Available online at www.sciencedirect.com
SCIENCE DIRECT®
C. R. Mecanique 332 (2004) 263–269
COMPTES RENDUS MECANIQUE

General minimum principles for quasilinear transport and bioheat equations

Józef Joachim Telega, Maciej Stańczyk

Polish Academy of Sciences, Institute of Fundamental Technological Research, 00-049 Warsaw, Poland

Received 20 October 2003; accepted 27 January 2004

Presented by Évariste Sanchez-Palencia

Abstract

The aim of this contribution is to derive minimum principles for quasi-linear linear transport (heat) equations in the steady and nonstationary case. Application to currently used nonstationary bioheat equations is sketched. ***To cite this article: J.J. Telega, M. Stańczyk, C. R. Mecanique 332 (2004).***

Résumé

Principes de minimum généraux pour équations du transport quasi linéaires et de la chaleur non stationaires en biomécanique. Le but de la contribution est de donner des principes de minimum pour l'equation du transport (de la chaleur) quasi lineéaire dans le cas stationaire et non stationaire. L'application aux équations de la chaleur, couramment utilisées en biomécanique est esquisée. ***Pour citer cet article : J.J. Telega, M. Stańczyk, C. R. Mecanique 332 (2004).***

Keywords: Heat transfer; Biomechanics; Quasilinear transport equations; Steady and nonstationary case; Bioheat equations; Minimum principles

Mots-clés : Transferts thermiques ; Biomécanique ; Cas stationnaire et non stationnaire ; Principes de minimum ; Equations de la chaleur en biomécanique

1. Introduction

A common conviction is that for nonpotential and initial-value problems, particularly for quasilinear heat equations, one cannot find minimum principles. However, by using simple methods of convex analysis one can derive such principles for a general type of elliptic, parabolic and hyperbolic differential inclusions. In fact, Brezis and Ekeland [1,2] derived minimum principles for classical parabolic heat equations. The approach was next extended by Auchmuty [3] and Telega [4].

E-mail addresses: jtelega@ippt.gov.pl (J.J. Telega), mstan@ippt.gov.pl (M. Stańczyk).

1631-0721/$ – see front matter
doi:10.1016/j.crme.2004.02.012

An interesting feature of the method is that the variational functionals to be minimised are bounded from below by zero.

In this contribution we derive minimum principles for quasilinear transport (heat) equations by exploiting the general method developed by Telega [4]. In essence, two approaches are proposed. One involves an inverse operator (Green's function). The second approach may be viewed as a generalization of the primal and dual problems expounded in [5]. It may be said that for nonpotential and initial-value conditions the primal and dual problems are inextricably coupled. Application to currently used nonstationary bioheat equations is sketched.

2. Stationary quasi-linear transport equation

The general setting was developed by Auchmuty [3] and Telega [4]. It has the form of the subdifferential inclusion

$$F(u) + f \in \partial\Phi(u)$$

In this section we apply the variational approach proposed in [3,4] to the transport Neumann boundary value problem

$$\begin{aligned} &-\operatorname{div}\big[\mathbf{a}(\mathbf{x}, u(\mathbf{x}))\nabla u(\mathbf{x})\big] = f \qquad \text{in } \Omega \\ &\big(\mathbf{a}\big(\mathbf{x}, u(\mathbf{x})\big)\nabla u(\mathbf{x})\big)\cdot\mathbf{n} = g \qquad \text{on } \Gamma \end{aligned} \tag{1}$$

Here Ω is a bounded, sufficiently regular domain in $\mathbb{R}^N$ (physically $N = 1, 2$ or 3), $\Gamma = \partial\Omega$ and $\mathbf{n}$ denotes the outward unit normal to Γ. For the assumptions on $\mathbf{a} = (a_{ij})$ and existence of solutions to (1) the reader is referred to the relevant references cited in [6].

Assume that $\mathbf{a}(\mathbf{x}, \boldsymbol{\xi}) = \mathbf{a}^0(\mathbf{x}) + \mathbf{a}^1(\mathbf{x}, \boldsymbol{\xi})$, $\mathbf{x} \in \overline{\Omega}$, $\boldsymbol{\xi} \in \mathbb{R}^N$ and that there exist positive constants $c_2 \geqslant c_1$ such that

$$c_1|\boldsymbol{\xi}|^2 \leqslant a^0_{ij}(x)\xi_i\xi_j \leqslant c_2|\boldsymbol{\xi}|^2$$

for each $\boldsymbol{\xi} \in \mathbb{R}^N$ and almost every (a.e.) $\mathbf{x} \in \bar{\Omega}$.

