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Stabilising effect of the Coriolis forces on a viscous liquid film flowing over a spinning disc

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Abstract

A model developed for the description of finite-amplitude wave regimes in a film flowing over a spinning disc is used to explain the stabilizing role of the Coriolis forces. As in the case of waves on a falling film, a linear relationship between the wave velocity and the maximum of the film thickness has been determined for axisymmetric waves in a film flowing over a spinning disc. ***To cite this article: G.M. Sisoev et al., C. R. Mecanique 332 (2004).***

Résumé

Effet stabilisant des forces de Coriolis sur un film liquide visqueux s'écoulant sur un disque tournant. Un modèle développé pour la description des régimes d'ondes d'amplitude finie dans un film s'écoulant sur un disque tournant est utilisé pour expliquer le rôle stabilisant des forces de Coriolis. Comme dans le cas d'ondes sur un film tombant, une relation linéaire entre la vitesse de l'onde et l'épaisseur maximale du film a été établie pour les ondes axisymétriques dans un film s'écoulant sur un disque tournant. ***Pour citer cet article : G.M. Sisoev et al., C. R. Mecanique 332 (2004).***

Keywords: Fluid mechanics; Waves; Bifurcations

Mots-clés : Mécanique des fluides ; Ondes ; Bifurcations

1. Introduction

Film flow over a spinning disc is widely applied in chemical engineering, in pharmaceutical production and for spray generation or atomization of liquids. Because this flow is similar to that down a vertical plane, the so-called falling film problem, theoretical studies are often based on theories developed for the latter flow. Flow over

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a spinning disc, however, is complicated by the presence of Coriolis forces and the variation of the body forces, centrifugal and Coriolis, along the radius. These two factors which are absent in falling films render the analysis of flow over a spinning disc more difficult, especially for the flow conditions realised in experiments.

In recent work [1,2], we have reviewed the relevant experimental and theoretical studies, with particular emphasis placed on the modelling of finite-amplitude wave regimes at moderate flow rates. The first strongly non-linear model was formulated in [3] for the case of negligible Coriolis forces, or, formally, large Eckman number. In this limit, the problem reduces to that of a falling film, which was experimentally confirmed in [4] in the case of relatively short waves; longer waves were calculated in [1] for conditions corresponding to the experimental data obtained in [5]. However, the recalculation of flow parameters carried out in [2] revealed that the parameter characterising the Coriolis forces, the Eckman number, is actually finite in the majority of cases and these forces must be incorporated into the theoretical analysis. The second important issue, also determined in [2], is the existence of regular wave regimes under flow conditions that correspond to falling films with a disordered film surface. As a result of detailed analysis of the flow parameters corresponding to axisymmetric wave regimes, a thin layer approximation to the full Navier–Stokes problem was employed. Further, the Galerkin method with a single basis function for each velocity component was applied and a system of evolution equations was derived for the film thickness and flow rates in the radial and azimuthal directions. Based on experimental observations demonstrating weak variation of the wave parameters along the radius, local bifurcation analysis was successfully applied to compute quasi-steady travelling waves for the cases measured in experiments [5]. Later, in [6], we confirmed that the local travelling waves are realised in transient numerical solutions of the nonlinear evolution equations.

In the present Note, we demonstrate that, despite essential deformation of the bifurcation structure with decreasing Eckman number, the relationship between the velocities and maximum heights of the waves in a film flowing over a spinning disc is linear, just as in the case of a falling film. We have restricted our investigation to the case of axisymmetric waves for which the model developed in [2] was validated against experimental data from numerous studies; these data comprised the shapes and velocities of axisymmetric waves, which accompany the flow over a spinning disc at low and moderate flowrates. The analysis presented here may be extended to nonaxisymmetric flows, and, in particular, to the case of spiral waves observed for relatively large flowrates [7].

2. Mathematical model

A system of nonlinear evolution equations for a viscous, incompressible liquid film, of kinematic viscosity ν, density ρ and surface tension, σ, flowing over a solid, impermeable disc rotating with angular velocity Ω was derived in [2]. To describe axisymmetric flow, the laboratory cylindrical coordinate system, $(\tilde{r}, \theta, \tilde{z})$ is used, along with the velocity components $(\tilde{u}_r, \tilde{u}_\theta, \tilde{u}_z)$; here, the tilde decoration designates dimensional quantities; the following dimensionless variables are introduced

$$\tilde{t} = \frac{\kappa \mathrm{E}}{\Omega} t_\kappa, \quad \tilde{r} = R_c\, \mathrm{e}^{\kappa x_\kappa}, \quad \tilde{z} = H_c z, \quad \tilde{u}_r = \frac{\Omega \tilde{r} u}{\mathrm{E}}, \quad \tilde{u}_\theta = \Omega \tilde{r}\left(1 + \frac{v}{\mathrm{E}}\right), \quad \tilde{h} = H_c h$$

where, $\tilde{h}$ denotes the local film thickness, $\tilde{t}$ represents time, R_c and H_c are scales for the radius and film thickness; parameters E and κ are defined below.

