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An asymptotic non-linear model for thin-walled rods

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Abstract

In this paper, we present a non-linear one-dimensional model for thin-walled rods with open strongly curved cross-section, obtained by asymptotic methods. A dimensional analysis of the non-linear three-dimensional equilibrium equations lets appear dimensionless numbers which reflect the geometry of the structure and the level of applied forces. For a given force level, the order of magnitude of the displacements and the corresponding one-dimensional model are deduced by asymptotic expansions. ***To cite this article: L. Grillet et al., C. R. Mecanique 332 (2004).***

Résumé

Un modèle asymptotique non-linéaire de poutres à parois minces. Dans cette Note, on présente un modèle non-linéaire de poutre voile valable pour les sections ouvertes fortement courbées, obtenu par méthodes asymptotiques. Une analyse dimensionnelle des équations d'équilibre tridimensionnelles fait naturellement apparaître des nombres sans dimension caractérisant la géométrie de la structure et les niveaux d'efforts exercés. Pour un niveau d'effort donné, l'ordre de grandeur des déplacements et le modèle asymptotique correspondant sont obtenus par développement asymptotique des équations. ***Pour citer cet article : L. Grillet et al., C. R. Mecanique 332 (2004).***

Keywords: Solids and structures; Thin-walled rod model; Non-linear elasticity; Asymptotic methods

Mots-clés : Solides et structures ; Modèle de poutre voile ; Élasticité non-linéaire ; Méthodes asymptotiques

Version française abrégée

Les poutres à paroi minces (ou poutres voiles) sont des structures très utilisées dans l'industrie car elle procurent un maximum de rigidité pour un minimum de poids. La géométrie de la poutre voile est caractérisée par deux petits paramètres : l'inverse de l'élancement η (rapport du diamètre de la poutre sur sa longueur) et la « finesse » ε (rapport de l'épaisseur de la poutre sur son diamètre). De ce fait, les équations d'équilibre tridimensionnelles peuvent être

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approchées par des modèles unidimensionnels. Le modèle linéaire unidimensionnel le plus utilisé a été développé par Vlassov [2], il est considéré comme un modèle « standard ». Comme ε et η sont petits, des hypothèses sur le déplacement tridimensionnel et sur l'état de contraintes sont postulées :

- le profil est indéformable dans le plan de la section de la poutre à paroi mince (c'est l'équivalent de l'hypothèse de Bernouilli pour les poutres pleines) ;
- il n'y a pas de déformation par distorsion du profil dans le plan tangent à la surface moyenne de la poutre voile ;
- les contraintes de traction et de cisaillement dans le plan de la section sont dominantes par rapport aux autres contraintes. En outre, la contrainte de traction est constante dans l'épaisseur et celle de cisaillement est affine.

Ces hypothèses ont été justifiées, dans le cadre linéaire, par méthode asymptotique à partir de l'élasticité tridimensionnelle linéaire [4,1].

Dans le cadre non linéaire, il n'existe pas de modèle « standard » de poutres voiles. On trouve dans la littérature une variété de modèles, qui se distinguent entre eux par les hypothèses simplificatrices utilisées par les différents auteurs [7]. Ces hypothèses généralisent, d'une part, au cas non linéaire, les hypothèses cinématique et statique de Vlassov, et d'autre part, contiennent des simplifications supplémentaires sur la forme de l'énergie de déformation.

Nous proposons dans cet article la construction d'un modèle asymptotique non linéaire, déduit des équations tridimensionnelles, généralisant ainsi au cas non linéaire les travaux [4] et [1]. Ce modèle qui est donné par les équations de la Proposition 4.1 généralise le modèle de Vlassov au cas non linéaire.

