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‘Les fleurs du mal’ – an adaptive wavelet method of arbitrary lines I: convection–diffusion problems

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Abstract

Baudelaire’s ‘les fleurs du mal’ refers to various new developments (‘les fleurs’) of the **m***ethod of* **a***rbitrary* **l***ines* (**mal**), since it was first published (in C. R. Acad. Sci. Paris, Sér. I, in 1991). Here we revisit the basic mal (semi-discretization) methodology for stationary convection–diffusion problems and develop an adaptive, wavelet-based solver that is capable of capturing the thin layers that arise in such problems. We show the efficacy and high accuracy of the wavelet-mal solver by applying it to a challenging 2D problem involving both boundary and interior layers. ***To cite this article: X. Ren, L.S. Xanthis, C. R. Mecanique 332 (2004).***

Résumé

« Les fleurs du MAL » – une méthode d’ondelettes adaptive de lignes arbitraires I : problèmes de convection–diffusion. « Les fleurs du MAL » font allusion aux quelques nouveaux développements (« les fleurs ») de la *méthode des lignes arbitraires* (MAL), obtenus depuis la première publication (dans C. R. Acad. Sci. Paris, Sér. I, dans 1991). Ici nous rappelons les bases de la méthodologie MAL (semi-discrétisation) appliquée aux problèmes de convection–diffusion stationnaire. Nous développons aussi un programme de résolution adaptatif, basé sur une analyse en ondelettes, capable de capter les situations avec des couches minces, apparaissant dans ce type de problèmes. Nous démontrons l’efficacité et la haute précision de ce programme de résolution par ondelettes, en l’appliquant à un problème bidimensionnel qui pose un vrai défi, traitant à la fois les couches limites et intérieures. ***Pour citer cet article : X. Ren, L.S. Xanthis, C. R. Mecanique 332 (2004).***

Keywords: Computational fluid mechanics; Singularly perturbed problems; Dimensional reduction; Semi-discretization; Adaptive wavelet solver; Boundary and interior layers

Mots-clés : Mécanique des fluides numérique ; Problèmes singulièrement perturbés ; Réduction dimensionelle ; Semi-discrétisation ; Programme de résolution adaptatif par ondelettes ; Couches frontières et intérieurs

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MNEMOSYNE, DÆDALEAN MUSE
Sir George Cayley [1773–1857]
and his towering legacy to Aeronautics
on the CENTENNIAL of the
first powered flight on 17.12.1903
by **Wilbur** and **Orville Wright**[1]

Prolegomena

The ***m**ethod of **a**rbitrary **l**ines* [1–5] yields the acronym, **mal**, whose *amphisemy*, double-meaning, alluding to its *homoechon* epithet, *mal*, featuring in the title of Baudelaire's celebrated poetry, *les fleurs du mal*, we find compelling to using, generically, metaphorically, in the title of this and several Notes to follow. These herald some new developments, *les fleurs*, stemming from mal in various fields. They show that mal can cope quite naturally, and in an elegant and uniform fashion, with the high demands of problems exhibiting strong anisotropic behaviour (e.g., steep gradients, interior- and boundary-layers, moving fronts, localization of deformation effects, inter-laminar stress singularities in composites). In fact, such problems other standard numerical methods either *mal*treat, or require the construction of complicated and *ad-hoc* recipes to achieve satisfactory results. Here mal blossoms in the rough, challenging field of singularly perturbed, convection–diffusion problems, fertilized by some attractive properties of adaptive, wavelet methods. The consilience of mal with these fields, each with their own 'internal' dynamics and vigorous international research, has nurtured the present 'fleur du mal'.

1. Introduction

The ***m**ethod of **a**rbitrary **l**ines* (**mal**) [1–5] constitutes a numerical dimensional reduction methodology, that uses semi-discretization to transform a general elliptic system in variational form, over arbitrary two- and three-dimensional domains, into a system of ordinary differential equations (ODEs) solved (along lines) by a state-of-the-art solver. The distinguishing feature and strength of mal becomes evident when applied to problems involving strong anisotropic behaviour (e.g., thin layers, moving fronts, stress singularities) (see, e.g., [1–5]). This recognizes the fact that high gradients in pluridimensional problems usually exhibit, *locally*, almost one-dimensional (1D) behaviour – that is more economical to simulate using 1D adaptation (we note that 1D theory is generally better understood than 2D and 3D).