We set

$$\Phi(u) = \frac{1}{2}\int_{\Omega} a^0_{ij}(\mathbf{x})u_{,i}u_{,j}\,\mathrm{d}\mathbf{x}, \qquad F(u) = \begin{bmatrix} \operatorname{div}[\mathbf{a}^1(\mathbf{x}, u(\mathbf{x}))\nabla u(\mathbf{x})] \\ -(\mathbf{a}^1(\mathbf{x}, \mathbf{u}(\mathbf{x}))\nabla u(\mathbf{x}))\cdot\mathbf{n}) \end{bmatrix} \begin{matrix} \Omega \\ \Gamma \end{matrix}$$

Then we have

$$\partial\Phi\big(u(\mathbf{x})\big) = \begin{bmatrix} -\operatorname{div}[\mathbf{a}^0(\mathbf{x}, u(\mathbf{x}))\nabla u(\mathbf{x})] \\ (\mathbf{a}^0(\mathbf{x}, \mathbf{u}(\mathbf{x}))\nabla u(\mathbf{x}))\cdot\mathbf{n}) \end{bmatrix} \begin{matrix} \Omega \\ \Gamma \end{matrix}$$

and

$$\Phi^\star(u^\star) = \sup_u\left\{\langle u^\star, u\rangle - \frac{1}{2}\int_{\Omega}\big(\mathbf{a}^0\nabla u\big)\cdot\nabla u\,\mathrm{d}\mathbf{x}\right\}$$

The duality pairing $\langle\cdot,\cdot\rangle$ is here to be understood in the sense of $[W^{1,p}(\Omega)]' \times W^{1,p}(\Omega)$, $p \geqslant 2$. The dual functional $\Phi^\star$ has the form

$$\Phi^\star(u^\star) = \frac{1}{2}\big\langle L^{-1}u^\star_1, u^\star_1\big\rangle_{|\Omega} + \frac{1}{2}\big\langle L^{-1}u^\star_2, u^\star_2\big\rangle_{|\Gamma}$$

where

$$\mathbf{u}^1 = \begin{bmatrix} u^\star_1 \\ u^\star_2 \end{bmatrix} \begin{matrix} \Omega \\ \Gamma \end{matrix}$$

Obviously u_2^* is to be taken in the sense of trace of a function on Γ. The linear operator L^{-1} is the inverse operator (linear) to the operator determined by the following Neumann problem

$$-\operatorname{div}\bigl(\mathbf{a}^0(\mathbf{x})\nabla u_1(\mathbf{x})\bigr) = u_1^\star \quad \text{in } \Omega; \qquad \bigl(\mathbf{a}^0(\mathbf{x})\nabla u_1(\mathbf{x})\bigr)\cdot\mathbf{n} = u_2^\star \quad \text{on } \Gamma$$

Here $u_1^\star$ and $u_2^\star$ are prescribed.

The minimum principle associated with (1) means evaluating

$$\alpha = \inf\bigl\{J(u) | u \in W^{1,p}(\Omega)\bigr\}$$

where

$$J(u) = \Phi(u) + \Phi^\star(u_1^\star, u_2^\star) - \langle u_1^\star, u\rangle_{|\Omega} - \langle u_2^\star, u\rangle_\Gamma$$

Here $u_1^\star = f + \operatorname{div}[\mathbf{a}^1(\mathbf{x}, u(\mathbf{x}))\nabla u(\mathbf{x})]$ in Ω, $u_2^\star = g - [\mathbf{a}^1(\mathbf{x}, u(\mathbf{x}))\nabla u(\mathbf{x})]\cdot\mathbf{n}$ on Γ. From Theorem 2 of the paper [4] we conclude that $\alpha \geqslant 0$ and $J(\hat{u}) = 0$ if and only if $\hat{u}$ is a solution to (1); $J(\hat{u}) = \min J(u)$.

