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# Parametric excitation of waves on a free boundary of a horizontal fluid layer ☆

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**Abstract**

We consider the dynamical stability of horizontal fluid layer, performing harmonic oscillations in vertical direction. The continued fractions approach allowed us to avoid the conventional restriction to the case of small viscosity and almost-resonant frequencies. Our numerical results cover a wide range of the parameters (viscosity, amplitude and frequency of the oscillation, and depth of the layer). ***To cite this article: V.I. Yudovich et al., C. R. Mecanique 332 (2004).***


**Résumé**

**L'exitation paramétrique des ondes sur la surface libre d'une couche horizontale de liquide.** Nous examinons l'instabilité dynamique d'une couche horizontale d'un liquide faisant des oscillations harmoniques verticales. L'utilisation des fractions continues nous a permis déviter les restrictions habituelles de petite viscosité et de fréquences presque résonnantes. Nous obtenons des résultats numériques pour un large domaine des paramètres (la viscosité, l'amplitude et la fréquence des oscillations, la profondeur de la couche). ***Pour citer cet article : V.I. Yudovich et al., C. R. Mecanique 332 (2004).***


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## 1. Problem formulation: Integro-differential equation

We consider a viscid fluid layer, bounded by a flat horizontal impermeable boundary (zero normal component of velocity). The second boundary condition can be a non-slip one (rigid wall) or zero tangent stresses ("soft wall"). We take our interest in flows caused by vertical wall oscillations, governed by the law $x_3 = Af(\widetilde{\omega}t)$ with given

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amplitude $A$ and frequency $\widetilde{\omega}$, the function $f$ is assumed to be $2\pi$-periodic. In Cartesian coordinate system, rigidly fixed with the lower boundary, the equations of motion are

$$\frac{\partial v}{\partial t} + (v, \nabla)v = -\frac{1}{\rho}\nabla p + \nu \triangle v + g(t)k, \quad \operatorname{div} v = 0 \tag{1}$$

Here $v$ is the relative velocity, $p$ is the pressure, $\rho$ is the liquid density, $\nu$ is the kinematical viscosity coefficient and $k$ is the normal unit vector of vertical axis $z$, which is directed downwards. We denote coordinates as $x_1,\ x_2,\ x_3$, and where it is convenient, as $x,\ y,\ z$; $g(t) = g_0 - A\widetilde{\omega}^2 f''(\widetilde{\omega}t)$ is the variable acceleration of gravity where $g_0$ is its mean value.

Let us further assume that $z = H$ is an impermeable and non-deformable free boundary ("soft wall"), on which the following conditions are satisfied:

$$v_3 = 0, \qquad \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} = 0, \qquad \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} = 0 \tag{2}$$

In assumption that the free boundary is defined by an equation $z = \xi(x_1, x_2, t)$ we write the boundary conditions on it as

$$v \cdot \ell = \frac{\partial \xi}{\partial t}, \qquad \ell = \left(-\frac{\partial \xi}{\partial x_1}, -\frac{\partial \xi}{\partial x_2}, 1\right), \qquad (p - p_0)n_i = \tau_{ij}n_j + \sigma_0 \Gamma n_i, \quad i = 1, 2, 3 \tag{3}$$

Here $\ell$ is the inner normal vector, $n = \ell/|\ell|$ is its unit vector; $p_0$ is the atmospheric pressure; $\tau_{ij}$ are the viscous stress tensor components; $\sigma_0$ is the coefficient of surface tension; $\Gamma$ is the mean curvature, so that

$$\Gamma = \frac{(1 + \xi_{x_1}^2)\xi_{x_2x_2} + (1 + \xi_{x_2}^2)\xi_{x_1x_1} - 2\xi_{x_1}\xi_{x_2}\xi_{x_1x_2}}{(1 + \xi_{x_1}^2 + \xi_{x_2}^2)^{3/2}} \tag{4}$$

In our paper we concentrate on fluid flows periodic in $x_1,\ x_2$ directions with periods $L_1$ and $L_2$, respectively. Besides, we assume that the mean depth of the layer is prescribed and equals to $H$, so that

$$\langle \xi \rangle = \frac{1}{L_1 L_2} \int_0^{L_1} \int_0^{L_2} \xi(x_1, x_2, t)\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0 \tag{5}$$

