Available online at www.sciencedirect.com
SCIENCE DIRECT®

C. R. Mecanique 332 (2004) 565–570

COMPTES RENDUS MECANIQUE

Symmetry-breaking in periodic gravity waves with weak surface tension and gravity-capillary waves on deep water

Rabah Aider *, Mohammed Debiane

U.S.T.H.B, faculté de physique, BP 32, El Alia 16111, Bab Ezzouar, Alger, Algeria

Received 22 October 2003; accepted after revision 10 February 2004

Available online 5 May 2004

Presented by Paul Clavin

Abstract

The method developed by Debiane and Kharif for the calculation of symmetric gravity-capillary waves on infinite depth is extended to the general case of non-symmetric solutions. We have calculated non-symmetric steady periodic gravity-capillary waves on deep water. It is found that they appear via bifurcations from a family of symmetric waves. On the other hand we found that the symmetry-breaking bifurcation of periodic steady class 1 gravity wave on deep water is possible when it approaches the limiting profile, if it is very weakly influenced by surface tension effects. ***To cite this article: R. Aider, M. Debiane, C. R. Mecanique 332 (2004).***

Résumé

Brisure de symétrie dans les vagues de gravité avec une faible tension de surface et les vagues de gravité-capillarité périodiques en eau profonde. La méthode développée par Debiane et Kharif pour le calcul des ondes de gravité-capillarité symétriques en profondeur infinie a été étendue aux cas de vagues à profils non-symétriques. Nous avons calculé des ondes de gravité-capillarité non-symétriques périodiques et de formes permanentes. Elles apparaissent via des bifurcations à partir d'une solution symétrique. D'autre part, nous avons trouvé qu'en présence d'une très faible tension de surface, la brisure de symétrie d'une onde de gravité périodique de classe 1 en profondeur infinie est possible à l'approche de sa forme limite. ***Pour citer cet article : R. Aider, M. Debiane, C. R. Mecanique 332 (2004).***

Keywords: Fluid mechanics; Water waves; Bifurcation; Symmetry-breaking; Infinite depth

Mots-clés : Mécanique des fluides ; Vagues ; Bifurcation ; Brisure de symétrie ; Profondeur infinie

* Corresponding author.
E-mail address: aider_rabah@yahoo.fr (R. Aider).

1631-0721/$ – see front matter © 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.
doi:10.1016/j.crme.2004.03.003

1. Introduction

A wave is said to be symmetric when, if $f(x)$ represents the shape of the wave, the origin of the horizontal axis can be chosen such that $f(x) = f(-x)$. Levi-Cevita [1] has demonstrated that progressive gravity waves of permanent form of small amplitudes cannot be non-symmetric, validating Stokes conjecture [2]. For finite amplitude Longuet-Higgins [3] has shown numerically that a class 1 gravity waves (one crest in window of extent equal to a wavelength) have no asymmetric bifurcation, and hence no bifurcation at all, over a certain range of the amplitude, confirming the calculus of Chen and Saffman [4]. Zufiria [5] is the first to obtain non-symmetrical gravity wave on infinite depth. It appears via symmetry-breaking bifurcations from a family of symmetric waves. One of the methods he used is based on the quadratic relations between the Stokes coefficients discovered by Longuet-Higgins [6]. The other method is based on the Hamiltonian structure of the water-wave problem. He found that for gravity waves six is the minimum number of crests (which correspond to class 6 wave) that is needed to have non-symmetric waves. The purpose of this work is first to show that a class 1 gravity wave in deep water can have an asymmetric bifurcation when very weak surface tension effects are taken into account. On the other hand we demonstrate numerically the existence of and calculate non-symmetric gravity-capillary waves on deep water. It is found that they appear via the same bifurcation scenario as the one found by Zufiria for pure gravity waves. The family of solutions from which they bifurcate is a class 6 symmetric wave. The method we used consists of determining the Fourier coefficients in Stokes expansion through a set of integral relations derived by Longuet-Higgins [6] for gravity waves. Then, following Debiane and Kharif [7], the term due to surface tension is reduced to a simple function of the slope of the local tangent to the profile of the free surface. Finally a set of nonlinear algebraic equations is derived and solved by Newton's method. Keller's method [8] is used to switch to a bifurcated branch.

