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Uniform strain fields and exact results in an elastoplastic fibre-reinforced composite ☆

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Abstract

This work is concerned with a two-phase material consisting of an elastoplastic matrix reinforced by linearly elastic fibres. It is first shown that uniform strain fields can be generated in this heterogeneous material. A return-mapping based algorithm is then proposed and used to find uniform strain loading paths. With the help of uniform strain fields, exact results, independent of the transverse geometry and arrangement of the fibres, are derived for the effective elastic properties and for the effective initial and current yield surfaces. ***To cite this article: Q.-C. He, H. Le Quang, C. R. Mecanique 332 (2004).***

Résumé

Champs de déformation uniformes et résultats exacts dans un composite élastoplastique renforcé par des fibres. Ce travail porte sur un matériau à deux phases, constitué d'une matrice élastoplastique renforcée par des fibres linéairement élastiques. Nous montrons d'abord que des champs de déformation uniformes peuvent être générés dans ce matériau hétérogène. Un algorithme basé sur le « return mapping » est ensuite proposé et utilisé afin de trouver des trajets de chargement produisant des champs de déformation uniformes. A l'aide de ceux-ci, des résultats exacts, indépendants de la géométrie et de la distribution des fibres dans leur plan transverse, sont établis pour les propriétés élastiques macroscopiques et pour les surfaces macroscopiques initiale et actuelle de seuil de plasticité. ***Pour citer cet article : Q.-C. He, H. Le Quang, C. R. Mecanique 332 (2004).***

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1. Introduction

The fact that uniform fields may be produced in heterogeneous media under certain conditions is of prime importance to micromechanics. Indeed, it allows us to derive microstructure-independent exact results for the effective properties of heterogeneous materials. The usefulness of such exact results turns out to be threefold. Firstly, they serve directly for the determination of effective properties. Secondly, they constitute benchmarks for analytical and numerical methods elaborated for the prediction of linear and nonlinear effective properties (see, e.g., Ponte Castañeda and Suquet, [1], Milton, [2]). Thirdly, they shed light on the dependence of effective properties on the microstructure.

In the pioneer works of Hill [3], Levin [4] and Cribb [5], the existence of uniform fields in heterogeneous materials was tacitly recognized and employed to deduce some today's well-known exact elastic and thermoelstic relations. For the last 15 years and especially since the important work of Dvorak [6], the concept of uniform fields has been systematically studied and exploited to obtain a number of exact results for the effective mechanical and physical properties of *linear* heterogeneous materials (see the book of Milton [2] and the relevant references cited therein). In the works of Milton [7], He [8] and He and Bary [9], exact results have also been established for the effective properties of *smooth nonlinear elastic and thermoelastic* heterogeneous materials by extending the concept of uniform fields to smooth nonlinear cases. However, to the authors' best knowledge, no works have not been reported about the existence of uniform fields in *non-smooth dissipative* heterogeneous materials and about the use of uniform fields to the derivation of exact results for them. The present paper deals with this open problem by considering an elastoplastic fibre-reinforced composite as a prototype of non-smooth dissipative heterogeneous materials.

The notation adopted in this work is as follows. Scalars are denoted by Greek letters, and vectors by bold-face minuscule Latin letters. Second- and fourth-order tensors are designated by light- and bold-face majuscule Latin letters, respectively. The components of a vector, second- or fourth-order tensor are represented by the corresponding light-face letter with a suitable number of subscripts.

2. Local constitutive laws

The two-phase material investigated in the present work is made up of aligned parallel continuous fibers embedded in a matrix. The fiber and matrix phases are assumed to be individually homogeneous and perfectly bonded together across their interface but no restrictions are imposed on the transverse geometry and distribution of the fibers. The composite under consideration is thus homogeneous along the fiber direction and heterogeneous in the transverse plane.

The matrix, referred to as phase 1, is assumed to have an elastoplastic behavior. Let $E^{(1)}$ and $S^{(1)}$ denote its strain and stress tensors, respectively. As usual in elastoplasticity, $E^{(1)}$ is decomposed into an elastic part $E_{\mathrm{e}}^{(1)}$ and a plastic part $E_{\mathrm{p}}^{(1)}$:

$$E^{(1)} = E_{\mathrm{e}}^{(1)} + E_{\mathrm{p}}^{(1)} \tag{1}$$

The elastic stress–strain relation is taken to be linear and isotropic:

$$S^{(1)} = \mathbf{L}^{(1)}\big(E^{(1)} - E_{\mathrm{p}}^{(1)}\big), \quad \mathbf{L}^{(1)} = \lambda^{(1)} I \otimes I + 2\mu^{(1)}\mathbf{1} \tag{2}$$