3. A setting avoiding the inverse operator

In the paper by the first author [4] the minimum principle for the following differential inclusion was derived

$$F(u) + f \in \partial\Phi_1(\Lambda u)$$

where Λ is a linear and continuous operator, cf. [5]. To apply the general formulation developed in [4] to the system (1) we take

$$\Phi_1(\mathbf{p}) = \frac{1}{2}\int_\Omega a_{ij}^0(\mathbf{x})\, p_i\, p_j \,\mathrm{d}\mathbf{x}$$

Standard calculation gives the conjugate functional [5,7]

$$\Phi_1^\star(\mathbf{p}^\star) = \frac{1}{2}\int_\Omega b_{ij}^0(\mathbf{x})\, p_i^\star\, p_j^\star \,\mathrm{d}\mathbf{x}$$

where $\mathbf{b}^0 = (\mathbf{a}^0)^{-1}$.

Now $\Lambda u = \nabla u$ and we calculate

$$\langle \Lambda u, \mathbf{p}^\star\rangle = \int_\Omega (\nabla u)\cdot\mathbf{p}^\star \,\mathrm{d}\mathbf{x} = -\int_\Omega (\operatorname{div}\mathbf{p}^\star) u \,\mathrm{d}\mathbf{x} + \int_\Gamma (\mathbf{p}^\star\cdot\mathbf{n}) u \,\mathrm{d}\mathbf{x} = \langle \Lambda^\star\mathbf{p}^\star, u\rangle$$

The adjoint operator $\Lambda^\star$ of Λ has thus the form

$$\Lambda^\star\mathbf{p}^\star = \begin{bmatrix} -\operatorname{div}\mathbf{p}^\star \\ \mathbf{p}^\star\cdot\mathbf{n} \end{bmatrix} \begin{matrix} \Omega \\ \Gamma \end{matrix}$$

By using the general results presented in Section 3.2 of the paper [4], we formulate the variational functional:

$$K_1(u,\mathbf{q}) = \Phi_1(\nabla u) + \Phi_1^\star(\mathbf{q}) - \bigl\langle \operatorname{div}\bigl(\mathbf{a}^1\nabla u\bigr) + f, u\bigr\rangle_{V^\star\times V} - \bigl\langle g - \bigl(\mathbf{a}^1\nabla u\bigr)\cdot\mathbf{n}, u\bigr\rangle_{V_\Gamma'\times V_\Gamma}$$

where $V = W^{1,p}(\Omega)$, $V_\Gamma = W^{1-1/p,p}(\Gamma)$ and V_Γ' is the dual of V_Γ. The minimum principle means evaluating

$$\begin{aligned} &\inf\bigl\{K_1(u,\mathbf{q}) \mid u\in V,\ \mathbf{q}\in L^q(\Omega),\ \operatorname{div}\mathbf{q}(x) = f(\mathbf{x}) + \operatorname{div}\bigl[\mathbf{a}^1\bigl(\mathbf{x}, u(\mathbf{x})\bigr)\nabla u\bigr] \text{ in } \Omega,\\ &\quad \mathbf{q}\cdot\mathbf{n} = g - \bigl(a^1\bigl(\mathbf{x}, u(\mathbf{x})\bigr)\nabla u(\mathbf{x})\bigr)\cdot\mathbf{n} \text{ on } \Gamma\bigr\} \end{aligned} \tag{2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 1. Minimization problem (2) couples the primal and dual problems. Indeed, if $\mathbf{a}^1 \equiv \mathbf{0}$, then (2) yields a classical pair of dual problems, now derived without using the duality theory expounded in Ekeland and Temam [5].

4. General setting for parabolic-type equations

Prior to the formulation of minimum principles for nonstationary bioheat equations we shall present two general schemes. Consider first the following parabolic-like differential inclusion:

$$\begin{cases} \kappa \dot{u}(\tau) + \partial \Phi\big(\tau, u(\tau)\big) \ni F\big(\tau, u(\tau)\big), \quad 0 < \tau < \tau_0 \\ u(0) = u^0 \end{cases} \tag{3}$$