Problem (1)–(5) has a solution, corresponding to a relative equilibrium

$$v^0 = 0, \qquad p^0 = \rho g(t)z + p_0, \qquad \xi^0 = 0 \tag{6}$$

We consider its stability with the use of the linearization method. Substituting $v = v^0 + u,\ p = p^0 + P,\ \xi = \xi^0 + \zeta$ and converting to the dimensionless variables, we obtain the following problem for the infinitesimal disturbances $u$, $P$, $\zeta$

$$\frac{\partial u}{\partial t} = -\nabla P + \delta \triangle u, \quad \operatorname{div} u = 0 \tag{7}$$

$$z = 0: \quad u_3 = \frac{\partial \zeta}{\partial t}, \quad \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = 0, \quad \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0 \tag{8}$$

$$-P + 2\delta \frac{\partial u_3}{\partial x_3} + C\triangle_1 \zeta - Q(\omega t)\zeta = 0 \tag{9}$$

$$z = h: \quad u_3 = 0, \quad \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = 0, \quad \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0 \tag{10}$$

The following dimensionless parameters and functions are used: $\delta = \nu T/L^2$ is the viscosity, $\omega = \widetilde{\omega}T$ is the frequency of modulation, $Q(\omega t) = Q_0 - a\omega^2 f''(\omega t)$ $(a = A/L$ is the amplitude of modulation) is the variable acceleration of gravity, $Q_0 = g_0 T^2/L$ is its mean value, $-a\omega^2 f''(\omega t)$ is its modulation with amplitude

$a\omega^2$, $C = \sigma_0 T^2/\rho L^3$ is the surface tension coefficient, $h = H/L$ is the mean layer depth, $L$ is the length scale and $T$ is the time scale.

Let us assume, for a moment, $\zeta(t)$ to be a known function, discard the dynamic boundary condition (9) and find the solution to the system (7), (8), (10). Then, substitution of the obtained velocity field $u$ and the pressure $P$ into (9), and separation of the variables ($\zeta(x_1, x_2, t) = \eta(t)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{\alpha},x)}$, $x = (x_1, x_2)$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ is the wave vector) leads to the integro-differential equation for determination of the motion of the free boundary, that is, of the amplitude function $\eta(t)$ (see [1,2] for more details). In the case of the layer of a finite depth, it has the form

$$\eta_{tt} + 4\mu\eta_t + \operatorname{th}\frac{\pi}{\gamma}\big(\alpha Q(\omega t) + C\alpha^3\big)\eta = \frac{8\mu^2}{\pi}\operatorname{th}\frac{\pi}{\gamma}\int_{-\infty}^{t} K(t-s)\eta_s\,\mathrm{d}s \tag{11}$$

$$K(s) = \gamma\sum_{m=1}^{\infty}\frac{m^2\gamma^2}{1+m^2\gamma^2}\,\mathrm{e}^{-\mu(1+m^2\gamma^2)s}, \qquad Q(\omega t) = Q_0 - a\omega^2 f''(\omega t)$$

Here $\mu = \delta\alpha^2$, $\gamma = \frac{\pi}{\alpha h}$, where $\alpha$ is the module of the wave vector.

For an infinitely deep layer the integro-differential equation is reduced to

$$\eta_{tt} + 4\mu\eta_t + \big(4\mu^2 + \alpha Q(\omega t) + C\alpha^3\big)\eta + 4\mu^{3/2}\int_0^{\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{-\mu t}\eta(t-\tau)]}{\sqrt{\pi\tau}}\,\mathrm{d}\tau = 0 \tag{12}$$

This equation was obtained by Cherepanov [3] for the case $f(\omega t) = \cos\omega t$ and has the form:

$$\eta_{tt} + 4\mu\eta_t + (4\mu^2 + \Omega^2 + 2q\cos\omega t)\eta + 4\mu^{3/2}\int_0^{\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{-\mu t}\eta(t-\tau)]}{\sqrt{\pi\tau}}\,\mathrm{d}\tau = 0 \tag{13}$$

Here $\Omega^2 = Q_0\alpha + C\alpha^3$ is the square of frequency of stationary gravitational-capillary waves and $2q = a\omega^2\alpha$.