where I is the second-order identity tensor, $\mathbf{1}$ is the fourth-order identity tensor on the space Sym of second-order symmetric tensors, $\mathbf{L}^{(1)}$ is the elastic stiffness tensor, $\lambda^{(1)}$ and $\mu^{(1)}$ are the constants of Lamé. In addition, the von Mises criterion with a linear isotropic hardening is adopted:

$$f\big(S^{(1)}, \alpha\big) = \big\|\operatorname{dev} S^{(1)}\big\| - \sqrt{\frac{2}{3}}\, g(\alpha) \leqslant 0 \tag{3}$$

$$g(\alpha) = \sigma_{\mathrm{e}} + k\alpha \tag{4}$$

where $\|\operatorname{dev} S^{(1)}\|$ represents the norm of the stress deviator $\operatorname{dev} S^{(1)} = (\mathbf{1} - I \otimes I/3)S^{(1)}$, α the isotropic hardening parameter, σ_{e} the initial yield stress, and k the material hardening constant. Using the normality rule, we have the following evolution laws:

$$\dot{E}_{\mathrm{p}}^{(1)} = \gamma \frac{\partial f}{\partial S^{(1)}} = \gamma \frac{\operatorname{dev} S^{(1)}}{\|\operatorname{dev} S^{(1)}\|}, \qquad \dot{\alpha} = -\gamma \frac{\partial f}{\partial g} = \sqrt{\frac{2}{3}}\gamma \tag{5}$$

$$\gamma \geqslant 0, \quad f \leqslant 0, \quad \gamma f = 0 \tag{6}$$

In order to obtain an explicit incremental stress–strain relation for the matrix, we first use (2)–(5) to calculate $\dot{f}(S^{(1)}, \alpha)$ as follows:

$$\dot{f}\big(S^{(1)}, \alpha\big) = 2\mu M : \dot{E}^{(1)} - (2\mu + 2k/3)\gamma \tag{7}$$

with

$$M = \frac{\operatorname{dev}(E^{(1)} - E_{\mathrm{p}}^{(1)})}{\|\operatorname{dev}(E^{(1)} - E_{\mathrm{p}}^{(1)})\|} \tag{8}$$

Making the assumption that $k > -3\mu^{(1)}$ and accounting for the requirement that $\gamma \geqslant 0$, we derive from the classical consistency condition $\dot{f} = 0$ that

$$\gamma = \frac{2\mu\langle M : \dot{E}^{(1)}\rangle_+}{2\mu + 2k/3} \tag{9}$$

where $\langle x\rangle_+$ is equal to 0 if $x < 0$ and to x if $x \geqslant 0$. Next, it is easy to obtain the incremental stress–strain relation for the matrix as follows:

$$\dot{S}^{(1)} = \mathbf{L}^{(1)}\dot{E}^{(1)} \quad \text{if } f < 0 \text{ or both } f = 0 \text{ and } \operatorname{dev}\big(E^{(1)} - E_{\mathrm{p}}^{(1)}\big) : \dot{E}^{(1)} \leqslant 0 \tag{10a}$$

$$\dot{S}^{(1)} = \left(\mathbf{L}^{(1)} - \frac{6\mu^{(1)}}{3 + k/\mu^{(1)}} M \otimes M\right)\dot{E}^{(1)} \quad \text{if both } f = 0 \text{ and } \operatorname{dev}\big(E^{(1)} - E_{\mathrm{p}}^{(1)}\big) : \dot{E}^{(1)} > 0 \tag{10b}$$

Note that the assumption $k > -3\mu^{(1)}$ does not exclude the description of softening elastoplastic materials (Nguyen and Bui [10]).

The fibers, called phase 2, are taken to be linearly elastic and isotropic. Thus, its stress–strain relation is simply given by

$$S^{(2)} = \mathbf{L}^{(2)}E^{(2)}, \quad \mathbf{L}^{(2)} = \lambda^{(2)} I \otimes I + 2\mu^{(2)}\mathbf{1} \tag{11}$$

where $\lambda^{(2)}$ and $\mu^{(2)}$ are the Lamé constants of the fibers.

3. Existence of uniform strain fields

Let Ω denote the closed domain occupied by a representative volume element of the two-phase material described above and let $\Omega^{(i)}$ stand for the corresponding closed sub-domain of phase i $(= 1, 2)$. The boundary of Ω is designated by $\partial\Omega$. If the fiber direction is described by a unit vector $\mathbf{n}$, then any unit vector $\mathbf{n}^{\perp}$ normal to the interface Γ between $\Omega^{(1)}$ and $\Omega^{(2)}$ is perpendicular to $\mathbf{n}$.