Les étapes de l'article sont les suivantes. On construit d'abord des nombres adimensionnels caractérisant la géométrie de la poutre ainsi que les efforts appliqués (expression (5)). Afin d'obtenir un problème mono-échelle, ces nombres sans dimension sont reliés au petit paramètre ε caractérisant la finesse. On cherche alors la solution du problème tridimensionnel sous forme d'une série formelle en ε. Pour des niveaux d'efforts modérés (relations (6)), on montre que le premier terme non nul du déplacement vérifie les hypothèses cinématiques (7) et (8) qui constituent une généralisation de celles de Vlassov au cas non linéaire. Les contraintes associées vérifient également cette généralisation (Éqs. (9)). On obtient alors un modèle asymptotique non linéaire de poutres voiles, décrit par un système d'équations différentielles fortement couplées, représentant le modèle de traction (Éq. (10)), le modèle de torsion (Éq. (11)) et le modèle de flexion (Éq. (12)).

1. Introduction

In non-linear elasticity, unlike in plate and shell theory, it does not seem to exist any classical model for thin-walled rods. Recently, Rodriguez and Viaño [1] have justified the linear elastic model of Vlassov [2] for thin-walled rods by asymptotic method.

We propose here to use an asymptotic approach [3] to deduce a non-linear model for thin-walled rods from three-dimensional equations. The approach used is based on a dimensional analysis of the three-dimensional non-linear equilibrium equations which let appear pertinent dimensionless numbers characterizing the geometry and the level of applied loads. These numbers are measurable and enable to define the domain of validity of the obtained model. The present work constitutes a generalization of [4,1] in non-linear elasticity.

2. The three-dimensional problem

Let ω_0^* be an open cylindrical surface of $\mathbb{R}^3$, (Oe_3) its axis, whose length is L_0 and diameter d_0. We note γ_{0g}^* and γ_{0d}^* its lateral boundary, $\gamma_{01}^* = \omega_0^* \times \{0\}$ and $\gamma_{02}^* = \omega_0^* \times \{L_0\}$ its extremities.

Let us consider now a thin-walled rod with open cross-section and $2h_0$ thickness, whose middle surface is ω_0^*. The thin-walled rod occupies the set $\overline{\Omega}_0^* = \overline{\omega}_0^* \times [-h_0, h_0]$ of $\mathbb{R}^3$ in its reference configuration.

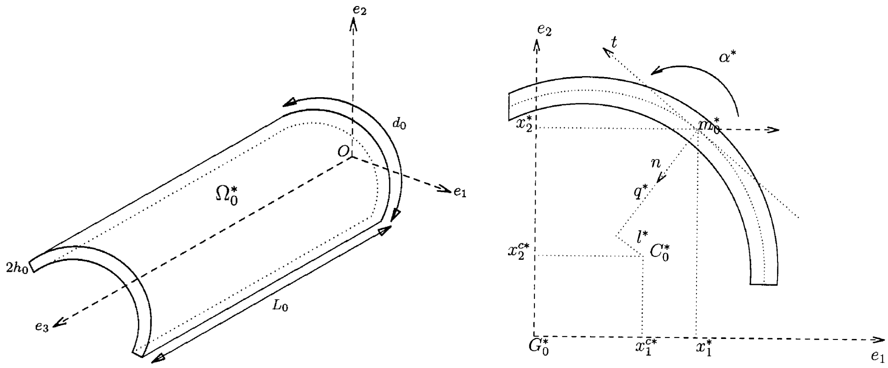

Fig. 1. The geometry of the rod.

Fig. 1. La géométrie de la poutre.

We call $\Gamma_{01}^* = \gamma_{01}^* \times]-h_0, h_0[$ and $\Gamma_{02}^* = \gamma_{02}^* \times]-h_0, h_0[$ the extreme faces, $\Gamma_{0g}^* = \gamma_{0g}^* \times]-h_0, h_0[$ and $\Gamma_{0d}^* = \gamma_{0d}^* \times]-h_0, h_0[$ the lateral faces, $\Gamma_{0\pm}^* = \omega_0^* \times \{\pm h_0\}$ the upper and lower faces.