The dynamics of the dimensionless film thickness, h, and flowrates in the radial and azimuthal directions, $q^{(u)}$ and $q^{(v)}$, respectively, given by

$$q^{(u)} \equiv \int_0^h u \,\mathrm{d}z, \qquad q^{(v)} \equiv \int_0^h v \,\mathrm{d}z$$

are governed by the system of equations

$$\frac{\partial h}{\partial t_\kappa}+\frac{\partial q^{(u)}}{\partial x_\kappa}+2\kappa q^{(u)}=0 \tag{1}$$

$$\frac{\partial q^{(u)}}{\partial t_\kappa}+\beta_{11}\frac{\partial}{\partial x_\kappa}\left(\frac{(q^{(u)})^2}{h}\right)+\frac{\kappa}{h}\left[\beta_{12}\left(q^{(u)}\right)^2-\beta_{13}\left(q^{(v)}\right)^2\right]$$
$$=\frac{1}{45\delta}\left[\mathrm{e}^{-2\kappa x_\kappa}h\frac{\partial}{\partial x_\kappa}\left(\mathrm{e}^{-2\kappa x_\kappa}\frac{\partial^2 h}{\partial x_\kappa^2}\right)-\beta_{14}\frac{q^{(u)}}{h^2}+h+\frac{2q^{(v)}}{\mathrm{E}}\right] \tag{2}$$

$$\frac{\partial q^{(v)}}{\partial t_\kappa}+\beta_{21}\frac{\partial}{\partial x_\kappa}\left(\frac{q^{(u)}q^{(v)}}{h}\right)+\beta_{22}\frac{\kappa q^{(u)}q^{(v)}}{h}=-\frac{1}{45\delta}\left(\beta_{23}\frac{q^{(v)}}{h^2}+\frac{2q^{(u)}}{\mathrm{E}}\right) \tag{3}$$

where the coefficients are

$$\beta_{11}=\frac{6}{5},\quad \beta_{12}=\frac{18}{5},\quad \beta_{13}=\frac{155}{126},\quad \beta_{14}=3,\quad \beta_{21}=\frac{17}{14},\quad \beta_{22}=\frac{34}{7},\quad \beta_{23}=\frac{5}{2}$$

The system (1)–(3) contains two similarity parameters,

$$\delta=\frac{1}{45\nu^2}\left(\frac{\rho\Omega^8R_c^4H_c^{11}}{\sigma}\right)^{1/3},\qquad \mathrm{E}=\frac{\nu}{\Omega H_c^2}$$

and a related parameter $\kappa=(45\delta\mathrm{E}^2)^{-1}$; here, the following film thickness scale has been used

$$H_c=\left(\frac{Q_c\nu}{2\pi\Omega^2R_c^2}\right)^{1/3}$$

The system (1)–(3) is derived from the full Navier–Stokes equations in two steps. Firstly, terms of $\mathrm{O}(H_c^2R_c^{-2}\kappa^{-2})$ are omitted; as shown in [2], this approximation is justified in all of the experimental work examined. This simplification, nevertheless, conserves all essential effects: viscosity, convective non-linearity, capillarity, centrifugal and Coriolis forces. Secondly, the Galerkin method with a single function each for the radial and azimuthal velocity components is applied. It is shown in [2] that the system (1)–(3) is reduced to the Shkadov system [8], widely applied for the modelling of a falling film [9,10], as $\mathrm{E}\to\infty$ or, in physical terms, at large radii, fast rotational speed or small flowrate.