In this paper we focus on the important class of singularly perturbed, convection–diffusion problems (1) – which model a diverse array of physical, chemical, biological and financial engineering processes. In the convection-dominated case, these problems develop sharp, boundary- and interior-layers which, despite recent intensive research, they still pose formidable difficulties to all numerical methods – owing to the rapid variation within the thin, especially curved- and interior-layers (see, e.g., the books [6–8] and the recent reviews [9,10]). This is precisely one class of problems to show the advantages of the mal-methodology.

We start with the formulation of (1) into the mal, ODE-setting (6), and then we address the main issue of solving accurately, and efficiently, the resulting ODE system. We do this (in Section 5) by exploiting some

[1] **Scholium.** We invoke MNEMOSYNE [archetypal image of cultural and intellectual memory, mother of the nine Muses] to record our homage to the DÆDALEAN MUSE [archetypal image of flight, inventiveness, ingenuity, arts and sciences, personified by the mythical Dædalus, the first (mortal) to create something out of nothing] who inspired the early aviation pioneers. The epic mission of mechanical flight, crowned with success 100 years ago at Kitty Hawk, North Carolina, USA, was a momentous event for the progress of civilization. Here we pay particular tribute to SIR GEORGE CAYLEY, the most innovative of all aviation pioneers (who, paradoxically, is not as widely known by the general public as the WRIGHT brothers): « La plus haute figure technique, dans le premier tiers du XIXe siècle, est celle de Sir George Cayley, le véritable inventeur de l'aéroplane »– Charles Dollfus and Henri Bouché (France) 1932; "The principle of the airplane, as we know it now, that of the rigid airplane, was first announced by Cayley" – Theodore von Kármán (USA) 1954. He also established in 1838 the (Royal, in 1839) Polytechnic Institution, the predecessor of the University of Westminster.

attractive properties of adaptive, wavelet approximation theory (see, e.g., [11–15]). We use the Deslauries–Dubuc interpolating wavelet expansion [16], and transform (6) into the (multiscale) mal-setting (10), for which we construct an adaptive, wavelet-mal solver. We note that in the mal-framework, this solver represents a 'second-generation' development – for, due to the space refinement property of the wavelet approximations, the grid on each *line* is naturally adapted *independently* of the rest.

As a numerical demonstration of the ability of the wavelet-mal methodology to capture boundary- and interior-layers we solve the challenging, convection-dominated, test-problem proposed by Hemker [17–19] (Section 6). We note that in treating such problems we need not know *a priori* the location and width of the layers – unlike, e.g., the layer-adapted, Shishkin-type methods, see, e.g., [9,10]). For mal even a rough idea of the layer-orientation is sufficent to produce the desired results (see Section 4).

Finally, we mention that the wavelet-mal methodology offers the possibility of treating higher-dimensional problems in general domains that hitherto elude other, wavelet-based methods (see, e.g., [20,12]).

2. Convection–diffusion problem

We consider the (singularly perturbed) convection–diffusion problem (see, e.g., [6])

$$\begin{aligned} -\varepsilon\triangle u+\boldsymbol{\beta}\cdot\nabla u+\sigma u&=f \quad \text{in } \Omega\\ u&=g \quad \text{on } \Gamma_D\\ \nabla u\cdot\mathbf{n}&=h \quad \text{on } \Gamma_N \end{aligned} \tag{1}$$

where the small parameter $\varepsilon>0$ is the diffusion coefficient, $\boldsymbol{\beta}=(\beta_1,\beta_2)$ is the given velocity field and $\sigma\geqslant 0$ is the absorption coefficient. The domain $\Omega\subset\mathbb{R}^2$ has a piecewise analytic boundary $\partial\Omega=\Gamma_D\cup\Gamma_N$ and $\Gamma_D\cap\Gamma_N=\emptyset$; $\mathbf{n}$ is the outward normal to Γ_N.