2. Formulation

We consider a bidimensional periodic gravity-capillary wave propagating with the velocity C on the surface of inviscid and incompressible fluid of infinite depth. The motion is assumed to be irrotational so that it can be represented by a velocity potential Φ. The free surface elevation is denoted by η. Units of length and time are chosen such that as the wave number and the body acceleration are equal to unity. In order to study waves of permanent form, we choose a Cartesian coordinate system (O, X, Y) moving with the velocity C in the horizontal direction OX. In this frame of reference the profile of non-symmetric wave of wavelength 2π can be represented by the parametric equations:

$$X(\varphi) = -\varphi - \sum_{n=1}^{\infty}\left[p_n \sin(n\varphi) + q_n \cos(n\varphi)\right], \quad \eta(\varphi) = \frac{p_0}{2} + \sum_{n=1}^{\infty}\left[p_n \cos(n\varphi) - q_n \sin(n\varphi)\right]$$
$$\text{with } \varphi = \frac{\Phi}{C} \tag{1}$$

Proceeding in the same way as Debiane and Kharif [7] we derive the following system of $2N + 1$ algebraic equations solved for the coefficients p_n $(n = 0, 1, 2, \ldots, N)$ and q_n $(n = 1, 2, \ldots, N)$:

$$\begin{cases} \Re_j(p_k, q_k) = p_j + \sum_{n=1}^{N} n p_n p_{|n-j|} + \sum_{n=1}^{N} n q_n q_{|n-j|} + j\kappa(c_j + s_j) = 0 \\ I_j(p_k, q_k) = q_j + \sum_{n=1}^{N} n q_n p_{|n-j|} - \sum_{n=1}^{N} n p_n q_{|n-j|} + j\kappa(r_j - d_j) = 0 \end{cases} \quad j = 1, 2, \ldots, N \tag{2}$$

$$Q = 1 + \eta(0) = 1 + \frac{p_0}{2} + \sum_{n=1}^{N} p_n \tag{3}$$

where $\kappa = k^2T/\rho g$ is the dimensionless capillary number (the inverse of the Bond number). Here ρg is the weight density, k the wave number and T the surface tension. Eq. (3) defines the continuation parameter Q, given by Longuet-Higgins [3], which is convenient for computing non-symmetric waves, since its behavior is monotonic. In the system (2) c_n, s_n, d_n and r_n are the Fourier coefficients of the function $\mathrm{e}^{\mathrm{i}\theta(\varphi)}$, where $\theta(\varphi) = \arctan(\eta_x)$ is the slope of the local tangent to the free surface:

$$\cos\theta + \mathrm{i}\sin\theta = \frac{c_0}{2} + \sum_{n=1}^{\infty}\big[c_n\cos(n\varphi) + s_n\sin(n\varphi)\big] + \mathrm{i}\left\{\frac{r_0}{2} + \sum_{n=1}^{\infty}\big[r_n\cos(n\varphi) + d_n\sin(n\varphi)\big]\right\} \tag{4}$$

The system (2), (3) can be solved by the iterative Newton's method. However, starting the iteration process by an infinitesimal profile leads to symmetric waves because the non-symmetric solution branches are not connected to the trivial solution. Nevertheless, the access to these branches can be achieved via bifurcation from symmetric ones.

The solutions describing the symmetric waves correspond to $q_n = 0$ for all n. Thus the coefficients r_n and d_n are all equal to zero and Eqs. (2) are reduced to:

$$p_j + \sum_{n=1}^{N} n p_n p_{|n-j|} + j\kappa(c_j + s_j) = 0, \quad j = 1, 2, \ldots, N \tag{5}$$

For this system one can use as parameter of continuation the parameter Q as well as the wave steepness ε, defined by:

$$\varepsilon = \frac{1}{2}\big(\eta(0) - \eta(\pi)\big) = \sum_{n=0}^{N} p_{2n+1} \tag{6}$$

The system (5), (6) is solved by Newton's method; the c_j and s_j are calculated by the algorithm of Fast Fourier Transform (F.F.T) and an infinitesimal wave is used as an initial iterate.

A property of the system (5) is that if a set $\{p_n, \kappa\}$ is a solution, then the set $\{p_n^{(m)}, \kappa'\}$ defined by:

$$\begin{aligned} &\kappa' = \frac{\kappa}{m^2}, \qquad p_0^{(m)} = \frac{p_0}{m} \\ &p_{nm}^{(m)} = \frac{p_n}{m}, \quad m, n \text{ positive integers}, \quad p_{nm+l}^{(m)} = 0, \quad l = 1, 2, \ldots, m-1 \end{aligned} \tag{7}$$

is also a solution. In reality the two sets describe the same wave in different scales. $\{p_n^{(m)}, \kappa'\}$ defines a class m wave, of wavelength L/m and which has the same steepness as the wave given by the set $\{p_n, \kappa\}$ of wavelength L. It presents m crests in horizontal window of extent L, and has the property that the mth harmonic is dominant.