Consider the case where the plastic strain $E_{\mathrm{p}}^{(1)}$ and the isotropic hardening parameter α of phase 1 are uniform in the latter and their values are given and frozen. We look for a macroscopic strain $\overline{E}$ such that the homogeneous boundary displacement condition

$$\mathbf{u}(\mathbf{x}) = \overline{E}\mathbf{x} \quad \text{on } \partial\Omega \tag{12}$$

induces a uniform strain field over Ω. If $E^{(i)}$ is the resulting uniform strain field over $\Omega^{(i)}$, this means that

$$E^{(1)} = E^{(2)} = \overline{E} \tag{13}$$

It is known from Dvorak [6] and He [8] that such a uniform strain field exists if and only if the resulting stress fields $S^{(i)}$ over $\Omega^{(i)}$ $(i = 1, 2)$ satisfy the interface stress continuity condition, i.e.,

$$S^{(1)}\mathbf{n}^{\perp} = S^{(2)}\mathbf{n}^{\perp} \tag{14}$$

for any unit vector $\mathbf{n}^{\perp}$ perpendicular to $\mathbf{n}$. By means of the orthogonal projection operators $\mathbf{P}^{\perp}$ and $\mathbf{P}$ introduced by He [8] as

$$\mathbf{P}^{\perp}(\mathbf{n}) = \mathbf{1} - \mathbf{P}(\mathbf{n}) = \mathbf{1} - \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n} \tag{15}$$

(14) can be written in the equivalent but more convenient form:

$$\mathbf{P}^{\perp}(\mathbf{n})\big(S^{(1)} - S^{(2)}\big) = 0 \tag{16}$$

Using (2), (11) and (13) in (16) and accounting for the plastic incompressibilty condition $\operatorname{tr} E_{\mathrm{p}}^{(1)} = 0$ due to $(5)_1$, we obtain a system of 5 non-homogeneous linear equations with the 6 matrix components of $\overline{E}$ as unknowns:

$$\mathbf{P}^{\perp}\big[\hat{\lambda}(\operatorname{tr}\overline{E})I + 2\hat{\mu}\overline{E}\big] = 2\mu^{(1)}\mathbf{P}^{\perp}E_{\mathrm{p}}^{(1)} \tag{17}$$

where $\hat{\lambda} = \lambda^{(1)} - \lambda^{(2)}$ and $\hat{\mu} = \mu^{(1)} - \mu^{(2)}$. Clearly, the non-homogeneous linear system (17) admits an infinite number of solutions which result from the superposition of the solutions of the associated homogeneous linear system with $E_{\mathrm{p}}^{(1)} = 0$ to a particular solution of (17) with $E_{\mathrm{p}}^{(1)} \neq 0$. The existence of solutions to (17) implies the existence of uniform strain fields in the elastoplastic composite under consideration.

To specify the solutions to (17) in an explicit component way, let an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be chosen with $\mathbf{e}_1$ coinciding with the fiber direction $\mathbf{n}$. Then, all the solutions of (17) take the following form:

$$\overline{E}_{11} = \varepsilon, \qquad \overline{E}_{22} = \frac{-\hat{\lambda}\hat{\mu}\varepsilon + (2\hat{\mu} + \hat{\lambda})\mu^{(1)}E_{\mathrm{p}22}^{(1)} - \hat{\lambda}\mu^{(1)}E_{\mathrm{p}33}^{(1)}}{2\hat{\mu}(\hat{\mu} + \hat{\lambda})} \tag{18a}$$

$$\overline{E}_{33} = \frac{-\hat{\lambda}\hat{\mu}\varepsilon + (2\hat{\mu} + \hat{\lambda})\mu^{(1)}E_{\mathrm{p}33}^{(1)} - \hat{\lambda}\mu^{(1)}E_{\mathrm{p}22}^{(1)}}{2\hat{\mu}(\hat{\mu} + \hat{\lambda})}, \qquad \overline{E}_{ij} = \frac{\mu^{(1)}}{\hat{\mu}}E_{\mathrm{p}ij}^{(1)} \quad (i \neq j) \tag{18b}$$

where $\varepsilon \in \,]-\infty, \infty[$ is a strain control parameter which can vary arbitrarily. When $E_{ij}^{\mathrm{p}} = 0$, (18a) and (18b) reduce to

$$\overline{E} = \operatorname{diag}(\varepsilon, \eta\varepsilon, \eta\varepsilon) \quad \text{with } \eta = -\frac{\hat{\lambda}/2}{\hat{\mu} + \hat{\lambda}} \tag{19}$$

This uniform strain is axisymmetric about the fiber direction.