## 2. Floquet solutions. Continued fractions method

From now on, we assume that $f(\omega t) = \cos\omega t$. First, we turn our attention to the layer of infinite depth. The Floquet solutions of equation (13) with multiplier $\sigma$ are searched for as an infinite sum

$$\eta(t) = \mathrm{e}^{\sigma t}\sum_{n=-\infty}^{+\infty} c_n\,\mathrm{e}^{\mathrm{i}n\omega t} \tag{14}$$

Substitution of (14) into (13) results in an infinite system of linear algebraic equations for determination of unknown coefficients $c_n$

$$M_n c_n = -q(c_{n-1} + c_{n+1}), \quad n = 0, \pm 1, \ldots \tag{15}$$

$$M_n = M_n(\sigma) = (\sigma + \mathrm{i}n\omega + 2\mu)^2 + \Omega^2 - 4\mu^{3/2}Q_n \tag{16}$$

$$Q_n = \sqrt[4]{(\sigma_r+\mu)^2 + (\sigma_i + n\omega)^2}\left(\cos\frac{\varphi}{2} + \mathrm{i}\sin\frac{\varphi}{2}\right), \quad \varphi = \operatorname{arctg}\frac{\sigma_i + n\omega}{\sigma_r + \mu} \tag{17}$$

Expression (16) gives the equality $\overline{M_n(\sigma)} = M_{-n}(\bar{\sigma})$, where the bar stands for complex conjugation.

In the case of a finite depth $h$ we can also obtain the same three-diagonal system (15), but with the expression for $M_n$ being

$$M_n(\sigma) = (\sigma + \mathrm{i}n\omega + 2\mu)^2\operatorname{cth}\frac{\pi}{\gamma} + \Omega^2 - 4\mu^{3/2}\sqrt{\sigma + \mathrm{i}n\omega + \mu}\,\operatorname{cth}\left(\beta_n\frac{\pi}{\gamma}\right) \tag{18}$$

where $\beta_n = \sqrt{(\sigma + \mathrm{i}n\omega + \mu)/\mu}$, $\mathrm{Re}\sqrt{\sigma + \mathrm{i}n\omega + \mu} > 0$. Expression (18) coincides with (16) when $\gamma \to 0$.

For three-diagonal systems it turns out to be possible to write a dispersion relation for $\sigma$ in the explicit form using continued fractions [4,5]:

$$-M_0 + \cfrac{-q^2}{-M_1 + \cfrac{-q^2}{-M_2 + \cdots}} = \cfrac{-q^2}{M_{-1} + \cfrac{-q^2}{M_{-2} + \cdots}} \tag{19}$$

If $\sigma = 0$ or $\sigma = \mathrm{i}\omega/2$, the equation (19) can be simplified to the real form. The case $\sigma = 0$ corresponds to disturbances of the period $2\pi/\omega$, and the dispersion equation is

$$\mathrm{Re}\,\cfrac{q^2}{M_1(0) - \cfrac{q^2}{M_2(0) + \cdots}} = \frac{\Omega^2}{2} \tag{20}$$

The case $\sigma = \mathrm{i}\omega/2$ corresponds to the neutral disturbances of the double period $4\pi/\omega$, and the corresponding transcendent equation is

$$\left| M_0 - \cfrac{q^2}{M_1 - \cfrac{q^2}{M_2 - \cdots}} \right|^2 = q^2 \tag{21}$$

The continued fractions approach can be applied directly to the system of differential equations, for example, when the lower boundary is a rigid wall, to obtain the similar three-diagonal system (15) with the same general properties. Only the expression for $M_n(\sigma)$ is slightly different.

## 3. High frequency asymptotics

Now we turn to a brief description of the results for the high frequency asymptotics ($\omega \gg 1$) for the layer of an infinite depth (see [1] for more details). We assume $a = \frac{b}{\omega}$ so that the vertical oscillations are governed by the law $z = \frac{b}{\omega}\cos\omega t$, where $b$ is independent of $\omega$. This results in the equation $\dot{z} = -b\sin\omega t$, so that the amplitude $b$ of oscillations of the velocity is fixed, and the amplitude of oscillation of the height is $\mathrm{O}(1/\omega)$ as $\omega \to \infty$. We introduce the new parameter $2\ell = b\alpha$. We assume that the length and time scales, as well as the parameters $\mu,\ \Omega^2$ and $\ell$ are independent of $\omega$. Integro-differential equation in this case is:

$$\eta_{tt} + 4\mu\eta_t + \left(4\mu^2 + \Omega^2 + 2\ell\omega\cos\omega t\right)\eta + 4\mu^{3/2}\int_0^\infty \frac{\frac{\mathrm{d}}{\mathrm{d}s}[\mathrm{e}^{-\mu s}\eta(t-s)]}{\sqrt{\pi s}}\,\mathrm{d}s = 0 \tag{22}$$