We define $H:=\{w\in H^1(\Omega)\colon w=g \text{ on } \Gamma_D\}$ and $H_0:=\{w\in H^1(\Omega)\colon w=0 \text{ on } \Gamma_D\}$, where H^1 is the standard Sobolev space. Then the weak form of (1) can be written: find $u\in H$ such that

$$B(u,v):=(\varepsilon\nabla u,\nabla v)+(\boldsymbol{\beta}\cdot\nabla u+\sigma u,v)=(f,v)+\int_{\Gamma_N}\varepsilon h v\,\mathrm{d}\Gamma=:F(v)\quad \forall v\in H_0 \tag{2}$$

where $(\nu,w)=\int_\Omega \nu w\,\mathrm{d}x$.

3. Mal semi-discretization

Here we reformulate (1) in the context of mal (cf. [1]). We partition the domain Ω into N curvilinear quadrilateral non-overlapping elements Ω_i, such that $\Omega=\bigcup_{i=1}^N\Omega_i$ and B, F in (2) are sums of element-contributions, i.e., $B(u,v)=\sum_{i=1}^N B^i(u,v)$ and $F(v)=\sum_{i=1}^N F^i(v)$. Under a mapping Φ_i, each element Ω_i is the image of the reference square $\widehat{\Omega}=(-1,1)^2$ with local coordinates $\hat{x}^1=\xi$ and $\hat{x}^2=\eta$. The associated metric tensor in $\{\hat{x}^i\}$ is defined by the components $g_{ij}=x_{,i}\,x_{,j}+y_{,i}\,y_{,j}$, where $(\,)_{,i}=\partial/\partial\hat{x}^i$. In a mal-element, we refer to $\xi=$ *const*. and $\eta=$ *const*. as *faces* and *lines*, respectively.

With mal we semi-discretize (2) and obtain a system of ODEs solved along *lines*. We first express B^i, F^i in the local coordinate system $\{\hat{x}^i\}$. Then we approximate the solution u by $(u|_{\Omega_i}\circ\Phi_i)(\xi,\eta)\in V_p(\widehat{\Omega})$, where

$$V_p\big(\widehat{\Omega}\big):=\left\{\sum_{j=0}^{p}X_j^i(\xi)P_j(\eta)\colon X_j^i\in H^1(-1,1),\ j=0,\dots,p\right\} \tag{3}$$

with $P_j(\eta)$ denoting polynomials of degree p. Likewise $(v|_{\Omega_i}\circ\Phi_i)$ is replaced by $\sum_{j=0}^{p}Y_jP_j$. Then we obtain

$$B^i(u,\nu)=\mathcal{B}^i(X,Y),\qquad F^i(\nu)=\mathcal{F}^i(Y)$$

where

$$\mathcal{B}^i(X,Y)=\int_{-1}^{1}\left(\dot{Y}^\top A_i\dot{X}^i-Y^\top G_i\dot{X}^i-\dot{Y}^\top C_iX^i+Y^\top B_iX^i\right)\mathrm{d}\xi \tag{4}$$

$$\mathcal{F}^i(Y)=\int_{-1}^{1}Y^\top\int_{-1}^{1}(f\circ\Phi_i)P|J|\,\mathrm{d}\eta\,\mathrm{d}\xi+\int_{-1}^{1}Y^\top(\varepsilon h\circ\Phi_i)\sqrt{g_{11}}\,\mathrm{d}\xi\,P|_{\eta=\pm1}$$
$$+\,Y^\top\int_{-1}^{1}(\varepsilon h\circ\Phi_i)P\sqrt{g_{22}}\,\mathrm{d}\eta|_{\xi=\pm1} \tag{5}$$

with $\top$ being the transpose, $\dot{(\ )}=\mathrm{d}/\mathrm{d}\xi$, $|J|=|g_{ij}|^{1/2}$ is the Jacobian determinant of Φ_i and A_i, B_i, C_i, G_i are $(p+1)\times(p+1)$ block matrices given below.

Integrating (4) and (5) by parts yields the following elemental ODE system together with the boundary conditions

$$-A_i\ddot{X}^i-(\dot{A}_i+G_i-C_i)\dot{X}^i+(\dot{C}_i+B_i)X^i=F_i,\quad \xi\in(-1,1)$$
$$A_i\dot{X}^i-C_iX^i=(q_i)^\pm,\quad \xi=\pm1 \tag{6}$$

where

$$A_i(\xi)=\int_{-1}^{1}\varepsilon PP^\top|J|^{-1}g_{22}\,\mathrm{d}\eta$$

$$B_i(\xi)=\int_{-1}^{1}\left[\varepsilon P'P'^\top|J|^{-1}g_{11}+(\beta_1y_{,1}-\beta_2x_{,1})PP'^\top+\sigma PP^\top|J|\right]\mathrm{d}\eta$$