3. Bifurcation

If we put $q_n = 0$ in the Jacobian matrix of the system (2), we obtain:

$$J(p_n, 0) = \begin{bmatrix} S(p_n, 0) & 0 \\ 0 & A(p_n, 0) \end{bmatrix} \quad \text{with } S = \frac{\partial \Re_j}{\partial p_n}(p_n, 0), \ A = \frac{\partial I_j}{\partial q_n}(p_n, 0) \tag{8}$$

A bifurcation from a family of symmetric solutions to a different family can take place if $J(p_n, 0)$ becomes singular. This can occur in two ways. The first is when the determinant of the sub-matrix S vanishes, and this may correspond to a bifurcation into another symmetric branch. The second way is that the bloc A becomes singular, in which case a bifurcation to a non-symmetric solution is possible. Once a bifurcation point is located, the tangent vector of a bifurcated branch is computed using the Keller's method [8].

4. Results

4.1. Gravity wave with surface tension

A branch of symmetric solutions of wavelength 2π is generated by solving the system of Eqs. (5), (6) with $\kappa = 3 \times 10^{-4}$, value which corresponds to a wavelength in water of 100 cm. The wave steepness ε is varied from 0.001 to 0.4435. The F.F.T algorithm used requires a number M, of intervals of the sampling of the variable φ, equal to a power of 2. For small amplitude $M = 1024$ ensures a good convergence. As the wave steepness increases, its crest develops a region of high curvature as the limiting wave is approached, and the influence of the surface tension manifests itself by the appearance of ripples. It ensues that the wave can be regarded as a class 1 gravity wave (Stokes wave) on which capillaries ride in the neighborhood of the crest (Fig. 1). The value $M = 4096$ is used for intermediate wave steepness and $M = 8192$ as the highest wave is approached. The latter was calculated by Debiane and Kharif [9] and is reached for $\varepsilon = 0.4439$. When we progress on the solution branch, we found that the matrix $S(p_n, 0)$ becomes singular for $\varepsilon = 0.4335$. This is not a bifurcation point but rather a limit point coinciding with a stationary wave speed. A second singularity have been detected in the neighborhood of the limiting wave, where the vanishing of the determinant of $A(p_n, 0)$ is located at $\varepsilon = 0.442297$. At this point the matrix $A(p_n, 0)$ has a null eigenvalue confirming the existence of a bifurcation into non-symmetric waves. The calculations at this critical point were carried out with $N = 702$ coefficients and a sampling $M = 16\,384$. The bifurcation persists for different pairs (N, M). For instance, for the combinations $(610, 8192)$ and $(502, 4096)$ the corresponding critical values of ε are 0.443205 and 0.443401. The bifurcating branch of non-symmetric solutions was followed by pseudo-arclength continuation. Fig. 2 shows the profile of a wave corresponding to this new family of solutions. Enlargements display the dissimilarity of the two sides of a crest, breaking the symmetry of the wave.

4.2. Gravity-capillary waves

Following Zufiria [5] who found, for pure gravity waves, that class 6 is the minimum class to have symmetry-breaking, we started computations with a class 6 wave. We then used the relations (7) with $m = 6$ and replaced

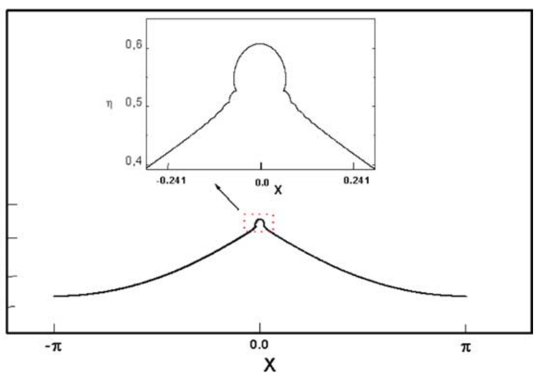

Fig. 1. Profile of periodic and symmetric gravity wave with weak surface tension on infinite depth.

Fig. 1. Profil d'une onde de gravité périodique et symétrique avec une faible tension de surface en profondeur infinie.