Let M_0^* be a generic point of the beam. We decompose the vector $\overrightarrow{OM_0^*}$ as follows:

$$\overrightarrow{OM_0^*} = x_3^* e_3 + \overrightarrow{G_0^* C_0^*} + \overrightarrow{C_0^* m_0^*} + r^* n \tag{1}$$

where x_3^* is the coordinate of the current cross-section containing M_0^* on the axis $(0x_3^*)$, G_0^* the point of intersection between the axis $(0x_3^*)$ and the current cross-section, C_0^* an arbitrary chosen point in the plane of the cross-section (see Fig. 1) located by its Cartesian coordinates (x_1^{c*}, x_2^{c*}), and r^* the thickness variable. We call $\mathcal{C}_0^*$ the intersection curve between ω_0^* and the cross-section. The orthogonal projection m_0^* of M_0^* on the middle surface is located by its Cartesian coordinates $x^* = (x_1^*, x_2^*)$ or by its curvilinear abscisse s^* along $\mathcal{C}_0^*$. The origin s_0^* of the curvilinear abscisse is an arbitrary chosen point of $\mathcal{C}_0^*$. We note n the unit normal and t the unit tangent vector of $\mathcal{C}_0^*$. Moreover, we call l^* and q^* the coordinates of the vector $\overrightarrow{C_0^* m_0^*}$ in the basis (t, n). Finally, we call α^* the angle (e_1, t) and c_0^* the curvature of the curve $\mathcal{C}_0^*$ (see Fig. 1 (right)).

In what follows, we consider only thin-walled rods such as $d_0/L_0 \ll 1$, $h_0/d_0 \ll 1$ and $h_0\|c_0^*\|_\infty \ll 1$. We assume that the rod is subjected to the applied body forces $f^* = f_t^* t + f_n^* n + f_3^* e_3 : \overline{\Omega}_0^* \to \mathbb{R}^3$ and to the applied surface forces $g^{*\pm} = g_t^{*\pm} t + g_n^{*\pm} n + g_3^{*\pm} e_3 : \overline{\Gamma}_{0\pm}^* \to \mathbb{R}^3$. Moreover, the rod is assumed to be clamped on its extremities Γ_{01}^* and Γ_{02}^*, and free on its lateral faces Γ_{0g}^* and Γ_{0d}^*. The unknown of the problem is then the displacement $U^* : \overline{\Omega}_0^* \to \mathbb{R}^3$. Within the framework of non-linear elasticity, the displacement U^* and the second Piola–Kirchhoff tensor Σ^* satisfy the non-linear equilibrium equations:

$$\begin{cases} \mathrm{Div}^*(\Sigma^* \overline{F}^*) = -f^* & \text{in } \Omega_0^* \\ (F^* \Sigma^*)\tilde{n} = g^{*\pm} & \text{on } \Gamma_{0\pm}^* \\ (F^* \Sigma^*)\tilde{t} = 0 & \text{on } \Gamma_{0g}^* \cup \Gamma_{0d}^* \\ U^* = 0 & \text{on } \Gamma_{01}^* \cup \Gamma_{02}^* \end{cases} \tag{2}$$

where the overbar denotes the transposition operator, $F^* = \partial\psi^*/\partial M_0^* = I + \partial U^*/\partial M_0^*$ the linear tangent map to the mapping function $M_0^* \to \psi^*(M_0^*) = M_0^* + U^*$, and $\tilde{n}$ (respectively $\tilde{t}$) the external unit normal to the upper and lower faces (respectively to the lateral faces). Limiting our study to Hookean materials, the constitutive relation is given by $\Sigma^* = \lambda \,\mathrm{Tr}\, E^* I + 2\mu E^*$ where $E^* = \frac{1}{2}(\overline{F}^* F^* - I)$ denotes the Green–Lagrange tensor and I the $\mathbb{R}^3$ identity tensor. As in linear elasticity [4], the boundary conditions on $\Gamma_{0g}^* \cup \Gamma_{0d}^*$ are considered on average upon the thickness, in order the twist to be of the same order as the bending in the asymptotic model of Proposition 4.1.