$$C_i(\xi)=\int_{-1}^{1}\varepsilon PP'^\top|J|^{-1}g_{12}\,\mathrm{d}\eta,\qquad G_i(\xi)=C_i^\top+\int_{-1}^{1}(\beta_1y_{,2}-\beta_2x_{,2})PP^\top\,\mathrm{d}\eta$$

$$F_i(\xi)=\int_{-1}^{1}(f\circ\Phi_i)|J|P\,\mathrm{d}\eta+\sqrt{g_{11}}(\varepsilon h\circ\Phi_i)P|_{\eta=\pm1}$$

$$(q_i)^\pm=\int_{-1}^{1}(\varepsilon h\circ\Phi_i)\sqrt{g_{22}}P\,\mathrm{d}\eta|_{\xi=\pm1}\quad\text{for }\Phi_i(\pm1,\eta)\in\Gamma_N$$

where $P'=\mathrm{d}P/\mathrm{d}\eta$.

Finally, by satisfying the inter-element continuity conditions [1] and assembling the elemental ODE (6) over all elements, a global ODE system is obtained, which we formally write in the matrix form

$$\begin{cases}\mathcal{A}\ddot{\mathrm{X}}(\xi)+\mathcal{B}\dot{\mathrm{X}}(\xi)+\mathcal{C}\mathrm{X}(\xi)=\mathcal{F}, & \xi\in(-1,1)\\ \mathcal{G}\dot{\mathrm{X}}(\xi)+\mathcal{H}\mathrm{X}(\xi)=\mathcal{P}, & \xi=\pm1\end{cases} \tag{7}$$

4. Mal-mesh design

To account for the anisotropic (basically 1D) nature of the thin layers we position *lines* approximately normal to the layers. Thus, the more difficult part of the solution (with lower regularity) is accurately captured along *lines* by an adaptive ODE solver, whereas for the smoother (regular) part, standard (hp-) finite elements are sufficient. We note that, generally, the location of the thin (especially interior) layers is unknown a priori, but the physics of the problem usually suggests their orientation – otherwise, a simple computational test can provide such information.

5. Adaptive wavelet-mal approximation

Here we briefly describe the wavelet collocation method [20] that we employ for solving the resulting ODE system (7).

We adopt the Deslauries–Dubuc interpolating wavelets defined by the auto-correlation function of the Daubechies compactly supported scaling function $\varphi(y)$ (with L filter coefficients), $\phi(\xi) = \int_{-\infty}^{\infty} \varphi(y)\varphi(y-\xi)\,\mathrm{d}y$ (see, e.g., [16,21,22]). The function $\phi(\xi)$ is modified to retain the property of interpolation at boundaries for the interval $[-1,1]$ [23]. Then, the expansion in the space $V_{j_0} = span\langle \phi_{j_0}^k(\xi) = \phi(2^{j_0}\xi - k),\ k = -2^{j_0}, \ldots, 2^{j_0}\rangle$ and a sequence of the complementary space $W_j = span\langle \psi_j^k(\xi) = \psi(2^j\xi - k),\ k = -2^j, \ldots, 2^j\rangle$ forms a multiscale representation for $\mathrm{X}(\xi) \in V_j = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus \cdots \oplus W_{j-1}$

$$\mathrm{X}(\xi) = I_{j_0}\mathrm{X}(\xi) + \sum_{j \geqslant j_0} \sum_{k=-2^j}^{2^j} \alpha_j^k \psi_j^k(\xi) \tag{8}$$

where $\psi_j^k(\xi) = \phi_{j+1}^{2k+1}(\xi)$ and $I_{j_0}\mathrm{X}(\xi) = \sum_{k=-2^{j_0}}^{2^{j_0}} \mathrm{X}(\xi_{j_0}^k)\phi_{j_0}^k(\xi)$ $(\xi_{j_0}^k = k \cdot 2^{-j_0})$.