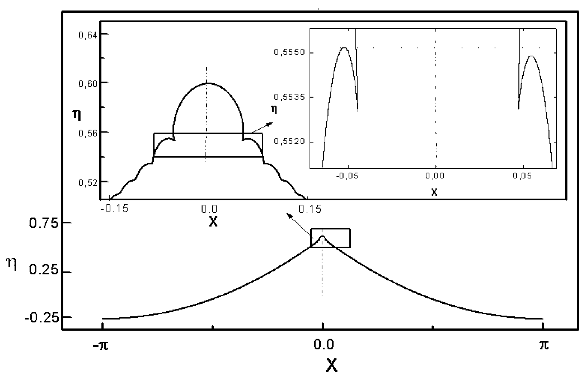

Fig. 2. Profile of non-symmetric gravity wave with weak surface tension on infinite depth.

Fig. 2. Profil d'une onde de gravité non-symétrique avec une faible tension de surface en profondeur infinie.

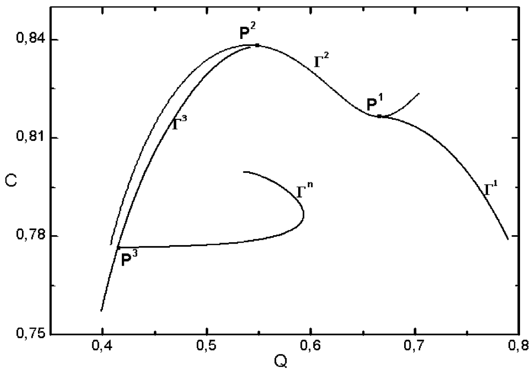

Fig. 3. Bifurcation diagram for periodic capillary-gravity waves on infinite depth for $\kappa = 1/12$.

Fig. 3. Diagramme de bifurcation d'ondes gravité-capillarité périodiques en profondeur infinie pour $\kappa = 1/12$.

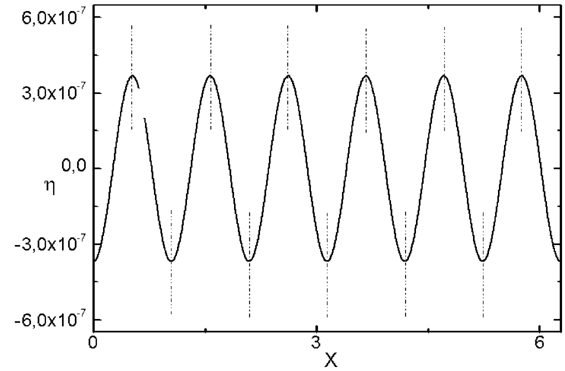

Fig. 4. Profile of periodic, symmetric and regular gravity-capillary wave of class 6 with wavelength $2\pi/6$.

Fig. 4. Profil d'une onde de gravité-capillarité de classe 6 périodique, symétrique et régulière de longueur d'onde $2\pi/6$.

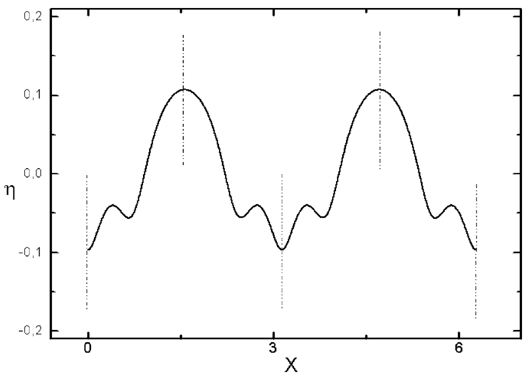

Fig. 5. Profile of periodic, symmetric and irregular gravity-capillary wave with wavelength π.

Fig. 5. Profil d'une onde de gravité-capillarité périodique, symétrique et irrégulière de longueur d'onde π.

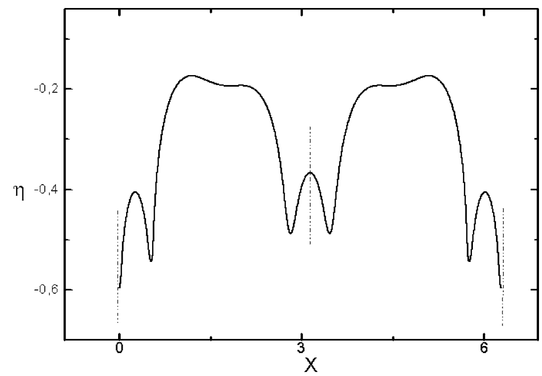

Fig. 6. Profile of periodic, symmetric and irregular gravity-capillary wave with wavelength 2π.