3. Dimensional analysis of equilibrium equations and reduction to a one-scale problem

First, we decompose the equations such as to separate the axial from the cross-section plane components. To do this, let us decompose U^* on Frenet basis (t, n, e_3) of the initial configuration as follows:

$$U^* = u_t^* t + u_n^* n + u_3^* e_3 \tag{3}$$

Then let us define the following dimensionless physical data and dimensionless unknowns of the problem:

$$\begin{aligned} &u_t = \frac{u_t^*}{u_{tr}}, \quad u_n = \frac{u_n^*}{u_{nr}}, \quad u_3 = \frac{u_3^*}{u_{3r}}, \quad x = \frac{x^*}{d_0}, \quad s = \frac{s^*}{d_0} \\ &x_3 = \frac{x_3^*}{L_0}, \quad r = \frac{r^*}{h_0}, \quad c_0 = \frac{c_0^*}{c_r}, \quad f_t = \frac{f_t^*}{f_{rt}}, \quad f_n = \frac{f_n^*}{f_{rn}} \\ &f_3 = \frac{f_3^*}{f_{r3}}, \quad g_t = \frac{g_t^*}{g_{rt}}, \quad g_n = \frac{g_n^*}{g_{rn}}, \quad g_3 = \frac{g_3^*}{g_{r3}} \end{aligned} \tag{4}$$

where the variables indexed by $({}_r)$ are the reference ones. The new variables which appear (without a star) are dimensionless. To avoid any assumption on the order of magnitude of the displacement components, the reference scales u_{tr}, u_{nr} and u_{3r} are firstly assumed to be equal to L_0. Thus we a priori allow large displacements. Introducing the dimensionnless variables previously defined in Eqs. (2) decomposed on Frenet basis, we obtain a new dimensionless problem involving the following dimensionless numbers which characterize the problem:

$$\varepsilon = \frac{d_0}{L_0}, \quad \eta = \frac{h_0}{d_0}, \quad \nu = h_0 c_r, \quad \mathcal{F} = \frac{h_0 f_r}{\mu}, \quad \mathcal{G} = \frac{g_r}{\mu} \tag{5}$$

The form ratios ε, η and ν characterize respectively the relative thickness, the inverse of the shooting pain and the shallowness of the rod. They are geometric data of the problem. The force ratios $\mathcal{F}$ and $\mathcal{G}$ represent respectively the ratio of the resultant on the thickness of the body forces (respectively of the surfaces forces) to μ considered as a reference stress. These numbers only depend on known physical quantities and must be considered as known data of the problem. As in [5,6], to obtain a one-scale problem, ε is chosen as the reference perturbation parameter and the other dimensionless numbers are linked to ε. In this paper, we will limit our study to strongly bent thin-walled rods, whose inverse of the shooting pain is of the same order as the relative thickness. Thus we have $\varepsilon = \eta = \nu$. Moreover, the rod is assumed to be subjected to moderate force levels such as:

$$\mathcal{F}_t = \mathcal{G}_t = \varepsilon^5, \qquad \mathcal{F}_n = \mathcal{G}_n = \varepsilon^5, \qquad \mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^4 \tag{6}$$

4. Asymptotic expansion of equations

Let us consider a thin-walled rod with open strongly bent cross-section subjected to force levels such as $\mathcal{F}_t = \mathcal{G}_t = \mathcal{F}_n = \mathcal{G}_n = \varepsilon^5$ and $\mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^4$. The standard asymptotic expansion method leads to postulate that the dimensionless solution $U = (V, u_3)$, with $V = (u_t, u_n)$, admits a formal expansion with respect to ε:

$$U = (V, u_3) = \left(V^0, u_3^0\right) + \varepsilon^1\left(V^1, u_3^1\right) + \varepsilon^2\left(V^2, u_3^2\right) + \cdots$$