The Krylov–Bogolubov averaging method is then applied to equation (22). We use two times: $t$ is slow and $\tau = \omega t$ is fast. The asymptotic solution $\eta = \eta(t,\tau)$ is represented as a sum of a smooth part (which depends only on the slow time) and of an oscillatory part (which depends on slow and fast times)

$$\eta(t,\tau) = \bar{\eta}(t) + \frac{1}{\omega}\tilde{\eta}(t,\tau) \tag{23}$$

The oscillatory part of the asymptotic solution is

$$\tilde{\eta}(t,\tau) = 2\ell\cos\tau\cdot\bar{\eta}(t) \tag{24}$$

Substituting (24) into (23) and then into (22) and averaging with respect to the fast time $\tau$ we get an autonomous integro-differential equation

$$\bar{\eta}_{tt} + 4\mu\bar{\eta}_t + \left(4\mu^2 + \Omega^2 + 2\ell^2\right)\bar{\eta} + 4\mu^{3/2}\int_0^\infty \frac{\frac{\mathrm{d}}{\mathrm{d}s}[\mathrm{e}^{-\mu s}\bar{\eta}(t-s)]}{\sqrt{\pi s}}\,\mathrm{d}s = 0 \tag{25}$$

It is natural to introduce in (25) a new parameter $\Omega_{\text{eff}}^2 = \Omega^2 + 2\ell^2$-the square of the effective frequency of the gravitational-capillary waves. $\Omega_{\text{eff}}^2$ can be both positive and negative (when $g_0 < 0$).

Closer investigation shows that the behavior of the system depends on the parameter $\kappa = \Omega_{\text{eff}}^2/\mu^2$, which has a critical value $\kappa_*$ such that if $\kappa > \kappa_*$, then there is oscillatory stability; if $0 < \kappa < \kappa_*$, then the stability is monotonic; if $\kappa < 0$, then there is a monotonic instability, and there exist one monotonically growing mode and $(-1 < \kappa < 0)$ one monotonically decreasing mode. Let $\sigma$ be an eigenvalue, corresponding to the former mode. Then $\sigma \to -\mu$ and converges to the continuous spectrum as $\kappa \to -1$.

## 4. Fluid movement on the ceiling

Now we discuss some results concerning the case $\kappa \leqslant 0$, which corresponds to the “upturned layer” (or flow on a ceiling), where the rigid (or soft) wall is on the upper boundary of the layer, and the free surface is its lower boundary. The expression for the parameter $\kappa$ is

$$\kappa = \frac{\Omega_{\text{eff}}^2}{\mu^2} = \frac{\Omega^2 + 2\ell^2}{\mu^2} = \frac{1}{\mu^2}\left(\frac{g_0 T^2}{L}\alpha + C\alpha^3 + \frac{b^2\alpha^2}{2}\right) \tag{26}$$

In the case of the normal layer, $\kappa$ is always positive. As $g_0 < 0$ in the case of upturned layer, there exists such an interval of parameter $\alpha$ that $\kappa$ is negative. But for any fixed value of parameter $\alpha$, when the intensity of vibration is so large that the inequality $b^2 > 2\,(\frac{|g_0|T^2}{L}\frac{1}{\alpha} - C\alpha)$ is satisfied $\kappa$ becomes positive, that is, the normal mode fades. One can see from this inequality that no vibration can fully stabilize an upturned layer, but it can stabilize all waves that are short enough, so that $\alpha > \alpha_*(b)$. In this case $\alpha_*$ can be determined from the same inequality, where the sign “greater than” is replaced by equality. The described procedure can be repeated for the oscillatory instability, and the corresponding condition $\kappa > \kappa_*$ for a given parameter $\alpha$ is $b^2 > 2\,(\frac{|g_0|T^2}{L}\frac{1}{\alpha} - C\alpha + \frac{\kappa_*}{\alpha^2})$.

## 5. Numerical results

Dispersion equations (20), (21) contain the following dimensionless parameters: $\mu = \delta\alpha^2$, $\gamma = \frac{\pi}{\alpha h}$, $\Omega^2 = \frac{g_0 T^2}{L}\alpha + C\alpha^3$, $\omega$ and $q = a\omega^2\alpha/2$. Further we present the results of numerical solution of Eqs. (20) and (21) for the layers of finite and infinite depth, finite viscosity and arbitrary frequency. The amplitude–frequency characteristics $(q,\omega)$ are obtained, as well as the neutral curves $(a,\omega)$ and $(a\omega,\omega)$ with all other parameters fixed. If $\mu = 0$, then Eq. (11) is the well known Mathieu equation. Its properties are described in sufficient details, for example, in [6]. The resonance points are: $\omega_{2n+1} = \frac{2\widetilde{\Omega}}{2n+1}$, $n = 0, 1, \ldots,$ and $\omega_{2n+2} = \frac{\widetilde{\Omega}}{n+1}$, $n = 0, 1, \ldots,$ $\widetilde{\Omega} = \sqrt{\operatorname{th}\frac{\pi}{\gamma}}\,\Omega$. One can easily see that resonant frequencies decrease when the depth decreases. Also, calculations show that the layer is stabilizing with the growth of viscosity (Fig. 1, $\Omega^2 = 1$, $\gamma = 0$). If the solid boundary is rigid, then the layer behaves more stably over the soft case. Further on the influence of vibration on the neutral curves (with viscosity fixed) is described. It turns out that if the frequency of vibration is small, the layer of a smaller depth is more stable, than the layer with a large depth. The neutral curves corresponding to $\sigma = 0$ and $\sigma = \mathrm{i}\omega/2$, rise as the depth of the layer decreases. As the frequency of vibration increases, the neutral curves approach to those for the case of an

