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Time derivative obtained by applying the Riemannian manifold of Riemannian metrics to kinematics of continua

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Abstract

The infinite dimensional Riemannian geometry of Riemannian metrics is employed to propose novel objective time derivative by means of covariant derivative. ***To cite this article: Z. Fiala, C. R. Mecanique 332 (2004).***

Résumé

Dérivée temporelle obtenue en utilisant la variété riemannienne des métriques riemanniennes aux cinématiques des milieux continus. La géometrie riemannienne des métriques riemanniennes, de dimension infinie, est employeé pour proposer une nouvelle derivée temporelle objective par la biais de la dérivation covariante. ***Pour citer cet article : Z. Fiala, C. R. Mecanique 332 (2004).***

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1. Introduction

In order to consistently describe the successive processes of deformation, *differential geometry* is employed, which offers significantly more refined tools for the description of finite deformations than matrix calculus. It was Noll who initiated a deep interest in the mathematical foundations of mechanics of continua, and, at present, *Riemannian geometry* has been employed in many papers concerned with theoretical aspects of the deformation of continua (Marsden and Hughes [1], Giessen and Kollmann [2], Stumpf and Hoppe [3]).

In addition to presenting this standard view, this Note briefly outlines more advanced aspects of the *kinematics of finite deformations*, as they have appeared in the literature, although without taking much notice. It is the book

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of Rougée [4] in particular, which offers novel key ideas for a proper understanding of finite deformations, and closely related mathematical papers on geometry of *infinite dimensional Riemannian spaces*. By combining the Rougée's approach with this general mathematics, a new objective time derivative will be derived at the end.

2. Preliminary information – continuum mechanics and Riemannian manifold

Classically, the kinematics of continua is described by tensor fields on 3D Euclidean space $\mathbb{E}^3$ using general curvilinear coordinate systems, but the more modern approach, considering Euclidean space as a Riemannian manifold, predominates. For our purposes it suffices to characterize the Riemannian manifold as a set of points, with no privileged coordinate system endowed with metric, which enters the Riemannian manifold via the tangent space; for more, see Frankel and Schutz [5]. Making use of the geometrical entities (such as pull-back, push-forward, Lie derivatives, covariant derivative etc.) enables us to identify the actual geometrical content of the kinematics of finite deformations. The following three paragraphs offer a brief outline of Riemannian geometry in continuum mechanics. For more details, see [1–3] and [6].

2.1. Basic notions

- A *configuration* of a simple body B, is a mapping $\tilde{\Phi}: I \times B \to \mathbb{E}^3$ indexed by time from the interval $I = [0, T]$. The configurations at time 0 or at actual time t, called *referential* $R = \tilde{\Phi}(0, B)$ or *spatial* $S = \tilde{\Phi}(t, B)$ configurations respectively, form Riemannian manifolds. The mapping $\tilde{\Phi}$ then induces a mapping $\Phi : I \times R \to S$. We denote by X points from R, and by x points from S.
- The *tangent space* $T_X R$ is a linearized, infinitesimal neighbourhood of a point $X \in R$. It is a linear, finite-dimensional real vector space of all 'infinitesimal *material line elements*' represented by *vectors* tangent at the point X to curves lying in R.
- The *cotangent space* $T_X^* R$ is again a linear, finite-dimensional real vector space of all 'infinitesimal *material surfaces*' represented by *covectors*, which are quantities intimately related to gradients to functions at the point X. The covectors a act as linear mappings $\langle a, u\rangle_{T_X R}$ of vectors u to real numbers $\mathbb{R}$, and so the cotangent space is the *dual space* to the tangent space. Unlike the classical approach, making use of the dual space enables us to define the tensors on manifolds more clearly, and distinguish between vectors and covectors, contravariant and covariant tensors, being considered here as different objects. As above and below, the same applies to the spatial configuration.
- $(p$–$q)$-*tensors* (p-contravariant, q-covariant) on a linear vector space V, with V^* being its dual, are elements of the sets $\mathbf{T}_q^p = \mathbf{T}^p \otimes \mathbf{T}_q = V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$ and $\mathbf{T}_0^0 = \mathbb{R}$. Here, V stands for some tangent (T) space $T_X R$ or $T_x S$, and V^* for its corresponding cotangent (C) space $T_X^* R$ or $T_x^* S$.
- The key notion of Riemannian geometry is the *metric*, a positive-definite symmetric 2-covariant tensor G defining the *scalar product* of two vectors $u, v : u \cdot v \equiv G(u, v)$.
- The metric G defines a mapping $\mathbf{G}: T_X R \to T_X^* R$ via the relation $\langle \mathbf{G}u, v\rangle_{T_X R} = G(u, v)$. It assigns an *associated* covector $u^\flat$ to a vector u: $u^\flat = \mathbf{G}u$ (and conversely an *associated* vector $a^\sharp$ to a covector a: $a^\sharp = \mathbf{G}^{-1}a$). The so-called *associated tensors* $t^\flat$, $t^\sharp$ to a (1–1)-tensor t are (2–0)- and (0–2)-tensors respectively, defined by extending the mapping $\mathbf{G}$ to tensors. These operations correspond to *raising and lowering indexes* of components of tensors, in classical approach. The metric G also induces the scalar product on covector space, defined by means of the associated 2-contravariant tensor $G^\sharp$: $a \cdot b \equiv G^\sharp(a, b) = G(a^\sharp, b^\sharp)$.
- A mapping $\Phi : I \times R \to S$ induces the *tangent mapping* (or deformation gradient $\mathbf{F}$ in other words) $T\Phi (\equiv \mathbf{F}) : T_X R \to T_x S$. The tangent mapping defines *push-forward* Φ_* and *pull-back* Φ^* operations between corresponding spaces of tensors. These then, in a simple way, couple the description of deformation and stress state in the referential and spatial configurations: in fact, *the description of the motion in the* referential (spatial) *picture is obtained by* pull-back (push-forward) *of the* spatial (referential) *picture*.