This representation expresses $\mathrm{X}(\xi)$ in terms of the coarsest approximation in the space V_{j_0} and detail corrections in the complementary space W_j. Moreover, $\alpha_j^k = \mathrm{X}(\xi_{j+1}^{2k+1}) - I_j\mathrm{X}(\xi_{j+1}^{2k+1})$, provides a (transparent) local measure of the quality of the approximation of $\mathrm{X}(\xi)$ by $I_j\mathrm{X}(\xi)$ [16]. Then we can adaptively define the grids (or basis functions) based on the magnitude of wavelet coefficients at various scales, i.e., we begin with an approximate solution (8) spanned by wavelets in V_j and generate new non-uniform grids for the solution in V_{j+1}

$$G_{j+1} = \{\xi_{j_0}^k\} \cup \{\xi_{j+1}^{2k+1},\ (j,k) \in \Lambda_j\} \tag{9}$$

where $\Lambda_j \subset \{(j,k),\ j \geqslant j_0,\ -2^j \leqslant k \leqslant 2^j\}$ is a subset of dyadic points composed of points selected by removing points (for $|\alpha_j^k| < \delta^-$) and adding neighbouring points (for $|\alpha_j^k| > \delta^+$), with $\delta^- < \delta^+$ being the prescribed tolerances.

Finally, by satisfying (7) at the selected dyadic collocation points $\xi_c \in G_{j+1}$, we obtain the following linear, wavelet-mal system

$$\begin{aligned} &\sum_{-2^{j_0}}^{2^{j_0}} \mathrm{X}(\xi_{j_0}^n)\lambda_{j_0}^n(\xi_c) + \sum_{j \geqslant j_0}\sum_n \alpha_j^n \lambda_{j+1}^{2n+1}(\xi_c) = \mathcal{F}(\xi_c), \quad \xi_c \in (-1,1) \\ &\sum_{-2^{j_0}}^{2^{j_0}} \mathrm{X}(\xi_{j_0}^n)\mu_{j_0}^n(\xi_c) + \sum_{j \geqslant j_0}\sum_n \alpha_j^n \mu_{j+1}^{2n+1}(\xi_c) = \mathcal{P}(\xi_c), \quad \xi_c = \pm 1 \end{aligned} \tag{10}$$

where

$$\begin{aligned} \lambda_{j_0}^n(\xi_c) &= \mathcal{A}(\xi_c)2^{2j_0}\ddot{\phi}_{j_0}^n(\xi_c) + \mathcal{B}(\xi_c)2^{j_0}\dot{\phi}_{j_0}^n(\xi_c) + \mathcal{C}(\xi_c)\phi_{j_0}^n(\xi_c) \\ \lambda_{j+1}^{2n+1}(\xi_c) &= \mathcal{A}(\xi_c)2^{2(j+1)}\ddot{\phi}_{j+1}^{2n+1}(\xi_c) + \mathcal{B}(\xi_c)2^{(j+1)}\dot{\phi}_{j+1}^{2n+1}(\xi_c) + \mathcal{C}(\xi_c)\phi_{j+1}^{2n+1}(\xi_c) \end{aligned}$$

$$\mu_{j_0}^{n}(\xi_c) = \mathcal{G}(\xi_c) 2^{j_0} \dot{\phi}_{j_0}^{n}(\xi_c) + \mathcal{H}(\xi_c) \phi_{j_0}^{n}(\xi_c)$$
$$\mu_{j+1}^{2n+1}(\xi_c) = \mathcal{G}(\xi_c) 2^{(j+1)} \dot{\phi}_{j+1}^{2n+1}(\xi_c) + \mathcal{H}(\xi_c) \phi_{j+1}^{2n+1}(\xi_c)$$

Remark 1. The *space refinement* property of wavelet approximations (see, e.g., [24]) enables the grid on each line to be *naturally* adapted *independently* of the rest – thus rendering a more efficient mal solver.

Remark 2. Due to the hierarchical nature of wavelet bases, there are less computations involved in the reassembling of matrices at each level of refinement (we need compute only the entries in the rows and columns corresponding to the new grid points). Further, since the function $\phi(\xi)$ is compactly supported, $\operatorname{supp}\phi = [-L+1, L-1]$, the summation over n in (10) needs to be taken only for the grid points $\xi_j^n \in \operatorname{supp}\phi_j^n(\xi_c)$.

6. Numerical example: The Hemker test-problem

Here we show the efficacy and high accuracy of the wavelet-mal methodology applied to a *non-trivial* model problem proposed in 1996 by Hemker [17] (see also [18,19]).