Here $\dot{u} = \frac{\mathrm{d}u}{\mathrm{d}\tau}$ and we use the standard notation $u(\tau) = \{u(x, \tau) \mid x \in \Omega\}$ where x is a spatial variable. We assume that the coefficient κ may depend on the spatial variable, but not on the time variable though such an extension is possible. We introduce the set of admissible functions u:

$$\mathrm{K} = \big\{v(\tau),\ \tau \in [0,\ \tau_0] \mid v(0) = u^0\big\}$$

and the variational functional [4]:

$$\begin{aligned} J(v) = \int_0^{\tau_0} \big\{ \Phi\big(\tau, v(\tau)\big) + \Phi^*\big[\tau, F\big(t, v(\tau)\big) - \kappa \dot{v}(\tau)\big] - \big\langle F\big(\tau, v(\tau)\big), v(\tau)\big\rangle \big\}\, \mathrm{d}\tau \\ + \frac{1}{2}\big(\big\|\kappa v(\tau_0)\big\|^2 - \big\|\kappa u^0\big\|^2\big) \end{aligned} \tag{4}$$

Here $\|\cdot\|$ stands for a properly defined norm, cf. [4]. In practice, if Φ is a quadratic functional then Φ^* is defined by an inverse operator (Green's function).

The minimum principle associated with problem (3) means evaluating

$$(\mathcal{P}) \quad \alpha = J(u) = \inf\big\{J(v) \mid v \in \mathrm{K}\big\}$$

Under physically plausible, rather weak assumptions, $\alpha \geqslant 0$ and $J(u) = 0$ if and only if u is a solution to (3). One of the important assumption requires $\Phi(\tau, \cdot)$ to be a convex and proper function for each $\tau \in [0, \tau_0]$, cf. [3,4]

Consider now a more general setting, allowing one to avoid the inverse operator, proposed in [4],

$$\begin{cases} \kappa \dot{u}(\tau) + \partial \Psi\big(\tau, \Lambda u(\tau)\big) \ni F\big(\tau, u(\tau)\big), \quad 0 < \tau < \tau_0 \\ u(0) = u_0 \end{cases} \tag{5}$$

Here Λ is a continuous linear operator, $\Lambda : V \to Y$; Λ^* is its adjoint.

Since $(\Psi\Lambda)^* = \Lambda^*\Psi^*$ and

$$(\Lambda^*\Psi^*)(v^*) = \inf\big\{\Psi^*(q^*) \mid \Lambda^* q^* = v^*,\ q^* \in Y^*\big\}$$

it can be shown that the minimum principle takes the form

$$\begin{aligned} (\mathcal{P}_1) \quad & \inf_{v, q^*} \big\{ \textstyle\int_0^{\tau_0} \big[\Psi\big(\tau, \Lambda v(\tau)\big) + \Psi^*\big[\tau, q^\star(\tau)\big] - \big\langle F\big(\tau, v(\tau)\big), v(\tau)\big\rangle\big]\, \mathrm{d}\tau \\ & \quad + \tfrac{1}{2}\big(\big\|\kappa v(\tau_0)\big\|^2 - \big\|\kappa u^0\big\|^2\big) \mid v(\tau) \in \mathrm{K}, q^*(\tau) \in Y^*, \Lambda^* q^*(\tau) = F\big(\tau, v(\tau)\big) - \kappa \dot{v}(\tau)\big\} \end{aligned}$$

5. Application to bioheat equations

Having developed the general scheme of formulation of minimum principles for parabolic-type problems we pass to the application to important bioheat equations. For relevant references on various bioheat equations and their applications the reader is referred to our paper [9] .

5.1. Minimum principles for quasilinear nonstationary heat equation: general model for bioheat equations

Consider now the following quasilinear heat equation in a bounded, sufficiently regular domain $\Omega \subset \mathbb{R}^3$:

$$\begin{aligned} &\kappa \dot{T} - \operatorname{div}\big[\mathbf{a}(\mathbf{x},\tau,T)\nabla T\big] = f(\mathbf{x},\tau) && \text{in } \Omega \times (0,\tau_0)\\ &T(\mathbf{x},\tau) = 0 && \text{on } \Gamma_0 \times (0,\tau_0)\\ &\big(\mathbf{a}(\mathbf{x},\tau,T)\nabla T\big)\cdot \mathbf{n} = g(\mathbf{x},\tau) && \text{on } \Gamma_1 \times (0,\tau_0)\\ &T(\mathbf{x},0) = T^0(\mathbf{x}) && \text{on } \Omega \end{aligned} \tag{6}$$

Here $\dot{T} = \frac{\partial T}{\partial \tau}$ stands for the temperature rate, $\Gamma = \partial\Omega,\ \Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$ whilst $\tau \in [0,\tau_0]$ denotes time. We assume that

$$\mathbf{a}(\mathbf{x},\tau,T) = \mathbf{a}^0(\mathbf{x},\tau) + \mathbf{a}^1(\mathbf{x},\tau,T) \quad \text{and} \quad \exists c_1 \geqslant c_0 > 0,\ \forall e \in \mathrm{R}^3 \quad c_0|e|^2 \leqslant a^0_{ij}(x,\tau)e_i e_j \leqslant c_1|e|^2$$

for each $\tau \in [0,\tau_0]$; moreover $a^0_{ij} = a^0_{ji},\ i,j = 1,2,3$.