4. Determination of uniform strain loading paths

In the foregoing section, the plastic strain $E_{\mathrm{p}}^{(1)}$ and hardening parameter α of phase 1 are taken to be uniform, known and fixed; moreover, they are tacitly assumed to be compatible with the necessary and sufficient uniform strain condition (16). In fact, $E_{\mathrm{p}}^{(1)}$ and α are two unknowns and vary generally with the macroscopic strain $\overline{E}$. Consequently, the problem of determining uniform strain fields in an elastoplastic fibre-reinforced composite is fundamentally a nonlinear problem. Further, this problem cannot be generally solved analytically. In this section, we propose a modified return-mapping algorithm which allow us to numerically find out uniform strain loading paths when $E_{\mathrm{p}}^{(1)}$ and α change.

A comprehensive presentation of the classical return-mapping (or catching-up) algorithm of elastoplasticity can be found in the book of Simo and Hughes [11]. Compared with the classical problem of integrating an elastoplastic law, the problem of determining a uniform strain loading path includes the additional requirement that (17) must be satisfied. Hence, a macroscopic strain increment $\Delta\overline{E}_n$ at a generic step n cannot be imposed in an arbitrary way.

First, assume that, at step n, the total strain $\overline{E}_n$, the plastic strain $E^{(1)}_{\mathrm{p}n}$ and the hardening parameter α_n are uniform and completely defined. The corresponding phase stresses are calculated by using the formulae (2) and (11):

$$S^{(1)}_n = \mathbf{L}^{(1)}\big(\overline{E}_n - E^{(1)}_{\mathrm{p}n}\big), \qquad S^{(2)}_n = \mathbf{L}^{(2)}\overline{E}_n \tag{20}$$

Further, the criterion (3) and the condition (17) are assumed to be satisfied at step n, i.e.,

$$f_n \equiv 2\mu^{(1)}\big\|\mathrm{dev}\big(\overline{E}_n - E^{(1)}_{\mathrm{p}n}\big)\big\| - \sqrt{\frac{2}{3}}(k\alpha_n + \sigma_{\mathrm{e}}) \leqslant 0 \tag{21}$$

$$\mathbf{P}^{\perp}\big[\hat{\lambda}\big(\mathrm{tr}\,\overline{E}_n\big)I + 2\hat{\mu}\overline{E}_n\big] = 2\mu^{(1)}\mathbf{P}^{\perp}E^{(1)}_{\mathrm{p}n} \tag{22}$$

Next, we proceed to determine uniform strain increments $\Delta\overline{E}_n$ by solving the following nonlinear problem:

$$\mathbf{P}^{\perp}\big[\hat{\lambda}\big(\mathrm{tr}\,\Delta\overline{E}_n\big)I + 2\hat{\mu}\Delta\overline{E}_n\big] = \frac{\langle f^{\mathrm{trial}}_{n+1}\rangle_+}{1 + k/(3\mu^{(1)})}\mathbf{P}^{\perp}M_{n+1} \tag{23}$$

where

$$f^{\mathrm{trial}}_{n+1} \equiv 2\mu^{(1)}\big\|\mathrm{dev}\big(\overline{E}_n + \Delta\overline{E}_n - E^{(1)}_{\mathrm{p}n}\big)\big\| - \sqrt{\frac{2}{3}}(k\alpha_n + \sigma_{\mathrm{e}}) \tag{24}$$

$$M_{n+1} = \frac{\mathrm{dev}(\overline{E}_n + \Delta\overline{E}_n - E^{(1)}_{\mathrm{p}n})}{\|\,\mathrm{dev}(\overline{E}_n + \Delta\overline{E}_n - E^{(1)}_{\mathrm{p}n})\|} \tag{25}$$

These formulae are obtained by using the basic idea of the return-mapping algorithm (see, e.g., Moreau, [12], Simo and Hughes [11]) and accounting for the requirement (17). Finally, for any solution $\Delta\overline{E}_n$ to (23) together with (24) and (25), we can calculate the uniform strain, the plastic strain and the hardening parameter at step $n+1$:

$$\overline{E}_{n+1} = \overline{E}_n + \Delta\overline{E}_n \tag{26}$$

$$\Delta\gamma_{n+1} = \frac{\langle f^{\mathrm{trial}}_{n+1}\rangle_+}{2\mu^{(1)} + 2k/3}, \quad E^{(1)}_{\mathrm{p}(n+1)} = E^{(1)}_{\mathrm{p}n} + \Delta\gamma_{n+1}M_{n+1}, \quad \alpha_{n+1} = \alpha_n + \sqrt{\frac{2}{3}}\Delta\gamma_{n+1} \tag{27}$$

Thus, we can compute all uniform strain loading paths for the elastoplastic composite and the resulting plastic strain and hardening parameter of phase 1.

We remark that the initial assumption that the plastic strain tensor $E^{(1)}_{\mathrm{p}}$ and hardening parameter α of phase 1 are uniform in the latter is a posteriori justified by the fact that a uniform loading path results in uniform stresses and, consequently, uniform plastic strains in phase 1.