2.2. Dual stress and strain tensors, dual time derivatives

The various stress and strain tensors, and their objective time derivatives can be related to each other (Hill [7], Haupt and Tsakmakis [8]) via the *stress power density*:

$$\pi_t \equiv \langle \sigma_t^\sharp, d_t^\flat \rangle_{TS^*} = \langle \sigma_t^\flat, d_t^\sharp \rangle_{TS} = \sigma_t \cdot d_t \tag{1}$$

where σ_t is the Cauchy stress (1–1)-tensor and d_t is the *rate-of-deformation* (1–1)-tensor.

Hill's result is obtained by *pulling-back* the spatial picture to the referential configuration, so that the *referential stress power density* can be written:

$$\pi_t^{\mathrm{ref}} = \begin{Bmatrix} \langle P_t^\sharp, \partial E_t^\flat \rangle_{TR^*} \\ \langle K_t^\flat, \partial H_t^\sharp \rangle_{TR} \end{Bmatrix} = \begin{cases} P_t \cdot \partial E_t \\ K_t \cdot \partial H_t \end{cases} \qquad \text{where } \partial \text{ stands for the \textit{material time derivative.}}$$

In the above, the following two relations, playing the key role in the next paragraph, were employed:

$$\Phi^* d^\flat = \partial E^\flat = \tfrac{1}{2} \partial C^\flat, \qquad \Phi^* d^\sharp = -\partial H^\sharp = -\tfrac{1}{2} \partial B^\sharp \tag{2}$$

By *pushing-forward* the Hill's result back to spatial configuration, Haupt and Tsakmakis obtained:

$$\pi_t^{\mathrm{ref}} = \begin{cases} \langle S_t^\sharp, L_{\mathbf{F}} e_t^\flat \rangle_{TR^*} & (L_{\mathbf{F}} e^\flat)_{ij} = \dot{e}_{ij} + (d-w)_{ik} g^{kl} e_{lj} + e_{ik} g^{kl} (d+w)_{lj} \\ -\langle S_t^\flat, L_{\mathbf{F}} h_t^\sharp \rangle_{TR} & (L_{\mathbf{F}} h^\sharp)^{ij} = \dot{h}^{ij} - (d+w)^{ik} g_{kl} h^{lj} - h^{ik} g_{kl} (d-w)^{lj} \end{cases}$$

$L_{\mathbf{F}} = \Phi_* \circ \partial \circ \Phi^*$ is the so-called *Lie derivative* (again $\mathbf{F} \equiv T\Phi$), and w is the vorticity. This *dual time derivative* (Oldroyd derivative), obtained from material derivative exactly the same way as the corresponding dual stress and strain tensors, is naturally *objective*. $S = J\sigma$ is the weighted Cauchy (or Kirchhoff) stress (1–1)-tensor, and the Jacobian J (being scalar) is the determinant of the tangent mapping transformation $J = \det(\partial\Phi/\partial X)\sqrt{\det(g)/\det(G)}$. G and g denote the metric on R and S, respectively (i.e., the scalar product on $T_X R$ and $T_x S$, respectively). Below, T stands for tangent, C for cotangent spaces.