To formulate the variational functional of type (4) we set

$$\Phi(\tau,T) = \frac{1}{2}\int_{\Omega} a^0_{ij}(\mathbf{x},\tau)T_{,i}T_{,j}\,\mathrm{d}\mathbf{x}$$

Here $T_{,i} = \partial T/\partial x_i$. Then $\Phi^*(\tau,T^*) = \frac{1}{2}\langle GT^*, T^*\rangle$ The operator G is the Green operator solving the following boundary-value problem

$$\begin{cases} -\operatorname{div}\big[\mathbf{a}^0(\mathbf{x},\tau)\nabla T\big] = T^*_{\Omega}(\mathbf{x},\tau) & \text{in } \Omega \times (0,\tau_0)\\ \big(\mathbf{a}^0(\mathbf{x},\tau)\nabla T\big)\cdot \mathbf{n} = T^*_{\Gamma_1}(\mathbf{x},\tau) & \text{on } \Gamma_1 \times (0,\tau_0) \end{cases}$$

We observe that now $T^* = (T^*_{\Omega}, T^*_{\Gamma_1})$. Furthermore we set

$$\begin{aligned} &\mathrm{K} = \big\{T(\mathbf{x},\tau) \mid T(\mathbf{x},\tau) = 0 \text{ on } \Gamma_0 \times (0,\tau_0)\big\}\\ &T^*_{\Omega}(\mathbf{x},\tau) = f(\mathbf{x},\tau) + \operatorname{div}\big[a^1(\mathbf{x},\tau,T)\nabla T(\mathbf{x},\tau)\big] - \kappa(x)\dot{T}(\mathbf{x},\tau)\\ &T^*_{\Gamma_1}(\mathbf{x},\tau) = g(\mathbf{x},\tau) - \big[a^1(\mathbf{x},\tau,T)\nabla T(\mathbf{x},\tau)\big]\cdot \mathbf{n} \end{aligned}$$

The minimum principle is a particular case of problem $(\mathcal{P})$ and the variational functional takes now the form:

$$\begin{aligned} J(T) \;=\; &\tfrac{1}{2}\int_0^{\tau_0}\!\!\int_{\Omega}\big[\tfrac{1}{2}a^0_{ij}(\mathbf{x},\tau)T_{,i}T_{,j} + G(\mathbf{x},\tau)T^*_{\Omega}(\mathbf{x},\tau)T^*_{\Omega}(\mathbf{x},\tau)\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau\\ &+ \tfrac{1}{2}\int_0^{\tau_0}\!\!\int_{\Gamma_1} G(\mathbf{x},\tau)T^*_{\Gamma_1}(\mathbf{x},\tau)T^*_{\Gamma_1}(\mathbf{x},\tau)\,\mathrm{d}\Gamma\,\mathrm{d}\tau - \int_0^{\tau_0}\!\!\int_{\Omega} T^*_{\Omega}(\mathbf{x},\tau)T(\mathbf{x},\tau)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau\\ &- \int_0^{\tau_0}\!\!\int_{\Gamma_1} T^*_{\Gamma_1}(\mathbf{x},\tau)T_{\Gamma_1}(\mathbf{x},\tau)\,\mathrm{d}\Gamma\,\mathrm{d}\tau + \tfrac{1}{2}\int_{\Omega}\big[\big(\kappa(\mathbf{x})T(\mathbf{x},\tau)\big)^2 - \big(\kappa(\mathbf{x})T^0(\mathbf{x})\big)^2\big]\,\mathrm{d}\mathbf{x} \end{aligned}$$

Let us pass now to the formulation of the principle of type $(\mathcal{P}_1)$ for the some nonstationary quasilinear initial-boundary value problem (6). In this case $\Lambda T = \nabla T$ and

$$\Lambda^*\mathbf{p}^* = \begin{bmatrix} -\operatorname{div}\mathbf{p}^* \\ \mathbf{p}^*\cdot\mathbf{n} \end{bmatrix} \begin{matrix} \Omega \\ \Gamma_1 \end{matrix}$$