5. Microstructure-independent exact results

Denoting the volume average of the local stress tensor field by $\overline{S}$, the effective stiffness tensor by $\bar{\mathbf{L}}$, and the macroscopic plastic strain by $\overline{E}_{\mathrm{p}}$ (which is not the simple volume average of the local plastic strain field), the effective elastic stress–strain relation of the composite takes the form

$$\overline{S} = \bar{\mathbf{L}}(\overline{E} - \overline{E}_{\mathrm{p}}) \tag{28}$$

As no limitations are imposed on the transverse geometry and distribution of the fibres, the effective stiffness tensor $\bar{\mathbf{L}}$ is generally monoclinic with respect to a transverse plane of the fibres. Now, with the help of uniform strain fields, we proceed to establish exact relations between $\bar{\mathbf{L}}$, $\mathbf{L}^{(1)}$ and $\mathbf{L}^{(2)}$.

Let us define the orthogonal projection operator $\mathbf{Q}$ by

$$\mathrm{Ker}\big[\mathbf{P}^{\perp}\big(\mathbf{L}^{(1)}-\mathbf{L}^{(2)}\big)\big]=\{\overline{E}\colon\ \overline{E}=\mathbf{Q}E,\ E\in Sym\} \tag{29}$$

where Ker stands for the kernel of a linear operator. The complementary orthogonal projection operator $\mathbf{Q}^{\perp}$ is given by

$$\mathbf{Q}^{\perp}=\mathbf{1}-\mathbf{Q} \tag{30}$$

In fact, the operator $\mathbf{Q}$ characterizes the solutions of (17) with $E_{\mathrm{p}}^{(1)}=0$. By (19), we obtain

$$\mathbf{Q}=\mathbf{1}-\mathbf{Q}^{\perp}=H\otimes H\quad \text{with } H=\frac{1}{\sqrt{1+2\eta^2}}\,\mathrm{diag}(1,\eta,\eta) \tag{31}$$

Next, using a procedure owing to He and Bary [9] (see also Chen and Zheng [13]), we can show that there are the following exact relations:

$$\mathbf{Q}\big[\hat{\mathbf{L}}\big(\mathbf{Q}^{\perp}\hat{\mathbf{L}}\mathbf{Q}^{\perp}\big)^{-1}-\mathbf{1}\big]\big(\bar{\mathbf{L}}-\langle\mathbf{L}\rangle\big)\mathbf{Q}=\mathbf{0} \tag{32a}$$

$$\mathbf{Q}\big[\hat{\mathbf{L}}\big(\mathbf{Q}^{\perp}\hat{\mathbf{L}}\mathbf{Q}^{\perp}\big)^{-1}-\mathbf{1}\big]\big(\bar{\mathbf{L}}-\mathbf{L}^{(2)}\big)\mathbf{Q}^{\perp}=\mathbf{0} \tag{32b}$$

Above, $\hat{\mathbf{L}}$ is the difference $\hat{\mathbf{L}}=\mathbf{L}^{(1)}-\mathbf{L}^{(2)}$, $(\mathbf{Q}^{\perp}\hat{\mathbf{L}}\mathbf{Q}^{\perp})^{-1}$ is the inverse to be understood in the sense that $(\mathbf{Q}^{\perp}\hat{\mathbf{L}}\mathbf{Q}^{\perp})^{-1}(\mathbf{Q}^{\perp}\hat{\mathbf{L}}\mathbf{Q}^{\perp})=(\mathbf{Q}^{\perp}\hat{\mathbf{L}}\mathbf{Q}^{\perp})(\mathbf{Q}^{\perp}\hat{\mathbf{L}}\mathbf{Q}^{\perp})^{-1}=\mathbf{Q}^{\perp}$, $\langle\cdot\rangle$ is the volume average, and $\langle\mathbf{L}\rangle=c^{(1)}\mathbf{L}^{(1)}+c^{(2)}\mathbf{L}^{(2)}$ with $c^{(i)}$ being the volume fraction of phase i. We remark that the coordinate-free exact results (32a) and (32b) are identical to the relevant ones given by Dvorak [6] in matrix component forms.

