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Modelling of unsaturated water flow in soils with highly permeable inclusions

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Abstract

In this paper the mathematical macroscopic modeling of unsaturated water flow in a porous medium (soil) with highly permeable porous inclusions is presented. It is supposed that water flow in each sub domain can be described by the strongly non-linear Richards' equation. Gravity effects are considered. The upscaling process of this stiff problem is performed using the homogenization method of periodic structures with asymptotic expansions. The resulting non-linear macroscopic description is a one equation model, revealing the local equilibrium of the capillary pressure head. The effective water retention capacity was found to be the volume average of the water retention capacities of the two porous sub-domains. The effective conductivity tensor is obtained from a linear and non-stiff boundary value problem at the heterogeneity scale. ***To cite this article: J. Lewandowska, J.-L. Auriault, C. R. Mecanique 332 (2004).***

Résumé

Modélisation de l'écoulement dans les sols non saturés avec inclusions très perméables. Nous étudions dans cette Note le modèle mathématique macroscopique de l'écoulement non-saturé d'eau dans un milieu poreux contenant des inclusions poreuses très conductrices. L'écoulement est décrit dans chaque milieu par l'équation fortement non-linéaire de Richards. La gravité est prise en considération. La macroscopisation de ce problème raide est conduite au moyen de la méthode d'homogénéisation de structures périodiques. Le modèle résultant à l'échelle macroscopique est un modèle non-linéaire à une équation qui traduit l'équilibre local de la pression capillaire. La rétention d'eau effective est la moyenne de volume des rétentions locales. Le tenseur effectif de conductivité est obtenu à partir de la solution d'un problème local linéaire non raide. ***Pour citer cet article : J. Lewandowska, J.-L. Auriault, C. R. Mecanique 332 (2004).***

Keywords: Granular media; Soils; Porous media

Mots-clés : Milieux granulaires ; Sols ; Milieux poreux

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L'objectif de ce travail est le développement d'un modèle mathématique décrivant l'écoulement non saturé dans un sol hétérogène. Nous considérons un milieu poreux périodique (sol) contenant des inclusions isolées très perméables. Nous supposons donc qu'il est composé de deux sous-domaines poreux dont les propriétés hydrodynamiques sont très contrastées. On note Ω la période, Ω_1 et Ω_2 les deux domaines poreux, et Γ leur interface. Le domaine Ω_2 est connexe et le milieu 2 est beaucoup moins perméable que le milieu 1 (inclusions). Le modèle macroscopique est obtenu par la méthode des développements asymptotiques (Sanchez-Palencia [4], Bensoussan et al. [5]). Le point de départ de notre analyse est l'équation de Richards' [3] qui décrit l'écoulement d'eau dans un sol non saturé pour les domaines 1 et 2, Éq. (2). Dans cette équation C est la capacité de rétention d'eau et K la conductivité hydraulique. Sur l'interface Γ on suppose la continuité des pressions capillaires h_1 et h_2 et des flux d'eau, Éq. (3). Les rétentions d'eau $\theta_1(h)$ et $\theta_2(h)$ et les conductivités $K_1(h)$ et $K_2(h)$ sont fortement non linéaires et sont supposées connues. Pour estimer les ordres des grandeurs des différents termes de la description locale, nous avons choisi comme temps caractéristique le temps d'observation de l'écoulement de l'eau dans le milieux 2, $T = C_{2c}L^2/K_{2c}$, où l'indice c indique une propriété caractéristique et L est la longueur caractéristique macroscopique. Le contraste entre les paramètres est le suivant : $K_{2c}/K_{1c} = \mathcal{O}(\varepsilon)$ et $C_{2c}/C_{1c} = \mathcal{O}(1)$, où $\varepsilon = l/L$ (1) et l est la taille de la période. Enfin, le terme capillaire est supposé dominant par rapport au terme gravitaire : $h_c/l = \mathcal{O}(\varepsilon)$. En utilisant ces estimations, la description microscopique est donnée sous forme adimensionnelle par (6)–(8). C'est un problème raide fortement non linéaire.