This model describes the convection-dominated flow around a cylinder and calls for the solution of the singularly perturbed, convection–diffusion equation (1). By symmetry, we model only the half-space, exterior to the unit circle $\Omega = \{(x, y)\colon x^2 + y^2 \geqslant 1\}$, truncated at $A(-4, 0)$, $B(0, 4)$, $C(6, 0)$, with the boundary conditions, as shown in Fig. 1. This problem exhibits both boundary- and interior-layers; when $\varepsilon = 0.04$, $\boldsymbol{\beta} = (1, 0)$, $\sigma = 0$, $f = 0$; the analytic solution (numerically evaluated) is known [17]. Fig. 1 shows 7 mal-elements, each with polynomial degree $p = 3$; the thick lines illustrate the boundary- and interior-layers. We use transfinite blending to model the geometry. For the adaptive wavelet approximation we use: $L = 10$, $j_0 = 4$, $\delta^- = 10^{-6}$, $\delta^+ = 10^{-5}$ (see Section 5). Fig. 2 shows the non-uniform wavelet-mal grids. Fig. 3 portrays the excellent agreement between the analytic solution [17] and the wavelet-mal solution (at scale $j = 7$).

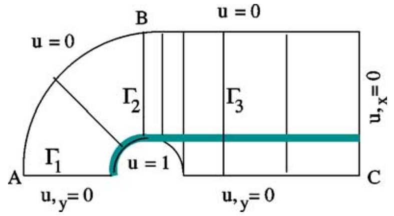

Fig. 1. Model problem. Flow around cylinder (darker line depicts boundary- and interior-layers).

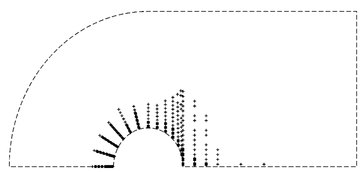

Fig. 2. Non-uniform wavelet-mal grids.

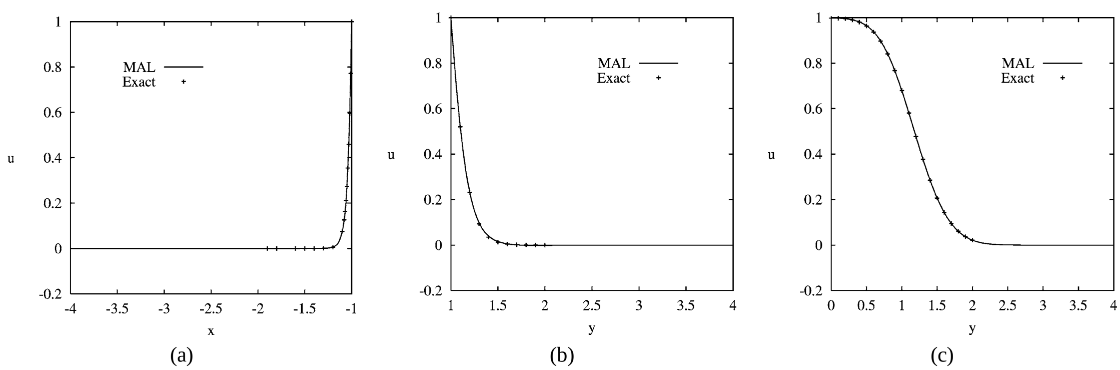

Fig. 3. Typical wavelet-mal solution on (a) Γ_1, (b) Γ_2 and (c) Γ_3.

7. Epilegomena

Finally, we remark that this model has engaged (or perhaps discouraged!?) a number of researchers exploring different numerical schemes without as yet *anyone* rendering a (published) method that is *robust* with respect to ε (see, e.g., [18,19]). Is this because this goal is unattainable, leading to an *aporia*, in which case one must pursue alternative goals – as the authors in [18], p. 354, seem to ponder? Here Baudelaire offers (as Hemker will agree!) a powerful verse for reflection ("*Le Voyage*" in: "*Les fleurs du mal*"):

Singulière fortune où le but se déplace,
Et n'étant nulle part, peut être n'importe où !
Où l'Homme, dont jamais l'espérance n'est lasse,
Pour trouver le repos court toujours comme un fou !

We address such issues in a forthcoming communication where we also present comprehensive numerical results that establish the reliability of the wavelet-mal methodology.

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