Problem (6) has now the form (5) where

$$\Psi(\tau,\Lambda T) = \frac{1}{2}\int_{\Omega}\big(\mathbf{a}^0(\mathbf{x},\tau)\Lambda T\big)\cdot\Lambda T\,\mathrm{d}\mathbf{x}$$

The variable conjugate to ΛT is the flux $\mathbf{q} = (q_i)$. We find

$$\Psi^*(\tau,\mathbf{q}) = \sup\left\{\int_{\Omega} q_i p_i\,\mathrm{d}\mathbf{x} - \Psi(\tau,\mathbf{p}) \mid \mathbf{p} \in Y\right\} = \frac{1}{2}\int_{\Omega} b_{ij}(\mathbf{x},\tau)q_i q_j\,\mathrm{d}\mathbf{x}$$

where $\mathbf{q} \in Y^*,\ \mathbf{b} = (\mathbf{a}^0)^{-1}$, and $Y = Y^* = L^2(\Omega)^3$.

The minimum principle $(\mathcal{P}_1)$ means now evaluating:

$$\begin{aligned}
\alpha = \inf_{T,\mathbf{q}}\Big\{ & \int_0^{\tau_0}\int_\Omega \tfrac{1}{2}\Big[\big(\mathbf{a}^0(\mathbf{x},\tau)\nabla T(\mathbf{x},\tau)\big)\cdot\nabla T(\mathbf{x},\tau) + \big(\mathbf{b}(\mathbf{x},\tau)\mathbf{q}(\mathbf{x},\tau)\big)\cdot\mathbf{q}(\mathbf{x},\tau)\Big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau \\
& - \int_0^{\tau_0}\int_\Omega \Big[f(\mathbf{x},\tau) + \operatorname{div}\big(\mathbf{a}^1(\mathbf{x},\tau)\nabla T(\mathbf{x},\tau,T)\big)\Big]T(\mathbf{x},\tau)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau \\
& - \int_0^{\tau_0}\int_{\Gamma_1}\Big[g(\mathbf{x},\tau) - \big(\mathbf{a}^1(\mathbf{x},\tau,T)\nabla T(\mathbf{x},\tau)\big)\cdot\mathbf{n}\Big]T(\mathbf{x},\tau)\,\mathrm{d}\Gamma\,\mathrm{d}\tau \\
& + \tfrac{1}{2}\int_\Omega\Big\{\big[\kappa(\mathbf{x})T(\mathbf{x},\tau_0)\big]^2 - \big[\kappa(\mathbf{x})T^0(\mathbf{x})\big]^2\Big\}\,\mathrm{d}\mathbf{x} \mid T \in \mathrm{K},\ \mathbf{q}(\cdot,\tau) \in L^2(\Omega)^3, \\
& - \operatorname{div}\mathbf{q}(\mathbf{x},\tau) = f(\mathbf{x},\tau) + \operatorname{div}\big(\mathbf{a}^1(\mathbf{x},\tau,T)\nabla T\big) - \kappa(x)\dot{T} \text{ in } \Omega\times(0,\tau_0), \\
& \mathbf{q}(\mathbf{x},\tau)\cdot\mathbf{n} = \mathbf{g}(\mathbf{x},\tau) - \mathbf{a}^1(\mathbf{x},\tau,T)\cdot\mathbf{n} \text{ on } \Gamma_1\times(0,\tau_0)\Big\}
\end{aligned} \tag{7}$$

Remark 2. We observe that if $\mathbf{a}^1 \equiv \mathbf{0}$ and the heat transfer problem is stationary ($\dot{T} = 0$) then (7) yields the standard pair of dual extremum problems. Recall that $\alpha = 0$. If $\mathbf{a}^1 \equiv \mathbf{0}$ and the problem is nonstationary then the primal problem (minimum over T) and the dual problem (minimum over **q**) remain coupled through the constraint:

$$-\operatorname{div}\mathbf{q} = f - \kappa\dot{T} \quad \text{in } \Omega\times(0,\tau_0)$$

5.2. Important bioheat equations

We are going to outline the formulation of minimum principles for currently used bioheat equations.