Premier ordre d'approximation $h^{(0)}$

Après introduction du développement asymptotique (9), nous obtenons au premier ordre un champ de pression capillaire macroscopique de la forme $h_1^{(0)} = h_2^{(0)} = h^{(0)}(\mathbf{x}, t^*)$.

Deuxième ordre d'approximation $h^{(1)}$

L'identification des termes à l'ordre suivant conduit à (13), (14), (16), dont la solution est (15) pour le domaine 1 et (17) pour le domaine 2. Le problème local définissant le champ vectoriel $\boldsymbol{\chi}(\mathbf{y})$, linéaire et non-raide, est le système (18), qui présente certaines similitudes avec le problème de filtration décrit dans Levy [1].

Description macroscopique

Afin de trouver la description macroscopique, nous considérons les Éqs. (19) et (20) avec la condition (21) sur l'interface, ainsi que la condition de périodicité. On intègre l'Éq. (19) sur Ω_1 et l'Éq. (20) sur Ω_2. Après quelques transformations, la somme membres à membres des deux équations résultant de cette opération nous donne l'équation recherchée (24). Les paramètres effectifs rentrant dans cette description sont la capacité effective de rétention d'eau $C^{\text{eff}}(h^{(0)})$ (23), et la conductivité effective $K^{\text{eff}}(h^{(0)})$ (25). L'Éq. (24) est fortement non linéaire. Nous pouvons vérifier que cette description est valable dans le cas de plusieurs inclusions présentes dans la période et/ou dans le cas $K_{2c}/K_{1c} = \mathcal{O}(\varepsilon^p)$, $p > 1$. Cette équation reste aussi valable pour $K_{2c}/K_{1c} = \mathcal{O}(1)$ et $C_{1c}/C_{2c} = \mathcal{O}(\varepsilon^p)$, $p \geqslant 1$. Enfin $K_{2c}/K_{1c} = \mathcal{O}(1)$, $C_{2c}/C_{1c} = \mathcal{O}(1)$ conduit au cas classique d'homogénéisation.

1. Introduction

To our knowledge, the mathematical problem of unsaturated water flow in a porous medium (soil) with highly permeable inclusions has not received much attention in the literature. However, this is a problem of practical interest in soil hydrology of vadose zone. The presence of highly permeable inclusions can considerably accelerate the water and pollutant transfers. In the literature two mathematically alike though linear problems can be found. The first one by Levy [1] addresses the saturated flow through non deformable porous fissured rocks. In the second paper, Moctezuma-Berthier et al. [2], the effect of porosity distribution on the electrical conductivity in reconstructed bimodal porous media is investigated.

In this paper the modelling of water flow in soils of double porosity (soil matrix with highly permeable inclusions) in the vadose zone is presented. Due to the non-linearity of the hydrodynamic parameters, the stiff problem into consideration becomes strongly non-linear. The modeling includes the derivation of the macroscopic description and the effective hydrodynamic parameters, together with the definition of the domain of its validity. The effective conductivity is obtained from a non-stiff linear boundary value problem.

2. General assumptions

We consider a porous medium (soil) with highly permeable porous inclusions. Let us assume that this medium is periodic and there exists the following separation of scales

$$\varepsilon = \frac{l}{L} \ll 1 \tag{1}$$

where l is the period length (the characteristic microscopic length) and L is the characteristic macroscopic length (e.g., the dimension of the macroscopic domain). We note Ω the period, Ω_1 and Ω_2 the two porous domains, respectively, and Γ the interface between them. For the moment, the period Ω contains a single inclusion Ω_1. We also assume that the domain Ω_2 is continuously connected and that the porous medium 2 is much less conducting than the porous medium 1.