T–R:	$C^\flat = \Phi^* g$	the associated RIGHT CAUCHY–GREEN *deformation* (0–2)-tensor	(3)
	$E = \frac{1}{2}(C - I)$	the GREEN–ST.VENENT *strain* (1–1)-tensor (Lagrangian strain tensor)	
	$P^\sharp = \Phi^* S^\sharp$	the associated SECOND PIOLA–KIRCHHOFF *stress* (2–0)-tensor	
T–S:	$c^\flat = \Phi_* G$		
	$e = \frac{1}{2}(i - c)$	the ALMANSI–HAMEL *strain* (1–1)-tensor (Eulerian strain tensor)	
	$S^\sharp$	the (contravariant) WEIGHTED CAUCHY *stress* (2–0)-tensor	

C–R:	$B^\sharp = \Phi^* g^\sharp$		(4)
	$H = \frac{1}{2}(B - I)$	the PIOLA *strain* (1–1)-tensor	
	$K^\flat = -\Phi^* S^\flat$	the associated NEGATIVE CONVECTED *stress* (0–2)-tensor	
C–S:	$b^\sharp = \Phi_* G^\sharp$	the associated LEFT CAUCHY–GREEN *deformation* (2–0)-tensor	
	$h = \frac{1}{2}(i - b)$	the FINGER *strain* (1–1)-tensor	
	$-S^\flat$	the (covariant) NEGATIVE WEIGHTED CAUCHY *stress* (0–2)-tensor	

3. Advanced information – continuum and Riemannian manifold of Riemannian metrics

In summary, here are the main points of the previous paragraphs required later on:

– Finite deformations of the continua at the referential point X are described by any of two *deformation* tensors $C^\flat$ (3) (cf. Ciarlet and Laurent [9]) or $B^\sharp$ (4).

– Their time derivatives $\partial C^\flat$ or $\partial B^\sharp$, in the progress of deformation, are obtained by pulling-back the corresponding associated tensors of the *rate-of-deformation tensor* $d^\flat$ or $d^\sharp$ (see (2)).

At this point, some comments are still in order on why the *deformation tensors*, *instead of strain tensors*, *are more fitting for the description of the process of finite deformations*. The answer reflects the very nature of the difference between finite and small deformations: provided we split the deformation $x \equiv \Phi(X) = X + u(X)$, for two successive deformations $X \to x_1 \to x_2$, the following relation holds $x_2 = \Phi_2 \circ \Phi_1(X) = \Phi_2(x_1) = \Phi_2(X + u_1(X)) = X + u_1(X) + u_2(X + u_1(X))$. In case of small deformations one neglects all the terms of the second order in magnitude, and so the relation takes the form $x_2 \approx X + u_1(X) + u_2(X)$, i.e., the diffeomorphism Φ acts as identity mapping $x = \Phi(X) \approx X$, and the concept of diffeomorphisms changes into that of fields. Similarly, for the deformation gradient $T\Phi \equiv I + Tu \approx I$, and for transformations of vectors and covectors: $v = T\Phi(V) \approx V$ and $A = T\Phi^*(a) \approx a$. The concept of small deformations thus identifies tangent and cotangent spaces in referential and spatial configurations. In particular, the metric tensors are equal $g \approx G$, and the objective time derivative is replaced by the simple material time derivative $L_{\mathbf{F}} = \Phi_* \circ \partial \circ \Phi^* \approx \partial$. Infinitesimal variation $u(X)$ around identity mapping $\Phi(X) = x \approx X$ at the point $x = \Phi(X)$ (i.e., linearization of mapping Φ in other words) results in substituting fields for diffeomorphisms, and enters the theory of small deformations via infinitesimal variation of the metric $g = G$. It is the strain tensors $e \approx E,\ h \approx H$ that represent this infinitesimal variation. Now, $c^\flat \approx C^\flat$ and $h^\sharp \approx H^\sharp$, and the relations (2) read: $\partial C^\flat = 2\partial E^\flat \approx 2d^\flat,\ \partial B^\sharp = 2\partial H^\sharp \approx -2d^\sharp$.

On the other hand, in case of finite deformations the deformation process no longer keeps moving inside the tangent linear space $T_{C^\flat}\mathbf{M}$ (see later on), as in the case of small deformations, and the finite difference between initial and terminal deformation tensors provides the same piece of information about deformation, as Euclidean distance between starting and ending points about the whole trajectory of a particle does: that means no information! Consequently, the deformation process at each material point X should be described not by time dependent strains, but by a trajectory in the *manifold* $\mathbf{M} = \mathrm{Met}(R)$ of all possible deformation tensors (relative to reference configuration).