Fig. 6. Profil d'une onde de gravité-capillarité périodique, symétrique et irrégulière de longueur d'onde 2π.

p_k by $p_k^{(6)}$ in the Jacobian (8). Because of the type of structure that we expect in the bifurcation diagram we have chosen the value $\kappa = 1/12$ corresponding to (6, 2) mode interaction. No symmetry-breaking was found when the computations were made using $\kappa = 1/6$ and $\kappa = 1/18$, values corresponding to (6, 1) and (6, 3) modes interaction.

Because we shall deal in our computations with families of waves for which the six crests are not equal, ε is not the true wave steepness and so the solution branches were followed using the parameter Q. The bulk of the computations have been performed using $N = 30$ and $M = 1024$. The bifurcation diagram in (Q, C) plane is represented in Fig. 3. From the trivial solution, we have followed the branch, called Γ^1, of symmetric regular waves of class 6 and of wavelength $2\pi/6$. The corresponding wave profile, shown in Fig. 4, is symmetrical about all crests and all troughs. These waves are called regular because all the crests and the troughs are equal. Γ^1 has a singularity for $Q = 0.666$. The treatment of this point, denoted P^1 on Fig. 3, leads to a bifurcation into a new branch, which we call Γ^2, corresponding to waves of wavelength π. This is a period-tripling bifurcation resulting from the interaction of the harmonics 6 and 2. Their profiles are not regular but remain symmetric with respect to the principal crests and the principal troughs (Fig. 5). During the continuation along of Γ^2 another bifurcation point, denoted P^2, have been detected for $Q = 0.55$ and the branch which emanates from it, called Γ^3, is generated. The corresponding

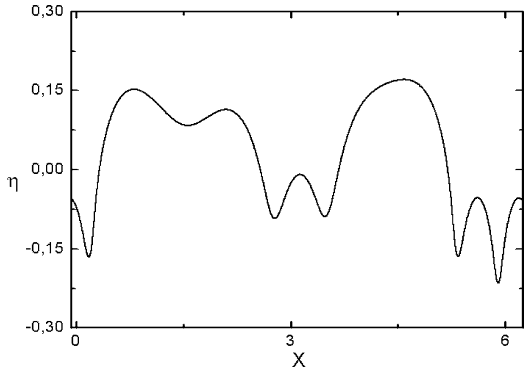

Fig. 7. Profile of periodic non-symmetric gravity-capillary wave on infinite depth.

Fig. 7. Profil d'une onde de gravité-capillarité périodique et non-symétrique en profondeur infinie.

waves are symmetric, irregular and of wavelength 2π. Their profile is shown in Fig. 6. The symmetries preserved by this period-doubling bifurcation are about one secondary crest and two principal troughs only. Taking Γ^3 in the direction of decreasing Q, a bifurcation to non-symmetric branch, denoted by Γ^n, is found at the point P^3 located at $Q = 0.4146$. Fig. 7 shows the shape of the wave described by this new solution. As can be seen the profile does not exhibit any symmetry axis. Note that the structure of bifurcation tree is the same as the one found by Zufiria for pure gravity waves.

References

[1] T. Levi-Cevita, Détermination rigoureuse des ondes permanentes d'ampleur finie, Math. Ann. 93 (1925) 264–314.

[2] G.G. Stokes, On the theory of oscillatory waves, Trans. Cambridge Philos. Soc. 8 (1847) 441–455.

[3] M.S. Longuet-Higgins, The bifurcation in gravity waves, J. Fluid Mech. 151 (1985) 457–475.

[4] B. Chen, P.G. Saffman, Numerical evidence for the existence of new type of gravity waves of permanent form on deep water, Stud. Appl. Math. 62 (1980) 1–21.

[5] J.A. Zufiria, Non-symmetric gravity waves on water of infinite depth, J. Fluid Mech. 181 (1987) 17–39.

[6] M.S. Longuet-Higgins, New integral relations for gravity waves of finite amplitude, J. Fluid Mech. 149 (1984) 457–475.

[7] M. Debiane, C. Kharif, A new way for the calculation of steady periodic capillary-gravity waves on deep water, Eur. J. Mech. B Fluids 16 (1997) 257–275.

[8] H.B. Keller, Numerical solution of bifurcation and nonlinear eigenvalue problems, in: Application of Bifurcation Theory, Academic Press, 1977, pp. 359–384.

[9] M. Debiane, C. Kharif, A new limiting form for steady periodic gravity waves with surface tension on deep water, Phys. Fluids 8 (1996) 2780–2782.