The unsaturated water flow in a homogenous rigid soil is often described by the strongly non-linear Richards' equation [3] which assumes that the air pressure in the soil is constant and equal to the atmospheric pressure during the whole flow process. Under isothermal conditions this equation is formulated in each porous subdomain for the capillary pressure h[L] being the water pressure head relative to the atmospheric pressure ($h \leqslant 0$) as follows:

$$C_\alpha(h_\alpha)\frac{\partial h_\alpha}{\partial t} - \operatorname{div}_X\big(\mathbf{K}_\alpha(h_\alpha)\,\mathbf{grad}_X(h_\alpha + X_3)\big) = 0, \quad \alpha = 1, 2, \text{ in } \Omega_\alpha \tag{2}$$

where $C(h) = \mathrm{d}\theta/\mathrm{d}h$ is the soil water retention capacity [L^{-1}], $\theta(h)$ is the volumetric water content [$\cdot$], $\mathbf{K}(h)$ is the hydraulic conductivity positive tensor [LT^{-1}], $\mathbf{X} = (X_1, X_2, X_3)$ denotes the physical spatial variable [L], where the axis X_3 is positively oriented upwards and t [T] is the time. This equation is strongly non-linear because of the non-linear functions $C(h)$ and $\mathbf{K}(h)$.

At the interface Γ, the capillary pressure head and the water flux are continuous:

$$h_1 = h_2, \quad \mathbf{K}_1(h_1)\,\mathbf{grad}_X(h_1 + X_3)\mathbf{N} = \mathbf{K}_2(h_2)\,\mathbf{grad}_X(h_2 + X_3)\mathbf{N}, \quad \text{on } \Gamma \tag{3}$$

where $\mathbf{N}$ is a unit vector normal to Γ. In Eqs. (2) and (3) the X_3 term is related to the gravity. We assume that the water retention curves $\theta_\alpha(h)$, as well as the conductivity curves $\mathbf{K}_\alpha(h)$, $\alpha = 1, 2$, are known and that hysteresis effects are negligible.

3. A priori estimation and dimensionless micro-description

Let us introduce dimensionless variables as follows:

$$h_\alpha = h_{\alpha c} h^*_\alpha, \quad \mathbf{K}_\alpha = K_{\alpha c}\mathbf{K}^*_\alpha, \quad C_\alpha = C_{\alpha c} C^*_\alpha, \quad \alpha = 1, 2, \quad t = T t^* \tag{4}$$

where the subscript c denotes a characteristic quantity (constant) and the asterisk denotes the corresponding dimensionless variable. We introduce also two dimensionless space variables $\mathbf{y} = \mathbf{X}/l$ and $\mathbf{x} = \mathbf{X}/L$, the microscopic dimensionless space variable and the macroscopic dimensionless space variable, respectively. Time T is the characteristic time of observation, chosen to be the time of the water flow in the medium 2 at the macroscopic scale L, $T = C_{2c}L^2/K_{2c}$.

We assume that medium 2 is much less conductive than medium 1 (therefore the problem (2), (3) is stiff) and that the two water retention capacities are of the same order of magnitude:

$$\frac{K_{2c}}{K_{1c}} = \mathcal{O}(\varepsilon), \qquad \frac{C_{2c}}{C_{1c}} = \mathcal{O}(1) \tag{5}$$

Finally, we notice that the ratio of characteristic length of the period l to the characteristic capillary pressure head h_c is of the order $\mathcal{O}(\varepsilon)$ in a period. It means that the capillary term is dominating over the gravity one.

By taking into account the above estimation, the local problem (2), (3) can be put in the following dimensionless form:

$$\varepsilon^3 C_1^* \frac{\partial h_1^*}{\partial t} - \mathrm{div}_y\big(\mathbf{K}_1^* \,\mathbf{grad}_y(h_1^* + \varepsilon y_3)\big) = 0, \quad \text{in } \Omega_1 \tag{6}$$

$$\varepsilon^2 C_2^* \frac{\partial h_2^*}{\partial t} - \mathrm{div}_y\big(\mathbf{K}_2^* \,\mathbf{grad}_y(h_2^* + \varepsilon y_3)\big) = 0, \quad \text{in } \Omega_2 \tag{7}$$

$$h_1^* = h_2^*, \quad \mathbf{K}_1^* \,\mathbf{grad}_y(h_1^* + \varepsilon y_3)\mathbf{N} = \varepsilon \mathbf{K}_2^* \,\mathbf{grad}_y(h_2^* + \varepsilon y_3)\mathbf{N}, \quad \text{on } \Gamma \tag{8}$$