A fundamental observation of Rougée [4] made it possible for him to significantly broaden the analysis of the process of finite deformations. He realized that the quantities $\partial C_t^\flat$ in fact constitute tangent vectors to the manifold $\mathbf{M}$ at the particular point $C_t^\flat$, chosen at the actual moment of time t. With the assistance of the relation $\partial C_t^\flat = 2\Phi_t^* d^\flat$ he introduced a scalar product on the *tangent space* $T_{C_t^\flat}\mathbf{M}$, so that the manifold $\mathbf{M}$ became Riemannian manifold. He managed to do this by extending the usual scalar product of vectors, defined on $T_x S$ by the metric g, to a scalar product of 2-tensors (see also (1)). In particular, for the rate-of-deformation tensor $d^\flat$ he obtained: $d^{1\flat} \cdot d^{2\flat}|_{g,x} \equiv g^{ik} g^{jl} d^1_{kj} d^2_{li}$. As the diffeomorphism Φ_t is actually an *isometry* (a metric preserving diffeomorphism between Riemannian spaces $(R, C_t^\flat = \Phi_t^* g)$ and (S, g)), he introduced the scalar product on the tangent space $T_{C_t^\flat}\mathbf{M}$ via the relation: $\partial C^{1\flat} \cdot \partial C^{2\flat}|_{C_t^\flat, X} \equiv\ \Phi_t^*(d^{1\flat} \cdot d^{2\flat})$, where $\partial C^{i\flat} \in T_{C_t^\flat}\mathbf{M},\ C_t^\flat = \Phi_t^* g,\ B_t^\sharp = (C_t^\flat)^{-1}$, and $\partial C^{i\flat} = 2\Phi_t^* d^{i\flat}$. Carrying out the pull-back operation, Rougée eventually obtained the *metric* on $\mathbf{M}$ (i.e., the scalar product on the tangent space $T_{C_t^\flat}\mathbf{M}$):

$$\partial C^{1\flat} \cdot \partial C^{2\flat}\big|_{C_t^\flat, X} = \frac{1}{4} B_t^{ik} B_t^{jl} \partial C^1_{kj} \partial C^2_{li} \tag{5}$$

Do not be confused by considering the deformation tensors $C_t^\flat$ as points of the Riemannian manifold $\mathbf{M}$ and their material time derivatives $\partial C^{i\flat} \in T_{C_t^\flat}\mathbf{M}$ as vectors lying in the corresponding tangent space $T_{C_t^\flat}\mathbf{M}$, at a particular point $C_t^\flat$ of $\mathbf{M}$! As we shall see in the next three paragraphs, such a viewpoint offers far-reaching implications for the description of kinematics of finite deformations.

First, one can define time derivative of a time-dependent tensor over S, via the *covariant derivative* of vector fields over $\mathbf{M}$. For a vector $V \in T_{C^\flat}\mathbf{M}$ and a vector field U over $\mathbf{M}$, the covariant derivative can be expressed:

$$(\nabla_V U)_{ij} = \left(\frac{\delta U}{\delta V}\right)_{ij} - \frac{1}{2}\big(V_{il}B_t^{lk}U_{kj} + U_{il}B_t^{lk}V_{kj}\big), \quad \text{where } \left(\frac{\delta U}{\delta V}\right)_{ij} \equiv \frac{d}{dq}U_{ij}\big(C_t^\flat + qV\big)\big|_{q=0}$$

Let $C_t^\flat : I \to \mathbf{M}$ denote a smooth curve, then the derivative of vector field U along the curve can be written:

$$\left(\frac{D}{Dt}U\right)_{ij} \equiv (\nabla_{\partial C_t^\flat}U)_{ij} = \partial U_{ij} - \frac{1}{2}\big((\partial C_t)_{il}B_t^{lk}U_{kj} + U_{il}B_t^{lk}(\partial C_t)_{kj}\big), \quad \text{since } \frac{\delta U}{\delta(\partial C_t^\flat)} = \partial U$$

As now $\partial C_t^\flat = 2\Phi_t^* d^\flat$, pushing the above derivative forward to the spatial configuration, one obtains

$$\left(\frac{D}{Dt}u\right)_{ij} \equiv \big(\Phi_{t*}(\nabla_{\partial C_t^\flat}U)\big)_{ij} = (L_{\mathbf{F}}u)_{ij} - \big(d_{il}g^{lk}u_{kj} + u_{il}g^{lk}d_{kj}\big) = \big(\dot{u}^{ZJ}\big)_{ij} \tag{6}$$