Due to the small values of κ in the majority of the experiments conducted, the wave parameters vary slowly along the radius. To compute local travelling waves, the last term in Eq. (1), as well as the exponential multipliers in the first term on the right-hand side of Eq. (2) are omitted. Detailed bifurcation analysis of the resulting equations, given in [2], permitted the computation of 'families' of local travelling waves in the form

$$h=3^{\frac{1}{3}}h'(\eta'),\quad q^{(u)}=q'(\eta'),\quad q^{(v)}=3^{1/2}p'(\eta'),\quad \eta'=\omega_\kappa'(\lambda_\kappa'x_\kappa'-t_\kappa'),\quad x_\kappa=3^{1/9}x_\kappa',\quad t_\kappa=3^{4/9}t_\kappa'$$

A family is defined as a smooth set of solutions parameterised by an inner parameter, which is the wave frequency. A family is denoted by $\gamma_{\pm m,j}$ where the indices indicate the wave behaviour at small wavenumber. The positive sign is used for fast waves whose velocity is larger than the velocity of neutral perturbations computed in the framework of linear stability analysis, the negative sign is used for slow waves. The number of main oscillations in the wave shape is given by j ($j=1,2$) and the number of the family among the slow (or fast) one-humped (or two-humped) families is given by m; a detailed description of the bifurcation structure may be found in [2]. The governing parameters are also modified: $\delta'=3^{11/9}\delta$ and $\mathrm{E}'=3^{-1/6}\mathrm{E}$. Primed variables are introduced in order to facilitate comparisons with solutions for a falling film; below the primes are suppressed because only travelling waves will be discussed.

A nonlinear eigenvalue problem was given in [2] for the periodic functions $h(\eta)$ and $p(\eta)$, the average film thickness h_m and the inverse wave velocity λ_κ. Numerical investigation of travelling waves demonstrated the strong stabilizing effect of the Eckman number: decreasing E led to the narrowing of the band of unstable wavenumbers and the modification of the structure of the bifurcation diagram.

3. Height maxima and velocities of waves

In the case of a falling film, the principal experimental result of the linear relationship between the maximum height, $h_{\max}$, and wave velocity, c_κ, was observed for a wide interval of flow conditions [9]. This dependence means that all experimental data plotted on a plane $(h_{\max}, c_\kappa)$ are inside a band with almost straight, parallel borders. This observation was explained in [11] using the concept of *dominating waves* introduced in [12], which, for given flow conditions, have maximum velocity and height among all possible travelling waves. It was also shown in [12, 13] that the dominating wave is an attractor of the unsteady problem for spatially-periodic waves: the numerical solution from any initial conditions evolves into the dominating wave. The set of dominating waves parameterized by the wavelength consists of segments of different wave families, which, for the falling film, depend on the single governing parameter δ. In the vicinity of wavelength values where the attractor jumps from one family to another, oscillating wave regimes may also form.

The concept of dominating waves for a falling film was successfully applied to experimental data for film flow over a spinning disc [5] in [2]. Direct numerical solutions of the system (1)–(3) carried out in [6] confirmed that the quasi-steady travelling waves that form are very close to some of the dominating waves.

Here, the concept of dominating waves is used to analyse the relationship between c_κ and $h_{\max}$ for two cases, $\delta = 0.1$ and 0.5 with $\mathrm{E} = 5$. The results are shown in Fig. 1 where solutions with $\omega_\kappa \geqslant 0.1$ are presented (lower