4. Upscaling

In this study, the classical method of homogenization by formal asymptotic expansions was applied (Sanchez-Palencia [4], Bensoussan et al. [5]). The procedure adopted in this paper was presented in details by Auriault [6]. The dimensionless capillary pressure head h^* is looked for in the form of the following expansion

$$h^* = h^{(0)}(\mathbf{y}, \mathbf{x}, t^*) + \varepsilon h^{(1)}(\mathbf{y}, \mathbf{x}, t^*) + +\varepsilon^2 h^{(2)}(\mathbf{y}, \mathbf{x}, t^*) + \cdots, \tag{9}$$

which results in similar expansions for $\mathbf{K}$ and C, with $\mathbf{K}^{(0)} = \mathbf{K}(h^{(0)})$ and $C^{(0)} = C(h^{(0)})$.

4.1. First order variable $h^{(0)}$

After introducing the above expasions in Eqs. (6)–(8) and extracting like powers of ε, we obtain for the domain Ω_1:

$$\frac{\partial}{\partial y_i}\left(K_{1ij}^{(0)} \frac{\partial h_1^{(0)}}{\partial y_j}\right) = 0, \quad \text{in } \Omega_1, \qquad K_{1ij}^{(0)} \frac{\partial h_1^{(0)}}{\partial y_j} N_i = 0, \quad \text{on } \Gamma \tag{10}$$

which yields $h_1^{(0)} = h_1^{(0)}(\mathbf{x}, t^*)$, after noting that $\mathbf{K}^{(0)} = \mathbf{K}(h^{(0)})$ is a positive tensor. The boundary value problem in Ω_2 is in the form

$$\frac{\partial}{\partial y_i}\left(K_{2ij}^{(0)} \frac{\partial h_2^{(0)}}{\partial y_j}\right) = 0, \quad \text{in } \Omega_2, \qquad h_2^{(0)} = h_1^{(0)}, \quad \text{on } \Gamma \tag{11}$$

where $h_2^{(0)}$ is $\mathbf{y}$-periodic. We finally obtain:

$$h_2^{(0)} = h_1^{(0)} = h^{(0)}(\mathbf{x}, t^*) \tag{12}$$

4.2. Second order variable $h^{(1)}$

After using Eq. (12), we obtain the following boundary value problem for $h_1^{(1)}$:

$$\frac{\partial}{\partial y_i}\left[K_{1ij}^{(0)}\left(\frac{\partial h^{(0)}}{\partial x_j} + I_{3j}\right) + K_{1ij}^{(0)} \frac{\partial h_1^{(1)}}{\partial y_j}\right] = 0, \quad \text{in } \Omega_1 \tag{13}$$

$$\left[K_{1ij}^{(0)}\left(\frac{\partial h^{(0)}}{\partial x_j} + I_{3j}\right) + K_{1ij}^{(0)} \frac{\partial h_1^{(1)}}{\partial y_j}\right] N_i = 0, \quad \text{on } \Gamma \tag{14}$$

After multiplying the two members of Eq. (13) by $y_i(\partial h^{(0)}/\partial x_i + I_{3i})$, integrating by parts and using the boundary condition (14) and the positiveness of $\mathbf{K}_1^{(0)}$, we obtain:

$$\frac{\partial h^{(0)}}{\partial x_i} + I_{3i} + \frac{\partial h_1^{(1)}}{\partial y_i} = 0, \qquad h_1^{(1)} = -y_i \left(\frac{\partial h^{(0)}}{\partial x_i} + I_{3i}\right) + \bar{h}^{(1)}(\mathbf{x}, t^*) \tag{15}$$

where $\bar{h}^{(1)}$ is an arbitrary function. For $h_2^{(1)}$ we obtain:

$$\frac{\partial}{\partial y_i}\left[K_{2ij}^{(0)}\left(\frac{\partial h^{(0)}}{\partial x_j} + I_{3j} + \frac{\partial h_2^{(1)}}{\partial y_j}\right)\right] = 0, \quad \text{in } \Omega_2, \qquad h_2^{(1)} = h_1^{(1)}, \quad \text{on } \Gamma \tag{16}$$

where $h_2^{(1)}$ is $\mathbf{y}$-periodic. That yields:

$$h_2^{(1)} = -\chi_i\left(\frac{\partial h^{(0)}}{\partial x_i} + I_{3i}\right) + \bar{h}^{(1)}(\mathbf{x}, t^*) \tag{17}$$