Once a uniform strain loading path is determined by using the method presented in Section 3, we can exactly find out two points on the effective yield surface, even though the form of the latter is unknown. For simplicity, consider the particular case where the uniform loading path is axisymmetric with respect to the fibre direction $\mathbf{n}=\mathbf{e}_1$. Correspondingly, the components of $E_{\mathrm{p}}^{(1)}$ are such that $E_{\mathrm{p}22}^{(1)}=E_{\mathrm{p}33}^{(1)}$ and $E_{\mathrm{p}ij}^{(1)}=0$ for $i\neq j$, and Eqs. (18a) and (18b) reduce to

$$\overline{E}_{11}=\varepsilon,\quad \overline{E}_{22}=\overline{E}_{33}=\eta\varepsilon-\frac{\mu^{(1)}}{2(\hat{\mu}+\hat{\lambda})}\varepsilon_{\mathrm{p}},\quad \overline{E}_{ij}=0\quad (i\neq j) \tag{33}$$

where η is defined in Eq. (19) and $\varepsilon_{\mathrm{p}}=-2E_{\mathrm{p}22}^{(1)}=-2E_{\mathrm{p}33}^{(1)}$. Clearly, (33) includes (19) as a particular case. Next, using the formulae (2), (11) and (13) and accounting for the fact that $E_{\mathrm{p}11}^{(1)}=-(E_{\mathrm{p}22}^{(1)}+E_{\mathrm{p}33}^{(1)})=\varepsilon_{\mathrm{p}}$, we calculate the non-zero stress components of phases 1 and 2 as follows:

$$S_{11}^{(1)}=\left(2\mu^{(1)}+\frac{\hat{\mu}\lambda^{(1)}}{\hat{\mu}+\hat{\lambda}}\right)\varepsilon-\mu^{(1)}\left(2+\frac{\lambda^{(1)}}{\hat{\mu}+\hat{\lambda}}\right)\varepsilon_{\mathrm{p}} \tag{34a}$$

$$S_{11}^{(2)}=\left(2\mu^{(2)}+\frac{\hat{\mu}\lambda^{(2)}}{\hat{\mu}+\hat{\lambda}}\right)\varepsilon-\frac{\mu^{(1)}\lambda^{(2)}}{\hat{\mu}+\hat{\lambda}}\varepsilon_{\mathrm{p}} \tag{34b}$$

$$S_{22}^{(1)}=S_{33}^{(1)}=S_{22}^{(2)}=S_{33}^{(2)}=\frac{\mu^{(1)}\lambda^{(2)}-\mu^{(2)}\lambda^{(1)}}{\hat{\mu}+\hat{\lambda}}\varepsilon-\frac{\mu^{(1)}(\lambda^{(2)}+\mu^{(2)})}{\hat{\mu}+\hat{\lambda}}\varepsilon_{\mathrm{p}} \tag{34c}$$

The corresponding macroscopic stress tensor $\overline{S}$ is given by $\overline{S}=\langle S\rangle=c^{(1)}S^{(1)}+c^{(2)}S^{(2)}$, i.e.,

$$\overline{S}_{11}=\left(2\langle\mu\rangle+\frac{\hat{\mu}\langle\lambda\rangle}{\hat{\mu}+\hat{\lambda}}\right)\varepsilon-\left(2c^{(1)}\mu^{(1)}+\frac{\mu^{(1)}\langle\lambda\rangle}{\hat{\mu}+\hat{\lambda}}\right)\varepsilon_{\mathrm{p}} \tag{35a}$$

$$\overline{S}_{22}=\overline{S}_{33}=\frac{\mu^{(1)}\lambda^{(2)}-\mu^{(2)}\lambda^{(1)}}{\hat{\mu}+\hat{\lambda}}\varepsilon-\frac{\mu^{(1)}(\lambda^{(2)}+\mu^{(2)})}{\hat{\mu}+\hat{\lambda}}\varepsilon_{\mathrm{p}} \tag{35b}$$

$$\overline{S}_{ij}=0\quad \text{for } i\neq j. \tag{35c}$$

Introducing the expressions of the stress components $S_{ij}^{(1)}$ of phase 1 into the criterion (3) together with (4) and assuming that the elastic properties of phases 1 and 2 are well-ordered in the sense that

$$\hat{\lambda}\hat{\mu} = \left(\lambda^{(1)} - \lambda^{(2)}\right)\left(\mu^{(1)} - \mu^{(2)}\right) \geqslant 0 \tag{36}$$

we obtain

$$-\frac{(\hat{\mu}+\hat{\lambda})(k\alpha+\sigma_{\mathrm{e}})}{\mu^{(1)}(2\hat{\mu}+3\hat{\lambda})} + \left(1 - \frac{\mu^{(2)}}{2\hat{\mu}+3\hat{\lambda}}\right)\varepsilon_{\mathrm{p}} \leqslant \varepsilon \leqslant \frac{(\hat{\mu}+\hat{\lambda})(k\alpha+\sigma_{\mathrm{e}})}{\mu^{(1)}(2\hat{\mu}+3\hat{\lambda})} + \left(1 - \frac{\mu^{(2)}}{2\hat{\mu}+3\hat{\lambda}}\right)\varepsilon_{\mathrm{p}} \tag{37}$$