This implies an expansion of the mapping function ψ and the stresses Σ:

$$\begin{aligned} &\psi = \psi^0 + \varepsilon\psi^1 + \varepsilon^2\psi^2 + \cdots \\ &\Sigma = \Sigma^0 + \varepsilon\Sigma^1 + \varepsilon^2\Sigma^2 + \cdots \end{aligned}$$

Replacing U, ψ and Σ by their expansion in the dimensionless equilibrium equations and equating to zero the factors of successive powers of ε, we obtain coupled problems $\mathcal{P}_{-6}$, $\mathcal{P}_{-5}, \ldots$ corresponding respectively to the cancellation of the factors of ε^{-6}, $\varepsilon^{-5}, \ldots$ We then have the following result:

Proposition 4.1. (i) *The leading term U^0 of the displacement expansion is equal to zero. Thus $V^0 = u_3^0 = 0$.*

(ii) *The axial displacement is equal to zero at the first order*: $u_3^1 = 0$.

(iii) *The first non-zeroth term* (V^1, u_3^2) *of the expansion of the displacement satisfies the following equations*:

$$V^1 = V(x_3) + (R_\Theta - I_2)x \tag{7}$$

$$u_3^2 = u_3(x_3) - \overline{R}_\Theta x \frac{\mathrm{d}V}{\mathrm{d}x_3} - \omega_n \frac{\mathrm{d}\Theta}{\mathrm{d}x_3} \tag{8}$$

where $V(x_3)$ *and* $u_3(x_3)$ *denote the bending and the axial displacement of the point* G_0 *respectively,* I_2 *the identity of* $\mathbb{R}^2$, Θ *the rotation angle around the* $(C_0 e_3)$ *axis whose matrix of rotation is*

$$R_\Theta = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix} \quad \text{and } \omega_n \text{ the sectorial surface given by } \frac{\mathrm{d}\omega_n}{\mathrm{d}s} = -q$$

(iv) *The first non-zeroth term of the expansion of the stress tensor is* Σ^2. *It can be written, in the* (t, n, e_3) *basis as*

$$\Sigma^2 = \begin{bmatrix} 0 & 0 & \Sigma_{t3}^2 \\ 0 & 0 & 0 \\ \Sigma_{t3}^2 & 0 & \Sigma_{33}^2 \end{bmatrix}$$

with

$$\Sigma_{t3}^2 = -2r\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}, \quad \Sigma_{33}^2 = \frac{E}{\mu}\frac{\partial u_3^2}{\partial x_3} + \frac{E}{2\mu}\left\| \frac{\partial \psi^1}{\partial x_3} \right\|^2 \quad \text{and} \quad \psi^1 = V^1 + x \tag{9}$$

(v) *The unknowns* $V(x_3)$, $u_3(x_3)$ *and* $\Theta(x_3)$ *are solution of the following one-dimensional equilibrium equations*:

- *Traction equation*

$$\frac{E}{2\mu}\frac{\mathrm{d}}{\mathrm{d}x_3}\left[\int_{s_g}^{s_d}\int_{-1}^{1}\left(\left\|\frac{\partial V^1}{\partial x_3}\right\|^2 + 2\frac{\partial u_3^2}{\partial x_3}\right)\mathrm{d}r\,\mathrm{d}s\right] = -P_3 \tag{10}$$

with

$$P_3 = \int_{s_g}^{s_d}\int_{-1}^{1} f_3\,\mathrm{d}r\,\mathrm{d}s + \int_{s_g}^{s_d}[g_3^+ - g_3^-]\,\mathrm{d}s$$