The boundary value problem for the vectorial function $\boldsymbol{\chi}$ is a linear non-stiff problem:

$$\frac{\partial}{\partial y_i}\left[K_{2ij}^{(0)}\left(-\frac{\partial \chi_k}{\partial y_j} + I_{jk}\right)\right] = 0, \quad \text{in } \Omega_2, \quad \chi_k = y_k, \quad \text{on } \Gamma, \quad k = 1, 2, 3 \tag{18}$$

where $\boldsymbol{\chi}$ is $\mathbf{y}$-periodic.

4.3. Macroscopic modelling

In order to determine the macroscopic model we extract from the asymptotic expansions the following relations, where the results (12) and (15) are accounted for:

$$\begin{aligned}&\frac{\partial}{\partial y_i}\left[K_{1ij}^{(0)}\left(\frac{\partial h_1^{(2)}}{\partial x_j} + \frac{\partial h_1^{(3)}}{\partial y_j}\right) + K_{1ij}^{(1)}\left(\frac{\partial h_1^{(1)}}{\partial x_j} + \frac{\partial h_1^{(2)}}{\partial y_j}\right) + K_{1ij}^{(2)}\left(\frac{\partial h^{(0)}}{\partial x_j} + \frac{\partial h_1^{(1)}}{\partial y_j} + I_{3j}\right)\right] \\ &\quad + \frac{\partial}{\partial x_i}\left[K_{1ij}^{(0)}\left(\frac{\partial h_1^{(1)}}{\partial x_j} + \frac{\partial h_1^{(2)}}{\partial y_j}\right)\right] = C_1^{(0)}\frac{\partial h^{(0)}}{\partial t^*}, \quad \text{in } \Omega_1\end{aligned} \tag{19}$$

$$\begin{aligned}&\frac{\partial}{\partial y_i}\left[K_{2ij}^{(0)}\left(\frac{\partial h_2^{(1)}}{\partial x_j} + \frac{\partial h_2^{(2)}}{\partial y_j}\right) + K_{2ij}^{(1)}\left(\frac{\partial h^{(0)}}{\partial x_j} + \frac{\partial h_2^{(1)}}{\partial y_j} + I_{3j}\right)\right] \\ &\quad + \frac{\partial}{\partial x_i}\left[K_{2ij}^{(0)}\left(\frac{\partial h^{(0)}}{\partial x_j} + \frac{\partial h_2^{(1)}}{\partial y_j} + I_{3j}\right)\right] = C_2^{(0)}\frac{\partial h^{(0)}}{\partial t^*}, \quad \text{in } \Omega_2\end{aligned} \tag{20}$$

$$\begin{aligned}&\left[K_{1ij}^{(0)}\left(\frac{\partial h_1^{(2)}}{\partial x_j} + \frac{\partial h_1^{(3)}}{\partial y_j}\right) + K_{1ij}^{(1)}\left(\frac{\partial h_1^{(1)}}{\partial x_j} + \frac{\partial h_1^{(2)}}{\partial y_j}\right) + K_{1ij}^{(2)}\left(\frac{\partial h^{(0)}}{\partial x_j} + \frac{\partial h_1^{(1)}}{\partial y_j} + I_{3j}\right)\right]N_i \\ &\quad = \left[K_{2ij}^{(0)}\left(\frac{\partial h_2^{(1)}}{\partial x_j} + \frac{\partial h_2^{(2)}}{\partial y_j}\right) + K_{2ij}^{(1)}\left(\frac{\partial h^{(0)}}{\partial x_j} + \frac{\partial h_2^{(1)}}{\partial y_j} + I_{3j}\right)\right]N_i, \quad \text{on } \Gamma\end{aligned} \tag{21}$$