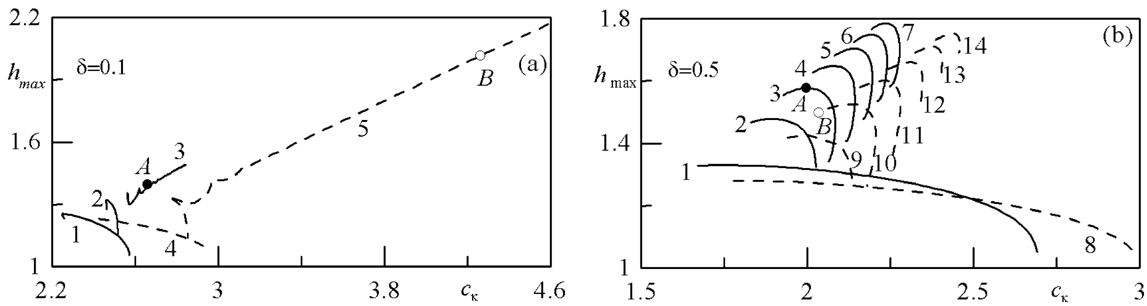

Fig. 1. Dominating waves. (a) Families for flow over a spinning disc for $\mathrm{E} = 5$: $\gamma_{-1,1}$ (curve 1), $\gamma_{-2,1}$ (2), $\gamma_{+1,1}$ (3). Families for a falling film: $\gamma_{-1,1}$ (4), $\gamma_{+1,1}$ (5). (b) Families for flow over a spinning disc for $\mathrm{E} = 5$: $\gamma_{-1,1}$ (1), $\gamma_{-2,1}$ (2), $\gamma_{-3,1}$ (3), $\gamma_{-4,1}$ (4), $\gamma_{-5,1}$ (5), $\gamma_{-6,1}$ (6), $\gamma_{-7,1}$ (7). Families for a falling film: $\gamma_{-1,1}$ (8), $\gamma_{-2,1}$ (9), $\gamma_{-3,1}$ (10), $\gamma_{-4,1}$ (11), $\gamma_{-5,1}$ (12), $\gamma_{-6,1}$ (13), $\gamma_{-7,1}$ (14). The wave profiles associated with points A and B are shown in Fig. 2.

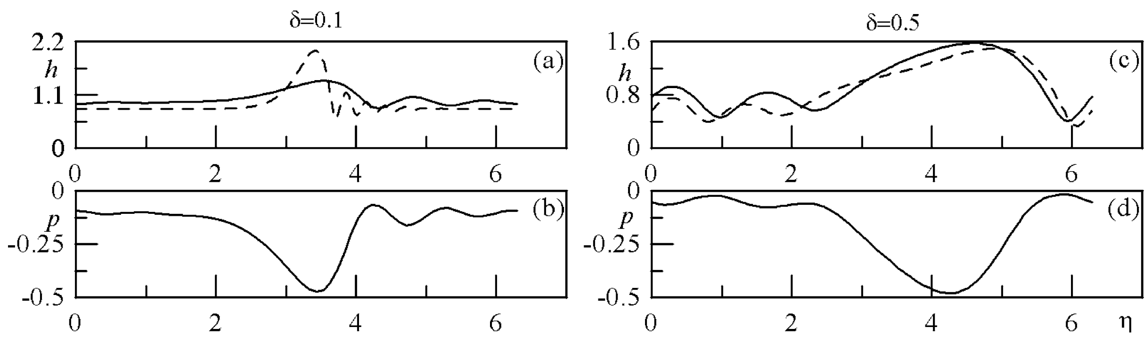

Fig. 2. Waves of the family $\gamma_{+1,1}$ at $\omega_\kappa = 0.5$: (a), film thickness for film flow over a spinning disc at $\mathrm{E} = 5$ (solid curve) and for a falling film (dashed curve); (b), azimuthal flow rate, p, for film flow over a spinning disc at $\mathrm{E} = 5$. Waves of the family $\gamma_{-3,1}$ at $\omega_\kappa = 1.1$: (c), film thickness for film flow over a spinning disc at $\mathrm{E} = 5$ (solid curve) and for a falling film (dashed curve); (d), azimuthal flow rate, p, for film flow over a spinning disc at $\mathrm{E} = 5$. Solid curves in panels (a), (b) and (c), (d) correspond to points A in panels (a) and (b) of Fig. 1, respectively. Dashed curves in panels (a) and (c) correspond to points B in panels (a) and (b) of Fig. 1, respectively.

frequencies must be considered separately due to comparability of the wavelength with the radial scale). It is seen that, as in the case of the falling film, there is a band in the plane $(h_{\max}, c_\kappa)$ where all the solutions lie. This band is shifted towards the $h_{\max}$ axis for the spinning disc: waves of the same amplitude travel more slowly than in the case of a falling film.

To illustrate the effect of decreasing the Eckman number, examples of wave profiles are shown in Fig. 2. It is seen that the difference in wave shapes, as well as wave parameters, between the spinning disc and falling film flows, denoted by points A and B in Fig. 1, respectively, is greater for $\delta = 0.1$ than for $\delta = 0.5$. Nevertheless, in both cases, the distribution of the azimuthal flow rate, p, is out of phase with the film thickness. The main term associated with the azimuthal flow rate is the Coriolis term, the last term on the right-hand side of Eq. (2), so the wave peaks are affected most by its retarding effect. This is in accordance with the general picture presented in Fig. 1.

4. Conclusion

Based on the concept of dominating waves, which has been confirmed by comparison with experimental measurements in [2] and transient numerical simulations in [6], the relationship between wave velocities and maximum heights for film flow over a spinning disc has been considered. It has been shown that this dependence is almost linear and similar to the case of a falling film. The dominating waves on the spinning disc, however, travel more slowly due to the presence of Coriolis forces, which are shown to provide a retarding effect.

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