- *Twist equations*

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\left[\int_{s_g}^{s_d}\int_{-1}^{1}\left(-r(1 - c_0 Q)\Sigma_{t3}^2 + \omega_N\frac{\partial \Sigma_{33}^2}{\partial x_3} + \Sigma_{33}^2\overline{\Lambda\psi^1}\frac{\partial\psi^1}{\partial x_3}\right)\mathrm{d}r\,\mathrm{d}s\right] = -M_t - \frac{\mathrm{d}M_{3t}}{\mathrm{d}x_3} \tag{11}$$

with

$$M_t = \int_{s_g}^{s_d}\overline{\Lambda\psi^1}\int_{-1}^{1} f\,\mathrm{d}r + [g^+ - g^-]\,\mathrm{d}s, \qquad M_{3t} = \int_{s_g}^{s_d}\int_{-1}^{1}\omega_N f_3\,\mathrm{d}r\,\mathrm{d}s + \int_{s_g}^{s_d}\omega_N[g_3^+ - g_3^-]\,\mathrm{d}s$$

$Q = -\mathrm{d}\omega_N/\mathrm{d}s$, $\omega_N = \omega_n - \overline{V}\Lambda R_\theta x$, *where* $\Lambda = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ *denotes the rotation of* $\pi/2$ *around* e_3.

• *Bending equations*

$$\frac{E}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}\left[\int_{s_g}^{s_d}\int_{-1}^{1}\left[\left\|\frac{\partial\psi^1}{\partial x_3}\right\|^2+2\frac{\partial u_3^2}{\partial x_3}\right]\psi^1\,\mathrm{d}r\,\mathrm{d}s\right]+\frac{\mathrm{d}}{\mathrm{d}x_3}\left[\int_{s_g}^{s_d}\int_{-1}^{1}2r^2c_0\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}T\,\mathrm{d}r\,\mathrm{d}s\right]=-P-\frac{\mathrm{d}M_{3f}}{\mathrm{d}x_3} \tag{12}$$

where $T=R_\Theta t$ *and*

$$P=\int_{s_g}^{s_d}\int_{-1}^{1}\Pi f\,\mathrm{d}r\,\mathrm{d}s+\int_{s_g}^{s_d}\Pi[g^+-g^-]\,\mathrm{d}s,\qquad M_{3f}=\int_{s_g}^{s_d}\int_{-1}^{1}\psi^1 f_3\,\mathrm{d}r\,\mathrm{d}s+\int_{s_g}^{s_d}\psi^1[g_3^+-g_3^-]\,\mathrm{d}s$$

with the associated boundary conditions

$$V(x_3)=\frac{\mathrm{d}V(x_3)}{\mathrm{d}x_3}=0 \quad \text{on } \gamma_{01}\cup\gamma_{02}$$
$$u_3(x_3)=0 \quad \text{on } \gamma_{01}\cup\gamma_{02}$$
$$\Theta(x_3)=\frac{\mathrm{d}\Theta(x_3)}{\mathrm{d}x_3}=0 \quad \text{on } \gamma_{01}\cup\gamma_{02}$$

For the force level chosen, the order of magnitude of the displacement components are h_0 for the axial displacement and d_0 for the other components.[1] Thus with this approach, the order of magnitude of the displacements and the associated asymptotic model are directly deduced from the level of applied forces.

On the other hand, the non-linear model obtained in Proposition 4.1 is not classical and does not seem to have any equivalent in the literature. It reduces to a system of four non-linear differential equations strongly coupled, which can be expressed in terms of the four unknowns[2] $V(x_3)$, $u_3(x_3)$, $\Theta(x_3)$. In particular, the twist equation contains cubic terms with respect to Θ as in the model of Gobarah and Tso [7] derived from a priori assumptions. Moreover the twist equation is coupled with the traction (and the bending) equations. This coupling characterizes the shortening effect observed experimentally for large rotations.

Finally, let us notice that the twist couples M_t and M_{3f} are expressed in the deformed configuration, whereas they are classically expressed in the initial configuration.

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[1] It would have been possible to define new reference scales for the components of the displacement, for the leading term of the expansion to be different from zero.

[2] The displacement $V(x_3)$ has two components in the plane of a section.