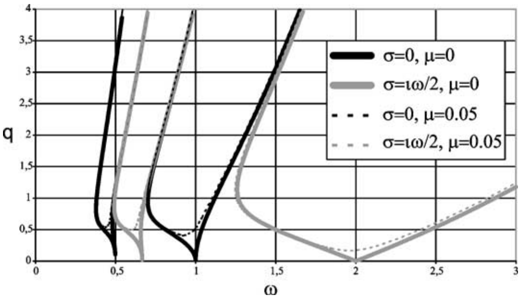


Fig. 1. Variation of amplitude of acceleration with the frequency of modulation.

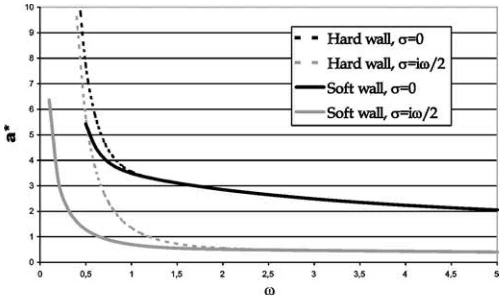


Fig. 2. Variation of the critical amplitude with the frequency of modulation.

infinite depth. Moreover, the increase of the frequency of vibration displays interesting effects on the layers with different types of solid wall. If the frequency of vibration is relatively small, the rigid wall layer is more stable than the soft wall one. However, as the frequency increases, the neutral curves of the rigid wall case become closer and closer to the neutral curves of the soft wall case (Fig. 2, $h = 1,\ \delta = 0.5,\ C = 1$). On Fig. 2 $a^*$ denotes critical amplitude that is minimized with respect to parameter $\alpha$.

Also the case of upturned layer, considered in [3], was studied. The curve corresponding to $\sigma = 0$ outlines the instability region which this time lies below the curve. The instability region increases if the depth decreases. Increase of the frequency of vibration displays stabilizing effects: the curves corresponding to $\sigma = \mathrm{i}\omega/2$ rise, and the curves corresponding to $\sigma = 0$ go down. As it was shown by Cherepanov [3], the amplitude on the neutral curve (critical amplitude) for $\sigma = 0$ grows asymptotically as $1/\alpha$ when $\alpha \to 0$, independently of the frequency of vibrations. Thus, if $\alpha$ is not bounded from below (that is, the wave length of disturbances is not bounded from above), then no vibration can stabilize the layer. For the case of small viscosity, and, respectively, small critical amplitudes and almost-resonant frequencies our results are in complete agreement with those in [3].

## References

[1] S.M. Zenkovskaya, V.I. Yudovich, Method of integro-differential equations in the free boundary problems and parametric wave exitation, Preprint, Part I, Dep. in VINITI 2587-B, 2001, p. 33.

[2] S.M. Zenkovskaya, V.I. Yudovich, Method of integro-differential equations and continued fractions in the problem of parametric wave exitation, submitted for publication.

[3] A.A. Cherepanov, Influence of variable fields on Rayleigh–Taylor instability, Preprint. Some of the problems of liquid boundary instability, Uralsk. Scientific center AN SSSR, Sverdlovsk, 1984, pp. 29–53.

[4] G.S. Markman, V.I. Yudovich, Numerical study of the onset of convection in fluid layer under influence of periodic in time external forces, Izv. Akad. Nauk SSSR Ser. Mech. Fluid and Gas 3 (1972) 81–86.

[5] V.I. Yudovich, in: Continued Fractions Method in Spectral Theory of Linear Differential Operators with Periodical Coefficients, Novgorodskiy Universitet, 2001, pp. 20–24.

[6] M.J.O. Strett, Lame, Mathieu and Related Functions in Physics and Technology, ONTI, Moscow, 1935.