Pennes equation [10] extended to anisotropic tissues assumes the following form:

$$\kappa_t(\mathbf{x})\dot{T}_t(\mathbf{x},\tau) = \operatorname{div}\big(\boldsymbol{\lambda}_t(\mathbf{x})\nabla T_t(\mathbf{x},\tau)\big) + w_b(\mathbf{x},\tau)c_b\big[T_a - T_t(\mathbf{x},\tau)\big] + q_v(\mathbf{x},\tau)$$

The subscripts a, b and t refer to artery, blood and tissue, respectively; w_b denotes the blood perfusion, c_b is the specific heat and q_v stands for the internal heat generation.

This model (in its isotropic form) was proposed by Pennes [10] on the basis of measurements of temperature in the resting human forearm. The fundamental underlying assumption of this model is that the blood arrives at the local tissue capillary bed at the arterial temperature and instantaneously equilibrates with the surroundings. This means that the blood–tissue heat exchange takes place only on the level of capillaries. This assumption was frequently criticized by numerous authors, see [9] for the discussion of this criticism. Nevertheless, the Pennes equation is by far the most popular and viable model nowadays, cf. [11].

In the case of Pennes model $\mathbf{a}^0 \equiv \boldsymbol{\lambda}_t,\ \mathbf{a}^1 \equiv \mathbf{0}$, and

$$f(\mathbf{x},\tau) = w_b(\mathbf{x},\tau)c_b\big[T_t(\mathbf{x},\tau) - T_a\big] + q_v(\mathbf{x},\tau)$$

Directed perfusion model [12] is described by the following bioheat equation:

$$\kappa_t(\mathbf{x})\dot{T}_t(\mathbf{x},\tau) = \operatorname{div}\big(\boldsymbol{\lambda}_t(\mathbf{x})\nabla T_t\big) - \rho_b c_b\mathbf{U}\cdot\nabla T_t + q_v$$

Now also $\mathbf{a}^0 \equiv \boldsymbol{\lambda}_t,\ \mathbf{a}^1 \equiv \mathbf{0}$ whilst $f(\mathbf{x},\tau) = q_v - \rho_b c_b\mathbf{U}\cdot\nabla T_t$. Here **U** denotes the Darcy filtration velocity assumed as known.

This model was proposed by Wulff in 1974 [12] as an alternative to the Pennes equation. The assumption used here is quite opposite to that used by Pennes. Namely one assumes that the blood-tissue heat transfer occurs on each level of the vascular tree and it is perfect, so that the blood is always thermally equilibrated with tissue. This fact gives rise to the convective term on the right-hand side of the equation.

Weinbaum and Jiji simplified bioheat equation [13]: The observation that most of the blood vessels exists only in the form of artery-vein pairs with countercurrent flow led Weinbaum and Jiji to suspect that the countercurrent heat exchange mechanism is of primary importance. Based on their previous anatomical observations they derived a model of a continuum which is approximately equivalent to the solid with the network of countercurrent vessels

embedded into it [13]. The vessels are characterized by their local direction: $\mathbf{l}(\mathbf{x}) = (l_i)$, and l_i $(i = 1, 2, 3)$ are the directional cosines of the vessel pairs:

$$\frac{\partial[\kappa(\mathbf{x}, T_t)T_t]}{\partial\tau} = \operatorname{div}\big(\boldsymbol{\lambda}^{\mathrm{eff}}(\mathbf{x}, T_t)\nabla T\big) + q_v(\mathbf{x}, T_t) - \big[\boldsymbol{\Phi}(\mathbf{x}, T_t)\cdot\nabla T_t(\mathbf{x}, \tau)\big]\operatorname{div}\big(\mathbf{l}(\mathbf{x})\big)$$

Here $\boldsymbol{\Phi}(\mathbf{x}, T) = \frac{\pi n a^2 \lambda_b Pe^2}{4\sigma\lambda_t^2}\mathbf{l}(\mathbf{x})$ and $\boldsymbol{\lambda}^{\mathrm{eff}}(\mathbf{x}, T) = \lambda_t(\mathbf{I} + \boldsymbol{\Phi}(\mathbf{x}, T)\otimes\mathbf{l}(\mathbf{x}))$, where n is the local number density of vessels, a is their radius, Pe is the Peclet number and σ is the conduction shape factor (which is the function of the vessel dimensions). For more details on the last bioheat equation the reader is referred to [9] or to the original paper [13].