$u = \Phi_{t*}U$ is any spatial 2-covariant symmetric tensor over S corresponding to a vector field U over $\mathbf{M}$. The resulting time derivative is the Zaremba–Jaumann derivative (Rougée [4]). If we interpret parameter t as time and the curve $C_t^\flat$ as a deformation process taking place at point X, the underlying mathematical structure of the Riemannian manifold $\mathbf{M}$, based on the metric (5), then unambiguously selects the only one objective time derivative (6). Note also that $\dot{g}^{ZJ} = 0$, or equivalently $DC_t^\flat/Dt = 0$.

Second, the geometrical structure of the manifold $\mathbf{M}$ enables us to clarify the geometrical meaning of *logarithmic strains* by relating them to *geodesics* (Rougée [4]).

Third, the manifold $\mathbf{M}$ can be split (Freed and Groisser [10]) into volumetric and shape submanifolds: $\mathbf{M} \cong \mathrm{Vol}(R) \times \mathrm{Met}_\mu(R)$. Whereas the space $\mathrm{Vol}(R)$ is *flat*, the space $\mathrm{Met}_\mu(R)$ has *nonzero curvature* (negative), resulting in the dependence of deformation processes on the trajectory $C_t^\flat$ in $\mathbf{M}$. In particular, here seems to lie the problems with the existing use of logarithmic strains in modelling of constitutive relations.

4. Discussion and proposal of novel time derivative

Provided we eliminate the restriction of deformation processes to a single material point $X \in R$, which is the case of Rougée [4], we have slightly to modify the above theory. Now, the Riemannian metric is a tensor field $C_t^\flat$ of deformation tensors over the referential configuration R, and the corresponding manifold $\mathbf{M}$ of such Riemannian metrics is an *infinite dimensional Riemannian manifold* (Freed and Groisser [10], Gill-Medrano and Michor [11], Kriegel and Michor [12]). The *metric* (5) should then be modified by

$$\big\langle U^1, U^2\big\rangle_{C_t^\flat} = \int_R U^1 \cdot U^2\big|_{C_t^\flat, X}\, \mathrm{dVOL}_X\big(C_t^\flat\big) = \int_R \frac{1}{4}B_t^{ik}B_t^{jl}U_{kj}^1U_{li}^2\sqrt{\det\big(C_t^\flat\big)}\big|_X\, \mathrm{d}X \tag{7}$$

Now, due to the additional multiplicative term $\sqrt{\det(C_t^\flat)}$ (which appears quite natural from the viewpoint of the relation (1) and that what immediately follows), the *covariant derivative* can be written:

$$(\nabla_V U)_{ij} = \left(\frac{\delta U}{\delta V}\right)_{ij} - \frac{1}{2}\big(V_{il}B_t^{lk}U_{kj} + U_{il}B_t^{lk}V_{kj}\big) + \frac{1}{4}\big(B_t^{kl}V_{lk}U_{ij} - B_t^{kl}V_{lo}B_t^{op}U_{pk}(C_t)_{ij} + V_{ij}B_t^{kl}U_{lk}\big)$$

and so for the objective time derivative of a spatial symmetric 2-covariant tensor field u one obtains:

$$\left(\frac{D}{Dt}u\right)_{ij} = \big(\dot{u}^{ZJ}\big)_{ij} + \frac{1}{2}\big(g^{kl}d_{lk}u_{ij} - g^{kl}d_{lo}g^{op}u_{pk}g_{ij} + d_{ij}g^{kl}u_{lk}\big) \tag{8}$$

5. Conclusion

The approach sketched above, initiated by Rougée [4], offers a great number of entirely novel ideas in the kinematics of finite deformations and deserves further scrutiny. To this end, the mathematical theory of infinite dimensional Riemannian manifolds of Riemannian metrics, as described in the papers citied in the previous paragraph, will no doubt proof helpful. As a starter, *new objective time derivative* (8) with clear geometrical origin has been proposed. As the time derivative should represent the *rate of change of quantities* attached to the points $X \in R$, the new objective time derivative seems to be promising. In fact, $Dg/Dt = 3/2 \cdot d^{\flat}$ (cf. $L_{\mathbf{F}} g = 2d^{\flat}$) and $DC_t^{\flat}/Dt = 3/4 \cdot \partial C_t^{\flat}$, contrary to the Zaremba–Jaumann derivative, for which these derivatives are zeros.

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