, The Theory of Catastrophe, Nauka, Moscow, 1990.
[7] A.V. Pogorelov, Bendings of Surfaces and Stability of Shells, Naukova Dumka, Kiev, 1998.
[8] T. Poston, I. Stewart, Catastrophe Theory and its Applications, Pitman, London, 1978.
[9] J. Baracs, H. Crapo, I. Rosenberg, W. Whiteley, Mathematiques et architecture, Topologie Structurale (Montreal) 41–42 (1978) 44–59.
[10] R. Connelly, An attack on rigidity, Preprint, Cornell University, 1974.
[11] H. Gluck, Almost all simply connected closed surfaces are rigid, in: Lecture Notes in Math., Vol. 438, 1975, pp. 225–240.
[12] A.L. Goldenvejzer, Mathematical rigidity of surfaces and physical rigidity of shells, Izv. Akad. Nauk SSSR. Mekh. Tverd. Tela 6 (1979) 66–77.