For any value of ε verifying (37), the corresponding macroscopic stress tensor calculated by (35a)–(35c) is inside or on the effective yield surface associated to a pair of values of ε_{p} and α which are compatible with the axisymmetric uniform loading path under consideration. In particular, when the minimum and maximum values of ε prescribed by (37) are taken, the formulae (35a)–(35c) give two macroscopic stress tensors $\overline{S}^{-}$ and $\overline{S}^{+}$ corresponding to two points on the effective yield surface. Precisely, the non-zero matrix components of $\overline{S}^{-}$ and $\overline{S}^{+}$ have the following expressions:

$$\overline{S}_{11}^{\pm} = \pm\frac{2\langle\mu\rangle(\hat{\mu}+\hat{\lambda}) + \langle\lambda\rangle\hat{\mu}}{\mu^{(1)}(2\hat{\mu}+3\hat{\lambda})}(k\alpha+\sigma_{\mathrm{e}}) - \mu^{(2)}\left(\frac{2\langle\mu\rangle+3\langle\lambda\rangle}{2\hat{\mu}+3\hat{\lambda}} - 2c^{(2)}\right)\varepsilon_{\mathrm{p}} \tag{38a}$$

$$\overline{S}_{22}^{\pm} = \overline{S}_{33}^{\pm} = \pm\frac{\mu^{(1)}\lambda^{(2)} - \mu^{(2)}\lambda^{(1)}}{\mu^{(1)}(2\hat{\mu}+3\hat{\lambda})}(k\alpha+\sigma_{\mathrm{e}}) - \frac{\mu^{(2)}(3\lambda^{(1)}+2\mu^{(1)})}{2\hat{\mu}+3\hat{\lambda}}\varepsilon_{\mathrm{p}} \tag{38b}$$

These formulae clearly show that ε_{p} affects the hardening of the composite. Setting $\varepsilon_{\mathrm{p}} = 0$ and $\alpha = 0$, the resulting stress tensors $\overline{S}^{-}$ and $\overline{S}^{+}$ given by (38a) and (38b) correspond to two points on the effective initial yield surface.

It should be emphasized that all foregoing results hold regardless of the transverse geometry and distribution of the fibres. In other words, these results are relevant only to the phase volume fractions and properties of the composite.

6. A numerical example

As an example of application, we consider a composite consisting of an aluminum matrix reinforced by boron fibres. The phase volume fractions and properties are given as follows:

- Aluminum: $c^{(1)} = 0.95$, $\lambda^{(1)} = 37$ MPa, $\mu^{(1)} = 21.1$ MPa, $\sigma_{\mathrm{e}} = 0.45$ MPa, $k = 1$ MPa;
- Boron: $c^{(2)} = 0.05$, $\lambda^{(2)} = 105.7$ MPa, $\mu^{(2)} = 158$ MPa.

Solving the nonlinear problem formulated by (23)–(25), an axisymmetric uniform strain loading path is found out and presented in Fig. 1(a). The resulting plastic strain ε_{p} along the fibre direction is illustrated in Fig. 1(b), and the corresponding macroscopic stress–strain relations are shown in Fig. 1(c) and (d).

In particular, for $\varepsilon_{\mathrm{p}} = \alpha = 0$, the two initial yield stress tensors S^{+} and S^{-} have the following numerical component values

$$S^{+} = \mathrm{diag}(0.757, 0.161, 0.161), \qquad S^{-} = \mathrm{diag}(-0.757, -0.161, -0.161)$$

When, for example, $\varepsilon_{\mathrm{p}} = \alpha = 0.03$, the numerical component values of the yield stress tensors S^{+} and S^{-} become

$$S^{+} = \mathrm{diag}(3.031, 1.685, 1.685), \qquad S^{-} = \mathrm{diag}(1.417, 1.341, 1.341)$$

Above, the unit of the stress components is MPa.