In this case we have to consider slightly more general version of Eq. $(6)_1$ where $\kappa\dot{T}$ is to be replaced by $\frac{\partial[\kappa(\mathbf{x},T_t)T_t]}{\partial\tau}$. Now $a^0_{ij}(x) = \lambda^0_t(x)\delta_{ij}$, $a^1_{ij}(\mathbf{x}, T_t) = \lambda_t\frac{\pi^2 n a^2\lambda_b^2(Pe)^2}{4\sigma\lambda_t^2}l_i l_j$, where $a^1_t(\mathbf{x}) > 0$, for all $\mathbf{x}\in\Omega$ and $a^0(\mathbf{x}) + a^1(\mathbf{x}, T_t) = a(\mathbf{x}, T_t)$. Moreover, $f(\mathbf{x}, \tau, T_t) = q_v(\mathbf{x}, T_t) - [\boldsymbol{\Phi}(\mathbf{x}, T_t)\cdot\nabla T_t]\operatorname{div}(\mathbf{l}(\mathbf{x}))$.

6. Final remarks

General approach developed by Telega in [4] and extended in Section 4 of the present paper enabled us to derive minimum principles for linear and quasilinear bioheat equations, including the equation studied in [14]. One can extend the procedure outlined in Sections 4 and 5 to strongly nonlinear equations, if such a need arises.

By generalizing the procedure outlined in Section 4 we can derive extremum principles for the coupled system of equations modelling polymerisation of PMMA in cement prostheses. For details on the relevant system the reader is referred to [8,15,16].

Acknowledgements

The authors were supported by EC through the project QLK6-CT-1999-02024. The second author was also supported by the Ministry of Science and Information Technology (Poland) through the grant No 4 T11F 009 24.

References

[1] H. Brezis, I. Ekeland, Un principe variationel associé à certaines équations paraboliques. Le cas indépendant du temps, C. R. Acad. Sci. Paris, Sér. A 282 (1976) 971–974.
[2] H. Brezis, I. Ekeland, Un principe variationel associé à certaines équations paraboliques. Le cas dépendant du temps, C. R. Acad. Sci. Paris, Sér. A 282 (1976) 1197–1198.
[3] G. Auchmuty, Variational principles for operator equations and initial value problems, Nonlinear Anal. 12 (1988) 531–564.
[4] J.J. Telega, Extremum principles for nonpotential and initial-value problems, Arch. Mech. 54 (2002) 663–690.
[5] J. Ekeland, R. Temam, Convex Analysis and Variational Problems, North-Holland, Amsterdam, 1976.
[6] A. Gałka, J.J. Telega, S. Tokarzewski, Nonlinear transport equation and macroscopic properties of microheterogeneous media, Arch. Mech. 49 (1997) 293–319.
[7] R.T. Rockafellar, R.J.-B. Wets, Variational Analysis, Springer, Berlin, 1998.
[8] M. Stańczyk, Modelling of PMMA cement polymerisation, J. Biomech. (2004), in press.
[9] M. Stańczyk, J.J. Telega, Modelling of heat transfer in biomechanics – a review. Part I. Soft tissues, Acta Bioeng. Biomech. 4 (2002) 31–61.
[10] H.H. Pennes, Analysis of tissue and arterial blood temperatures in the resting human forearm, J. Appl. Physiol. 1 (1948) 93–122.
[11] E.H. Wissler, Pennes' 1948 paper revisited, J. Appl. Physiol. 85 (1998) 35–41.
[12] W. Wulff, The energy conservation equation for living tissue, IEEE Trans. Biomed. Engrg. 21 (1974) 494–495.
[13] S. Weinbaum, L. Jiji, A new simplified bioheat equation for the effect of the local blood flow on local average tissue temperature, J. Biomech. Engrg. 107 (1985) 131–139.
[14] F. Bardati, G. Gerosa, On the solution of the nonlinear bioheat equation, J. Biomech. 23 (1990) 791–798.
[15] M. Stańczyk, J.J. Telega, Thermal problems in artificial joints: influence of bone cement polymerisation, Acta Bioeng. Biomech. 3 (2001) 489–496.
[16] M. Stańczyk, J.J. Telega, Modelling of heat transfer phenomena during cementation of femoral prosthesis, Acta Bioeng. Biomech. 4 (2002) 247–248.