Now, integrate Eqs. (19) and (20) over the domains Ω_1 and Ω_2, respectively, apply to both integrals the divergence theorem, and use the periodicity condition and the boundary condition (21). After dividing by $|\Omega|$ we get:

$$C^{\text{eff}}\frac{\partial h^{(0)}}{\partial t^*} = \frac{1}{|\Omega|}\frac{\partial}{\partial x_i}\left[\int_{\Omega_1} K_{1ij}^{(0)}\left(\frac{\partial h_1^{(1)}}{\partial x_j} + \frac{\partial h_1^{(2)}}{\partial y_j}\right)\mathrm{d}\Omega + \int_{\Omega_2} K_{2ij}^{(0)}\left(\frac{\partial h^{(0)}}{\partial x_j} + \frac{\partial h_2^{(1)}}{\partial y_j} + I_{3j}\right)\mathrm{d}\Omega\right] \tag{22}$$

where the effective water retention capacity C^{eff} is defined by:

$$C^{\mathrm{eff}} = \frac{1}{|\Omega|}\int_{\Omega_1} C_1^{(0)}\,\mathrm{d}\Omega + \frac{1}{|\Omega|}\int_{\Omega_2} C_2^{(0)}\,\mathrm{d}\Omega \tag{23}$$

Eq. (22) is independent of the micro-space variable $\mathbf{y}$. It represents a macroscopic equivalent model of the flow in the heterogeneous porous medium. After some analytical investigations, the macroscopic description (22) can be put in the simplified form

$$C^{\mathrm{eff}}\frac{\partial h^{(0)}}{\partial t^*} = \frac{\partial}{\partial x_i}\left[K_{ij}^{\mathrm{eff}}\left(\frac{\partial h^{(0)}}{\partial x_j} + I_{3j}\right)\right] \tag{24}$$

where the effective permeability tensor $\mathbf{K}^{\mathrm{eff}}$ is obtained from the solution $\boldsymbol{\chi}$ of the linear non-stiff boundary value problem (18), by the surface integral:

$$K_{ij}^{\mathrm{eff}} = \left\langle K_{2ik}^{(0)}\left(-\frac{\partial \chi_j}{\partial y_k} + I_{kj}\right)\right\rangle_{\Sigma_i}, \qquad \langle\cdot\rangle_{\Sigma_i} = \frac{1}{|\Sigma_i|}\int_{\Sigma_i}\cdot\,\mathrm{d}\Gamma \tag{25}$$

where Σ_k is the surface $y_k = l_k^* = l_k/l$ of the period and l_k is the dimension of the period in the direction $\mathbf{e}_k$. It can be demonstrated that the tensor $\mathbf{K}^{\mathrm{eff}}$ is positive symmetric.

5. Concluding remarks

At the macroscopic scale, the unsaturated water flow in double porosity soils with highly permeable inclusions is described by a one equation model with two effective parameters. The macroscopic model is non-linear, since both its effective parameters depend on the capillary pressure. The effective conductivity can be calculated as the solution of a linear non-stiff problem by using numerical methods. The presence of very permeable inclusions increases the effective conductivity and influences the macroscopic behaviour through the modified retention capacity.

The above analysis can be shown to apply in case when the period contains several inclusions or (and) when $K_{2c}/K_{1c} = \mathcal{O}(\varepsilon^p)$, $p > 1$, where it yields similar results. The case when $K_{2c}/K_{1c} = \mathcal{O}(1)$, $C_{1c}/C_{2c} = \mathcal{O}(\varepsilon^p)$, $p \geqslant 1$, is also described by the above macroscopic model. It can be obtained by following the analysis in Lewandowska et al. [7]. Finally, note that the case $K_{2c}/K_{1c} = \mathcal{O}(1)$, $C_{2c}/C_{1c} = \mathcal{O}(1)$ yields the classical homogenization case.

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