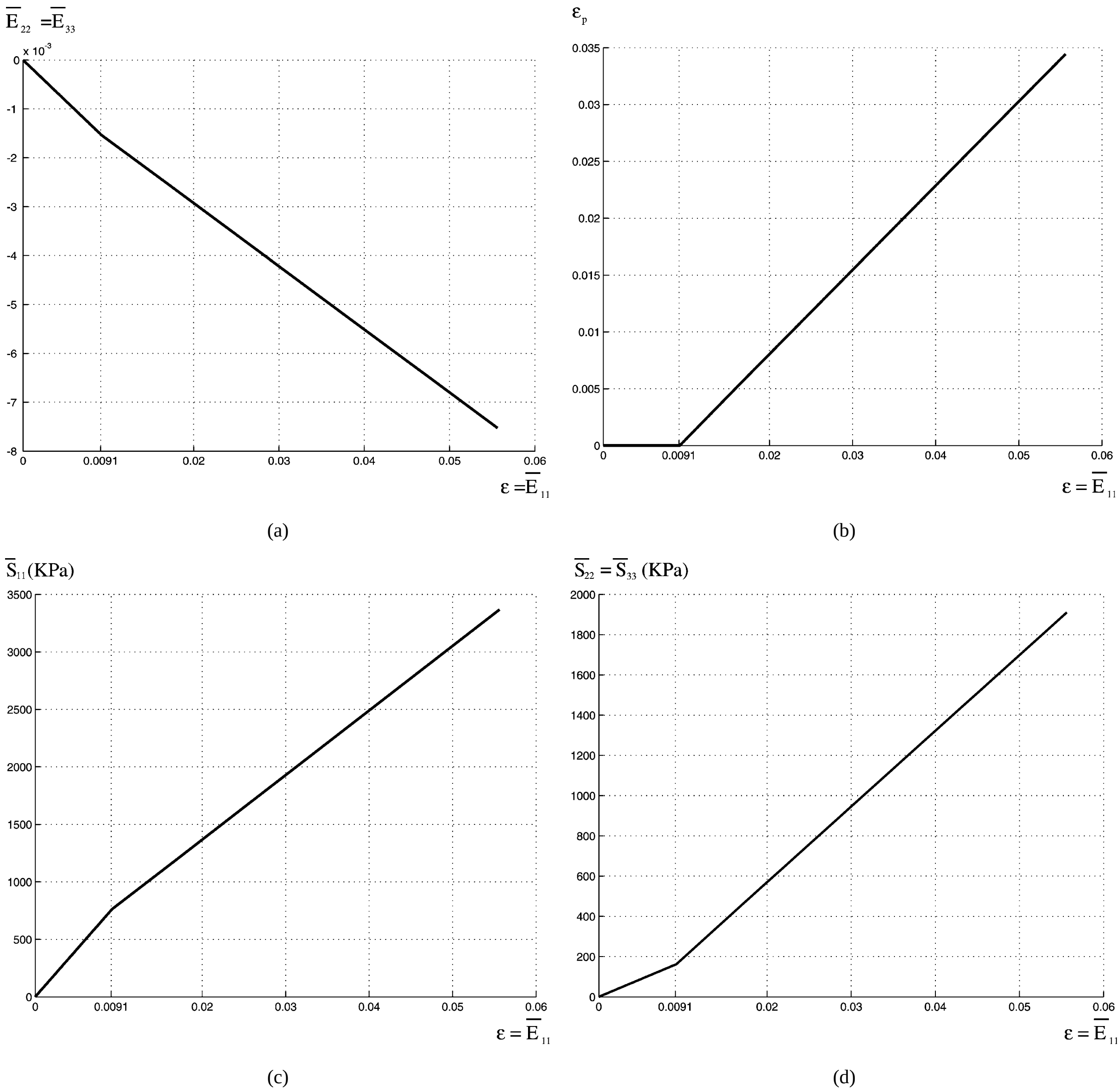

Fig. 1. (a) An axisymmetric uniform strain loading path; (b) the resulting plastic strain along the fibre direction; (c) the macroscopic stress–strain relation $\overline{S}_{11} - \overline{E}_{11}$; (d) the macroscopic stress–strain relation $\overline{S}_{22} - \overline{E}_{11}$ or $\overline{S}_{33} - \overline{E}_{11}$.

7. Final remarks

As in the case of a linearly or nonlinearly elastic fibre-reinforced composite, the existence of uniform strain fields in an elastoplastic fibre-reinforced composite is due to the fact that it is homogeneous along the fibre direction. However, since the behavior of the constituents of the latter is loading-path-dependent and non-smooth, the determination of uniform strain fields is much more difficult. In particular, the theorem of implicit functions, which is the key to extending the concept of uniform fields to smoothly nonlinear elastic heterogeneous materials (He [8], He and Bary [9]), is no longer applicable. The return-mapping based algorithm of Section 3 is an efficient way to find out loading paths generating uniform strain fields in the considered elastoplastic fibre-reinforced composite. This algorithm can be generalized to more complicated elastoplastic fibre-reinforced composites. In Section 4, a uniform loading path axisymmetric about the fibre direction was considered and exploited to derive exact results for the effective yield surface. The possibility of having non-axisymmetric uniform loading paths should be examined in a forthcoming work. In fact, for a linearly or nonlinearly elastic fibre-reinforced composite consisting of isotropic phases, the uniform strain fields can be only axisymmetric with respect to the fibre direction. It seems that this would not be the only possibility for an elastoplastic fibre-reinforced composite.

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